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OSTROWSKI TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR GENERALIZED (s, m, φ)-PREINVEX FUNCTIONS

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ABSTRACT. In the present paper, the notion of generalized (s, m, φ)-preinvex function is introduced and some new integral inequalities for the left hand side of Gauss-Jacobi type quadrature formula involving generalized (s, m, φ)-preinvex functions along with beta function are given. Moreover, some generalizations of Ostrowski type inequalities for generalized (s, m, φ)-preinvex functions via Riemann-Liouville fractional integrals are established.

Key words and phrases: Ostrowski? type inequality, Hölder's inequality, power mean inequality, Riemann-Liouville fractional integral, s-convex function in the second sense, m-invex, P-function.

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1. INTRODUCTION

The following notation is used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a generic n-dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_\circ = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following result is known in the literature as the Ostrowski inequality (see [11]) and the references cited therein, which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$.

Theorem 1.1. *Let $f : I \longrightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I° of I, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \le M$ for all $x \in [a, b]$, then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad \forall x \in [a, b]. \tag{1.1}$$

For other recent results concerning Ostrowski type inequalities (see [11]) and the references cited therein, also (see [12]) and the references cited therein.

Fractional calculus (see [10]) and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.1. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ with $a \ge 0$ are defined by

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u}u^{\alpha-1}du$. Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.
In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Ostrowski type inequalities for functions of different classes (see [10]) and the references cited therein.
Now, let us recall some definitions of various convex functions.

Definition 1.2. (see [2]) A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_\circ$ is said to be P-function or P-convex, if

$$f(tx + (1-t)y) \le f(x) + f(y), \quad \forall x, y \in I,\ t \in [0, 1].$$

Definition 1.3. (see [3]) A function $f : \mathbb{R}_\circ \longrightarrow \mathbb{R}$ is said to be s-convex in the second sense, if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.2}$$

for all $x, y \in \mathbb{R}_\circ,\ \lambda \in [0, 1]$ and $s \in (0, 1]$.

It is clear that a 1-convex function must be convex on $\mathbb{R}_\circ$ as usual. The s-convex functions in the second sense have been investigated in (see [3]).

Definition 1.4. (see [4]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true. For more details please see (see [4], [5]) and the references therein.

Definition 1.5. (see [6]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0, 1]$, we have that

$$f\left(x + t\eta(y, x)\right) \leq (1 - t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^\star |f|, \tag{1.3}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^\star |f|$ (see [7]).
Recently, Liu (see [8]) obtained several integral inequalities for the left hand side of (1.3) under the Definition 1.2 of P-function.
Also in (see [9]), Özdemir et al. established several integral inequalities concerning the left-hand side of (1.3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of generalized (s, m, φ)-preinvex function is introduced and some new integral inequalities for the left hand side of (1.3) involving generalized (s, m, φ)-preinvex functions along with beta function are given. In Section 3, some generalizations of Ostrowski type inequalities for generalized (s, m, φ)-preinvex functions via fractional integrals are given. In Section 4, some applications to special means are given.

2. NEW INTEGRAL INEQUALITIES FOR GENERALIZED (s, m, φ)-PREINVEX FUNCTIONS

Definition 2.1. (see [1]) A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

Remark 2.1. In Definition 2.1, under certain conditions, the mapping $\eta(y, x, m)$ could reduce to $\eta(y, x)$.

We next give new definition, to be referred as generalized (s, m, φ)-preinvex function.

Definition 2.2. Let $K \subseteq \mathbb{R}^n$ be an open m-invex set with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}^n$ and $\varphi : I \longrightarrow \mathbb{R}$ a continuous increasing function. For $f : K \longrightarrow \mathbb{R}$ $x, y \in K$ and some fixed $s, m \in (0, 1]$, if

$$f\left(m\varphi(x) + \lambda\eta(\varphi(y), \varphi(x), m)\right) \leq m(1 - \lambda)^s f(\varphi(x)) + \lambda^s f(\varphi(y)) \tag{2.1}$$

is valid for all $x, y \in K, \lambda \in [0, 1]$, then we say that $f(x)$ is a generalized (s, m, φ)-preinvex function with respect to η.

