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STRONG CONVERGENCE THEOREM FOR A COMMON FIXED POINT OF AN INFINITE FAMILY OF J-NONEXPANSIVE MAPS WITH APPLICATIONS

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ABSTRACT. Let E be a uniformly convex and uniformly smooth real Banach space with dual space E^*. Let $\{T_i\}_{i=1}^{\infty}$ be a family of J-nonexpansive maps, where, for each $i,\ T_i$ maps E to 2^{E^*}. A new class of maps, *J-nonexpansive maps* from E to E^*, an analogue of nonexpansive self maps of E, is introduced. Assuming that the set of common *J-fixed points* of $\{T_i\}_{i=1}^{\infty}$ is nonempty, an iterative scheme is constructed and proved to converge strongly to a point x^* in $\cap_{n=1}^{\infty} F_J(T_i)$. This result is then applied, in the case that E is a real Hilbert space to obtain a strong convergence theorem for approximation of a common fixed point for an infinite family of nonexpansive maps, assuming existences. The theorem obtained is compared with some important results in the literature. Finally, the technique of proof is also of independent interest.

Key words and phrases: J-Fixed points; J-Pseudocontractive map; Monotone map; Strong convergence.

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1. INTRODUCTION

Let E be a real normed space with dual space E^*. A map $J : E \to 2^{E^*}$ defined by

$$Jx := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|.\|x^*\|, \ \|x\| = \|x^*\|\}$$

is called the *normalized duality map* on E. If E^* is strictly convex, J is single-valued. If in addition, E is reflexive, the inverse of J, $J^{-1} : E^* \to E$ exists.

A map $A : E \to 2^{E^*}$ is called *monotone* if for each $x, y \in E$, the following inequality holds:

$$\langle \eta - \nu, x - y \rangle \geq 0 \ \forall \ \eta \in Ax, \ \nu \in Ay. \tag{1.1}$$

It is called *maximal monotone* if, in addition, the graph of A is not properly contained in the graph of any other monotone map. Also, A is maximal monotone if and only if it is monotone and $R(J + \lambda A) = E^*$, $\lambda > 0$.

Monotone maps were first studied in Hilbert spaces by Zarantonello [51], Minty [32], Kačurovskii [23] and a host of other authors. Interest in such maps stems mainly from their usefulness in applications. In particular, monotone maps appear in convex optimization theory. Consider, for example, the following.

Let H be a real Hilbert space and $h : H \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The *subdifferential* of h, $\partial h : H \to 2^H$, is defined for each $x \in H$ by

$$\partial h(x) = \{x^* \in H : h(y) - h(x) \geq \langle y - x, x^* \rangle \ \forall \ y \in H\}.$$

It is easy to check that ∂h is a *monotone operator* on H, and that $0 \in \partial h(u)$ *if and only if* u *is a minimizer of* h. Setting $\partial h \equiv A$, it follows that solving the inclusion $0 \in Au$, in this case, is solving for a minimizer of h.
We now take a brief look at the following class of maps which are closely related to monotone maps.
A map $A : E \to 2^E$ is called *accretive* if for each $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle \eta - \nu, j(x - y) \rangle \geq 0, \ \eta \in Ax, \ \nu \in Ay. \tag{1.2}$$

A is called *m-accretive* if, in addition, the graph of A is not properly contained in the graph of any other accretive operator. It is m-accretive if and only if A is accretive and $R(I + tA) = E$ for all $t > 0$.

In a real Hilbert space, the normalized duality map is the identity map, and so, in this case, inequality (1.2) and inequality (1.1) coincide. Hence, *in Hilbert spaces, accretivity and monotonicity coincide*.

Accretive operators have been studied extensively by numerous mathematicians (see e.g., the following monographs: Berinde [8], Browder [9], Chidume [14], Reich [37], and the references contained in them).