Remark 2.2. In Definition 2.2, it is worthwhile to note that the class of generalized (s,m,φ)-preinvex function is a generalization of the class of s-convex in the second sense function given in Definition 1.3.

In this section, in order to prove our main results regarding some new integral inequalities involving generalized (s,m,φ)-preinvex functions along with beta function, we need the following lemma:

Lemma 2.1. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous increasing function. Assume that $f : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow \mathbb{R}$ is a continuous function on the interval of real numbers K° with respect to $\eta : K\times K\times(0,1] \longrightarrow \mathbb{R}$, for $\varphi(a),\varphi(b)\in K$, $a<b$ and $m\varphi(a) < m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Then for some fixed $m\in(0,1]$ and $p,q>0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$=\eta(\varphi(b),\varphi(a),m)^{p+q+1}\int_0^1 t^p(1-t)^q f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt.$$

Proof. It is easy to observe that

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$=\eta(\varphi(b),\varphi(a),m)\int_0^1 (m\varphi(a)+t\eta(\varphi(b),\varphi(a),m)-m\varphi(a))^p$$

$$\times(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-m\varphi(a)-t\eta(\varphi(b),\varphi(a),m))^q f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt$$

$$=\eta(\varphi(b),\varphi(a),m)^{p+q+1}\int_0^1 t^p(1-t)^q f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt.$$

∎

The following definition will be used in the sequel.

Definition 2.3. The Euler Beta function is defined for $x,y>0$ as

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 2.2. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous increasing function. Assume that $f : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow \mathbb{R}$ is a continuous function on the interval of real numbers K° with $\varphi(a),\varphi(b)\in K$, $a<b$ with $m\varphi(a) < m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Let $k>1$. If $|f|^{\frac{k}{k-1}}$ is a generalized (s,m,φ)-preinvex function on an open m-invex set K with respect to $\eta : K\times K\times(0,1] \longrightarrow \mathbb{R}$ for some fixed $s,m\in(0,1]$, then for some fixed $p,q>0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \frac{|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}}{(s+1)^{\frac{k-1}{k}}}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}}\left(m|f(\varphi(a))|^{\frac{k}{k-1}}+|f(\varphi(b))|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.$$

Proof. Since $|f|^{\frac{k}{k-1}}$ is a generalized (s,m,φ)-preinvex function on K, combining with Lemma 2.1, Definition 2.3 and Hölder inequality for all $t\in[0,1]$ and for some fixed $s,m\in(0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 t^{kp}(1-t)^{kq}dt\right]^{\frac{1}{k}}\left[\int_0^1\left|f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))\right|^{\frac{k}{k-1}}dt\right]^{\frac{k-1}{k}}$$

$$\leq|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}}$$

$$\times\left[\int_0^1\left(m(1-t)^s|f(\varphi(a))|^{\frac{k}{k-1}}+t^s|f(\varphi(b))|^{\frac{k}{k-1}}\right)dt\right]^{\frac{k-1}{k}}$$

$$=\frac{|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}}{(s+1)^{\frac{k-1}{k}}}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}}\left(m|f(\varphi(a))|^{\frac{k}{k-1}}+|f(\varphi(b))|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.$$

∎

Theorem 2.3. *Let $\varphi: I\longrightarrow\mathbb{R}$ be a continuous increasing function. Assume that $f: K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow\mathbb{R}$ is a continuous function on the interval of real numbers K° with $\varphi(a),\varphi(b)\in K,\ a<b$ with $m\varphi(a)<m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Let $l\geq 1$. If $|f|^l$ is a generalized (s,m,φ)-preinvex function on an open m-invex set K with respect to $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $s,m\in(0,1]$, then for some fixed $p,q>0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$

$$\times\Big[m|f(\varphi(a))|^l\beta(p+1,q+s+1)+|f(\varphi(b))|^l\beta(p+s+1,q+1)\Big]^{\frac{1}{l}}.$$