Accretive maps were introduced independently in 1967 by Browder [9] and Kato [25]. Interest in such maps stems mainly from their firm connection with the existence theory for nonlinear equations of evolution in real Banach spaces. Furthermore, it is known (see e.g., Zeidler [53]) that many physically significant problems can be modeled in terms of an initial-value problem of the form

$$0 \in \frac{du}{dt} + Au, \quad u(0) = u_0, \tag{1.3}$$

where A is a multi-valued accretive map on an appropriate real Banach space. Typical examples of such evolution equations are found in models involving the heat, wave or Schrödinger equations (see e.g., Browder [10], Zeidler [53]). Observe that in the model (1.3), if the solution u is independent of time (i.e., at the equilibrium state of the system), then $\frac{du}{dt} = 0$ and (1.3) reduces to

$$0 \in Au \tag{1.4}$$

whose solutions then correspond to the equilibrium state of the system described by (1.3). Solutions of equation (1.4) when A is accretive can also represent solutions of partial differential equations (see e.g., Benilan, Crandall and Pazy [7], Khatibzadeh and MoroÅ§anu [27], Khatibzadeh and Shokri [26], Showalter [43], Volpert [48], and so on).

In studying the equation $0 \in Au$, where A is a multi-valued accretive operator on a Hilbert space H, Browder introduced an operator T defined by $T := I - A$ where I is the identity map on H. He called such an operator *pseudo-contractive*. It is clear that solutions of $0 \in Au$, if they exist, correspond to fixed points of T.

Examples of pseudocontractive maps include nonexpansive maps. i.e., maps $T : K \to E$ such that $\|Tx - Ty\| \leq \|x - y\| \; \forall \, , x, y \in \; K$, where K in a nonempty subset of a real normed space, E.

Within the past 40 years or so, methods for approximating solutions of equation (1.4) when A is an accretive-type operator have become a flourishing area of research for numerous mathematicians. Several convergence theorems have been published in various Banach spaces and under various continuity assumptions. Many important theorems have been proved, thanks to geometric properties of Banach spaces developed from the mid 1980s to the early 1990s. The theory of approximation of solutions of the equation when A is of the accretive-type reached a level of maturity appropriate for an examination of its central themes. This resulted in the publication of monographs which presented in-depth coverage of the main ideas, concepts and most of the important results on iterative algorithms for approximation of fixed points of nonexpansive and pseudocontractive maps and their generalisations; approximation of zeros of accretive-type operators; iterative algorithms for solutions of Hammerstein integral equations involving accretive-type maps; iterative approximation of common fixed points (and common zeros) of families of these maps; solutions of equilibrium problems; and so on (see e.g., Agarwal *et al.* [1]; Berinde [8]; Chidume [14]; Reich [38]; Censor and Reich [13]; William and Shahzad [49], and the references contained in them). Typical of such theorems recently published is the following theorem.

Theorem 1.1 (Chidume, [15])**.** *Let E be a uniformly smooth real Banach space with modulus of smoothness ρ_E, and let $A : E \to 2^E$ be a multi-valued bounded $m-$accretive operator with $D(A) = E$ such that the inclusion $0 \in Au$ has a solution. For arbitrary $x_1 \in E$, define a sequence $\{x_n\}$ by,*

$$x_{n+1} = x_n - \lambda_n u_n - \lambda_n \theta_n (x_n - x_1), \; u_n \in Ax_n, \; n \geq 1,$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:
(i) $\lim_{n\to\infty}\theta_n = 0,\ \{\theta_n\}$ *is decreasing;* (ii) $\sum\lambda_n\theta_n = \infty;\ \sum\rho_E(\lambda_n M_1) < \infty$*, for some constant* $M_1 > 0$*;*
(iii) $\lim_{n\to\infty}\frac{\left[\frac{\theta_{n-1}}{\theta_n}-1\right]}{\lambda_n\theta_n} = 0$. *There exists a constant* $\gamma_0 > 0$ *such that* $\frac{\rho_E(\lambda_n)}{\lambda_n} \le \gamma_0\theta_n$. *Then, the sequence* $\{x_n\}$ *converges strongly to a zero of* A.

For nonexpansive maps, methods for approximating *a common fixed point* for a finite, infinite or countable family of nonexpansive maps, assuming existence, have been of interest to mathematicians. Some of the important theorems proved include the following.

Theorem BSK 1. *[Bauschke,* [5]*, Theorem 3.1] Let* K *be a nonempty closed convex subset of a Hilbert space* H *and* $T_1, T_2, ..., T_r$ *be a finite family of nonexpansive maps of* K *into itself with* $F := \cap_{i=1}^r Fix(T_i) \neq \emptyset$ *and* $F = Fix(T_r T_{r-1}...T_1) = Fix(T_1 T_r...T_2) = ... = F(T_{r-1}T_{r-2}...T_1T_r)$*. Let* $\{\lambda_n\}$ *be a real sequence in* $[0,1]$ *which satisfies* $C_1 : \lim\lambda_n = 0$*;* $C_2 : \sum\lambda_n = \infty$ *and* $C_3 : \sum_n|\lambda_n - \lambda_{n+r}| < \infty$*. Given points* $u, x_0 \in K$*, let* $\{x_n\}$ *be generated by*

$$x_{n+1} = \lambda_{n+1}u + (1-\lambda_{n+1})T_{n+1}x_n,\ n \ge 0, \tag{1.5}$$

where $T_n = T_{n \bmod r}$*. Then,* $\{x_n\}$ *converges strongly to* $P_F u$*, where* $P_F : H \to F$ *is the metric projection.*

A complementary result to this theorem of Bauschke, still in a Hilbert space was proved by O'Hara *et al* [36], where the condition C_3 was replaced by $C_4 : \lim\limits_{n\to\infty}\dfrac{\lambda_n}{\lambda_{n+r}} = 1$ or equivalently, $\lim\limits_{n\to\infty}\dfrac{\lambda_n - \lambda_{n+r}}{\lambda_{n+r}} = 0$. They proved the following theorem.

Theorem OPH 1 (O'Hara *et al.*, [36] Theorem 3.3)**.** *Let* $\{\lambda_n\} \subset (0,1)$ *satisfy* $\lim\lambda_n = 0$ *and* $\sum\lambda_n = \infty$*. Let* K *be a nonempty closed and convex subset of a Hilbert space* H *and let* $T_n : K \to K,\ n = 1, 2, ...$ *be nonexpansive maps such that* $F := \cap_{i=1}^\infty Fix(T_i) \neq \emptyset$*. Assume that* $V_1, V_2, ..., V_n : K \to K$ *are nonexpansive maps with the property: for all* $k = 1, 2, ..., N$ *and for any bounded subset* C *of* K*, there holds* $\lim\limits_{n\to\infty}\sup\limits_{x\in C}\|T_nx - V_k(T_nx)\| = 0$*. For* $x_0, u \in K$ *define*

$$x_{n+1} = \lambda_{n+1}u + (1-\lambda_{n+1})T_{n+1}x_n,\ n \ge 0. \tag{1.6}$$

Then, $x_n \to Pu$*, where* P *is the projection from* H.

Theorem OPH 2. *(O'Hara et al.,* [36]*, Theorem 3.3) Let* K *be a nonempty closed convex subset of a Hilbert space* H *and* $T_1, T_2, \cdots, T_N$ *be nonexpansive self-maps of* K *with* $F := \cap_{n=1}^N Fix(T_i) \neq \emptyset$*. Assume that*
$F = Fix(T_N\cdots T_1) = Fix(T_1T_N\cdots T_2) = \cdots = Fix(T_{N-1}T_{N-2}\cdots T_N)$*. Let* $\{\lambda_n\} \subset (0,1)$ *satisfy the following conditions:* (i) $\lim\limits_{n\to\infty}\lambda_n = 0$ (ii) $\sum\limits_{n=1}^\infty\lambda_n = \infty$ *and* (iii) $\lim\limits_{n\to\infty}\dfrac{\lambda_n}{\lambda_{n+N}} = 1$*.*
Given points $x_0, u \in K$*, the sequence* $\{x_n\}_{n=1}^\infty \subset K$ *is defined by*

$$x_{n+1} = \lambda_{n+1}u + (1-\lambda_{n+1})T_{n+1}x_n,\ n \ge 0. \tag{1.7}$$

Then, $x_n \to P_F u$*, where* P_F *is the projection of* u *onto* F.