Proof. Since $|f|^l$ is a generalized (s,m,φ)-preinvex function on K, combining with Lemma 2.1, Definition 2.3 and Hölder inequality for all $t\in[0,1]$ and for some fixed $s,m\in(0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$=\eta(\varphi(b),\varphi(a),m)^{p+q+1}\int_0^1\Big[t^p(1-t)^q\Big]^{\frac{l-1}{l}}\Big[t^p(1-t)^q\Big]^{\frac{1}{l}}f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt$$

$$\leq|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}}\left[\int_0^1 t^p(1-t)^q\left|f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))\right|^l dt\right]^{\frac{1}{l}}$$

$$\leq|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$

$$\times\left[\int_0^1 t^p(1-t)^q\left(m(1-t)^s|f(\varphi(a))|^l+t^s|f(\varphi(b))|^l\right)dt\right]^{\frac{1}{l}}$$

$$=|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$

$$\times\Big[m|f(\varphi(a))|^l\beta(p+1,q+s+1)+|f(\varphi(b))|^l\beta(p+s+1,q+1)\Big]^{\frac{1}{l}}.$$

∎

3. OSTROWSKI TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR GENERALIZED (s,m,φ)-PREINVEX FUNCTIONS

In this section, in order to prove our main results regarding some generalizations of Ostrowski type inequalities for generalized (s,m,φ)-preinvex functions via fractional integrals, we need the following lemma:

Lemma 3.1. *Let $\varphi: I \longrightarrow \mathbb{R}$ be a continuous increasing function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta: K\times K\times(0,1] \longrightarrow \mathbb{R}$ for some fixed $m\in(0,1]$ and let $\varphi(a),\varphi(b)\in K$, $a<b$ with $m\varphi(a) < m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Assume that $f: K \longrightarrow \mathbb{R}$ is a differentiable function on K° and f' is integrable on $[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$. Then for $\alpha>0$, we have*

$$\frac{\eta(\varphi(x),\varphi(a),m)^{\alpha}f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(b),\varphi(a),m)}$$
$$-\frac{\eta(\varphi(x),\varphi(b),m)^{\alpha}f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)}$$
$$-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}\Big[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a))-J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\Big]$$
$$=\frac{\eta(\varphi(x),\varphi(a),m)^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 t^{\alpha}f'(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))dt$$
$$-\frac{\eta(\varphi(x),\varphi(b),m)^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 t^{\alpha}f'(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))dt, \tag{3.1}$$

where $\Gamma(\alpha)=\int_0^{+\infty}e^{-u}u^{\alpha-1}du$ is the Euler Gamma function.

Proof. Denote

$$I=\frac{\eta(\varphi(x),\varphi(a),m)^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 t^{\alpha}f'(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))dt$$
$$-\frac{\eta(\varphi(x),\varphi(b),m)^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\int_0^1 t^{\alpha}f'(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))dt.$$

Integrating by parts, we get

$$I=\frac{\eta(\varphi(x),\varphi(a),m)^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\Bigg[t^{\alpha}\frac{f(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(x),\varphi(a),m)}\Bigg|_0^1$$
$$-\alpha\int_0^1\frac{t^{\alpha-1}f(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(x),\varphi(a),m)}dt\Bigg]$$
$$-\frac{\eta(\varphi(x),\varphi(b),m)^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\Bigg[t^{\alpha}\frac{f(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(x),\varphi(b),m)}\Bigg|_0^1$$
$$-\alpha\int_0^1\frac{t^{\alpha-1}f(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(x),\varphi(b),m)}dt\Bigg]$$

$$= \frac{\eta(\varphi(x),\varphi(a),m)^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\left[\frac{f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(x),\varphi(a),m)}\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{\eta(\varphi(x),\varphi(a),m)^{\alpha+1}}J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a))\right]$$
$$-\frac{\eta(\varphi(x),\varphi(b),m)^{\alpha+1}}{\eta(\varphi(b),\varphi(a),m)}\left[\frac{f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(x),\varphi(b),m)}\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{\eta(\varphi(x),\varphi(b),m)^{\alpha+1}}J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]$$
$$= \frac{\eta(\varphi(x),\varphi(a),m)^{\alpha}f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(b),\varphi(a),m)}$$
$$-\frac{\eta(\varphi(x),\varphi(b),m)^{\alpha}f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)}$$
$$-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}\left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a))-J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right].$$

∎

Remark 3.1. Clearly, if we choose $m = 1$ and $\eta(\varphi(x),\varphi(y),m) = \varphi(x) - m\varphi(y),\ \varphi(x) = x,\ \forall x, y \in K$ in Lemma 3.1, we get Lemma (see [11], Lemma 1).

By using Lemma 3.1, one can extend to the following results.

Theorem 3.2. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous increasing function. Suppose $A \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $s, m \in (0,1]$ and let $\varphi(a), \varphi(b) \in A$, $a < b$ with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b),\varphi(a),m)$. Assume that $f : A \longrightarrow \mathbb{R}$ is a differentiable function on A°. If $|f'|^q$ is a generalized (s,m,φ)-preinvex function on $[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, $q > 1$, $p^{-1}+q^{-1} = 1$ and $|f'| \leq M$, then for $\alpha > 0$, we have*

$$\left|\frac{\eta(\varphi(x),\varphi(a),m)^{\alpha}f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(b),\varphi(a),m)}\right.$$
$$-\frac{\eta(\varphi(x),\varphi(b),m)^{\alpha}f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)}$$
$$\left.-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}\left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a))-J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\right|$$
$$\leq \frac{M}{(1+p\alpha)^{1/p}}\left(\frac{m+1}{s+1}\right)^{\frac{1}{q}}\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\right]. \tag{3.2}$$

Proof. Suppose that $q > 1$. Using Lemma 3.1, Definition 2.3, generalized (s,m,φ)-preinvexity of $|f'|^q$, Hölder inequality, the fact that $|f'| \leq M$ and taking the modulus, we have

$$\left|\frac{\eta(\varphi(x),\varphi(a),m)^{\alpha}f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(b),\varphi(a),m)}\right.$$
$$-\frac{\eta(\varphi(x),\varphi(b),m)^{\alpha}f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}\left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a))-J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\Bigg|$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 t^{\alpha}|f'(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|dt$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 t^{\alpha}|f'(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|dt$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left[\int_0^1\left(m(1-t)^s|f'(\varphi(a))|^q+t^s|f'(\varphi(x))|^q\right)dt\right]^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left[\int_0^1\left(m(1-t)^s|f'(\varphi(b))|^q+t^s|f'(\varphi(x))|^q\right)dt\right]^{\frac{1}{q}}$$

$$\leq \frac{M}{(1+p\alpha)^{1/p}}\left(\frac{m+1}{s+1}\right)^{\frac{1}{q}}\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\right].$$

∎

Theorem 3.3. *Let $\varphi : I \longrightarrow \mathbb{R}$ be a continuous increasing function. Suppose $A \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $s, m \in (0,1]$ and let $\varphi(a), \varphi(b) \in A$, $a < b$ with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Assume that $f : A \longrightarrow \mathbb{R}$ is a differentiable function on A°. If $|f'|^q$ is a generalized (s, m, φ)-preinvex function on $[m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)]$, $q \geq 1$ and $|f'| \leq M$, then for $\alpha > 0$, we have*

$$\Bigg|\frac{\eta(\varphi(x),\varphi(a),m)^{\alpha}f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{\eta(\varphi(x),\varphi(b),m)^{\alpha}f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}\left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a))-J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\Bigg|$$

$$\leq \frac{M}{(1+\alpha)^{1-\frac{1}{q}}}\left(m\beta(\alpha+1,s+1)+\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}}$$

$$\times\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\right]. \tag{3.3}$$

Proof. Suppose that $q \geq 1$. Using Lemma 3.1, Definition 2.3, generalized (s,m,φ)-preinvexity of $|f'|^q$, the well-known power mean inequality, the fact that $|f'| \leq M$ and taking the modulus, we have

$$\Bigg|\frac{\eta(\varphi(x),\varphi(a),m)^{\alpha} f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(b),\varphi(a),m)} - \frac{\eta(\varphi(x),\varphi(b),m)^{\alpha} f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)}$$
$$-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}\left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a)) - J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\Bigg|$$
$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 t^{\alpha}|f'(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|dt + \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 t^{\alpha}|f'(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|dt$$
$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}|f'(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}|f'(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left[\int_0^1 t^{\alpha}\bigg(m(1-t)^s|f'(\varphi(a))|^q + t^s|f'(\varphi(x))|^q\bigg)dt\right]^{\frac{1}{q}}$$
$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left[\int_0^1 t^{\alpha}\bigg(m(1-t)^s|f'(\varphi(b))|^q + t^s|f'(\varphi(x))|^q\bigg)dt\right]^{\frac{1}{q}}$$
$$\leq \frac{M}{(1+\alpha)^{1-\frac{1}{q}}}\left(m\beta(\alpha+1,s+1)+\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}} \times\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+1}+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+1}}{|\eta(\varphi(b),\varphi(a),m)|}\right].$$