Theorem S 1 (Suzuki [46])**.** *Let C be a convex compact subset of a Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be an infinite family of commuting nonexpansive maps on C. Fix $\lambda \in (0,1)$. Let $\{\alpha_n\}$ be a sequence in $[0, \frac{1}{2}]$ satisfying*

$$\liminf \alpha_n = 0, \quad \limsup \alpha_n > 0, and \lim(\alpha_{n+1} - \alpha_n) = 0.$$

Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$x_{n+1} = \lambda\Big(1 - \sum_{n=1}^{\infty} \alpha_n^k\Big) T_1 x_n + \lambda \sum_{n=2}^{\infty} \alpha_n^{k-1} T_k x_n + (1-\lambda)x_n \tag{1.8}$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a common fixed point of $\{T_n : n \in \mathbb{N}\}$.

In all the results involving *accretive-type* maps, or nonexpansive-type maps or *pseudo-contractive-type* maps, fixed point techniques are applicable because these operators *map a space into itself*.

Unfortunately, developing algorithms for approximating solutions of the very important case of equation (1.4) when $A : E \to 2^{E^*}$ is of *monotone-type* (for example, the case of the subdifferential, $\partial f : E \to 2^{E^*}$) has not been very fruitful. Part of the difficulty seems to be that fixed point techniques are not directly applicable because the operators involved map a space E to its dual space, E^*. Futhermore, the geometric properties of Banach spaces developed from the mid 1980s to the early 1990s which played a central role with accretive-type maps are not directly applicable when monotone-type maps from E to E^* are involved.

Fortunately, a new concept of fixed points for maps from a real normed space E to its dual, E^*, has now been introduced. Furthermore, Alber [2] (see also, Alber and Ryazantseva [4]) recently introduced a Lyapunov functional $\phi : E \times E \to \mathbb{R}$ which signalled the beginning of the development of new geometric properties of Banach spaces which are appropriate for studying iterative methods for approximating solutions of (1.4) when $A : E \to 2^{E^*}$ is of monotone-type. Geometric properties so far obtained have rekindled enormous research interest on iterative methods for approximating solutions of equation (1.4) where A is of the monotone-type, and other related problems (see e.g., Alber [2]; Alber and Guerre-Delabriere [3]; Chidume [15]; Chidume *et. al.* [17]; Diop *et. al.* [21]; Moudafi [34], Moudafi and Tera [35]; Reich [**?**]; Sow *et. al.* [45]; Takahashi [47]; Zegeye [52] and the references contained in them).

The following lemma will be needed in the sequel.

Lemma 1.2 (Alber, [4], p.45)**.** *Let X be a uniformly convex Banach space. Then, for any $R > 0$ and any $x, y \in X$ such that $\|x\| \leq R$, $\|y\| \leq R$, the following inequality holds:*

$$\langle Jx - Jy, x - y\rangle \geq (2L)^{-1}\delta_X(c_2^{-1}\|x-y\|),$$

where $c_2 = 2max\{1, R\}$, $1 < L < 1.7$.

In other to develop techniques analogous to the ones studied for accretive operators, the notion of$J-$*fixed point* of a map $T : E \to E^*$ has been introduced and studied (this notion has also been called *semi-fixed point* (Zegeye [52]), *duality fixed point* (Liu [30]).

A point $x^* \in E$ is called a $J-$*fixed point of* T if $Tx^* = J^*x$ and we denote by $F_J(T)$, the set of $J-$fixed points of T, i.e.,

$$F_J(T) := \{x \in E : Tx = Jx\}. \tag{1.9}$$

This is an analogue of the definition of a fixed point for a map T from a normed space E to itself.

Chidume and Idu in [19], studied a new class of maps called $J-$*pseudocontractions*.