∎

4. APPLICATIONS TO SPECIAL MEANS

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 4.1. (see [13]) A function $M : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:

(1) Homogeneity: $M(ax,ay) = aM(x,y)$, for all $a > 0$,
(2) Symmetry: $M(x,y) = M(y,x)$,
(3) Reflexivity: $M(x,x) = x$,
(4) Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x,y) \leq M(x',y')$,
(5) Internality: $\min\{x,y\} \leq M(x,y) \leq \max\{x,y\}$.

We consider some means for arbitrary positive real numbers α, β $(\alpha \neq \beta)$.

(1) The arithmetic mean:
$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}$$
(2) The geometric mean:
$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}$$
(3) The harmonic mean:
$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}$$
(4) The power mean:
$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$
(5) The identric mean:
$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$
(6) The logarithmic mean:
$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln|\beta| - \ln|\alpha|}; \quad |\alpha| \neq |\beta|, \quad \alpha\beta \neq 0.$$
(7) The generalized log-mean:
$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}, \quad \alpha \neq \beta.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(\varphi(a), \varphi(b)) : [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \times [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means and $\varphi : I \longrightarrow \mathbb{R}$ be a continuous increasing function, therefore one can obtain various inequalities using the results of Section 3 for these means as follows:
Replace $\eta(\varphi(y), \varphi(x), m)$ with $\eta(\varphi(y), \varphi(x))$ and setting $\eta(\varphi(a), \varphi(b)) = M(\varphi(a), \varphi(b))$ for $m = 1$ in (3.2) and (3.3), one can obtain the following interesting inequalities involving means:

$$\left|\frac{M(\varphi(a), \varphi(x))^\alpha f(\varphi(a) + M(\varphi(a), \varphi(x))) - M(\varphi(b), \varphi(x))^\alpha f(\varphi(b) + M(\varphi(b), \varphi(x)))}{M(\varphi(a), \varphi(b))}\right.$$
$$\left. - \frac{\Gamma(\alpha+1)}{M(\varphi(a), \varphi(b))}\left[J^\alpha_{(\varphi(a)+M(\varphi(a),\varphi(x)))-} f(\varphi(a)) - J^\alpha_{(\varphi(b)+M(\varphi(b),\varphi(x)))-} f(\varphi(b))\right]\right|$$
$$\leq \frac{M}{(1+p\alpha)^{1/p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{M(\varphi(a), \varphi(x))^{\alpha+1} + M(\varphi(b), \varphi(x))^{\alpha+1}}{M(\varphi(a), \varphi(b))}\right], \tag{4.1}$$

$$\left|\frac{M(\varphi(a), \varphi(x))^\alpha f(\varphi(a) + M(\varphi(a), \varphi(x))) - M(\varphi(b), \varphi(x))^\alpha f(\varphi(b) + M(\varphi(b), \varphi(x)))}{M(\varphi(a), \varphi(b))}\right.$$
$$\left. - \frac{\Gamma(\alpha+1)}{M(\varphi(a), \varphi(b))}\left[J^\alpha_{(\varphi(a)+M(\varphi(a),\varphi(x)))-} f(\varphi(a)) - J^\alpha_{(\varphi(b)+M(\varphi(b),\varphi(x)))-} f(\varphi(b))\right]\right|$$

(4.2)
$$\leq \frac{M}{(1+\alpha)^{1-\frac{1}{q}}}\left(\beta(\alpha+1, s+1)+\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}}\left[\frac{M(\varphi(a), \varphi(x))^{\alpha+1}+M(\varphi(b), \varphi(x))^{\alpha+1}}{M(\varphi(a), \varphi(b))}\right].$$

Letting $M(\varphi(a), \varphi(b)) = A, G, H, P_r, I, L, L_p$ in (4.1) and (4.2), we get the inequalities involving means for a particular choice of a differentiable generalized $(s, 1, \varphi)$-preinvex functions f. The details are left to the interested reader.

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