A map $T : E \to E^*$ is called $J-$*pseudo contractive* if

$$\langle Tx - Ty, x - y\rangle \leq \langle Jx - Jy, x - y\rangle \; \forall\, x, y \in E. \tag{1.10}$$

This notion had been called *duality pseudocontractive* in Liu, [30].

Remark 1.1. In theorem 1.3 below, $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying the following conditions.

(i) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$;

(ii) $\lambda_n M_0^* \leq \gamma_0\theta_n$; $\delta_E^{-1}(\lambda_n M_0^*) \leq \gamma_0\theta_n$,

(iii) $\frac{\delta_E^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right)}{\lambda_n\theta_n} \to 0,\; \frac{\delta_{E^*}^{-1}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right)}{\lambda_n\theta_n} \to 0,$ as $n \to \infty$,

(iv) $\frac{1}{2}\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}K\right) \in (0,1)$,

for some constants $M_0^* > 0$, $K > 0$ and $\gamma_0 > 0$; where $\delta_E : (0,\infty) \to (0,\infty)$ is the modulus of convexity of E.

Real sequences that satisfy the conditions $(i)^* - (iv)^*$ are the following:

$$\lambda_n = (n+1)^{-a} \text{ and } \theta_n = (n+1)^{-b},\; n \geq 1, \tag{1.11}$$
$$0 < b < \frac{1}{p}\cdot a,\;\; a + b < 1/p.$$

For example, one can choose $a := \frac{1}{(p+1)}$ and $b := \frac{1}{2p(p+1)}$.

Verification that these choices satisfy conditions (i) to (ii) above can be found in Chidume and Idu, [19].

With these conditions, Chidume and Idu proved the following theorem.

Theorem 1.3 (Chidume and Idu, [19])**.** *Let E be a uniformly convex and uniformly smooth real Banach space and let E^* be its dual. Let $T : E \to 2^{E^*}$ be a J-pseudocontractive and bounded map such that $(J - T)$ is maximal monotone. Suppose $F_E^J(T) := \{v \in E : Jv \in Tv\} \neq \emptyset$. For arbitrary $x_1,\, u \in E$, define a sequence $\{x_n\}$ iteratively by:*

$$x_{n+1} = J^{-1}\left[(1-\lambda_n)Jx_n + \lambda_n\eta_n - \lambda_n\theta_n(Jx_n - Ju)\right],\; \eta_n \in Tx_n,\; n \geq 1,. \tag{1.12}$$

Then, the sequence $\{x_n\}$ converges strongly to a J-fixed point of T.

Theorem 1.3 is an analogue of theorem 1.1 for bounded *maximal monotone* maps which is also a complement of the *proximal point algorithm* of Martinet [31] and Rockafellar [42] which has also been studied by numerous authors (see e.g., Bruck [12]; Chidume [16]; Chidume [15]; Chidume and Djitte [18]; Kamimura and Takahashi [24]; Lehdili and Moudafi [28]; Reich [39]; Reich and Sabach [40, 41]; Solodov and Svaiter [44]; Xu [50] and the references contained in them). Furthermore, the authors applied this analogue to approximate solutions of Hammerstein integral equations and to convex optimization problems.

It is our purpose in this paper to first introduce the notion of *J-nonexpansive map* and then prove that if $T : E \to E^*$ is J-nonexpansive, then it is J-pseudocontractive. Furthermore, in the case that E is a uniformly convex and uniformly smooth real Banach space and $\{T_i\}_{i=1}^{\infty}$ is an infinite family of J-nonexpansive maps with a common J-fixed point, we construct an iterative sequence in E which converges strongly to some $x^* \in \cap_{i=1}^{\infty} F_J(T_i)$. Finally, this result is applied in the case that E is a real Hilbert space to obtain a convergence theorem for approximating a common fixed point for an infinite family of nonexpansive maps. Our theorem is then compared with some important results in the literature.

2. PRELIMINARIES

Let E be a real normed space of dimension ≥ 2. The *modulus of smoothness* of E , $\rho_E : [0, \infty) \to [0, \infty)$, is defined by:

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau,\ \tau > 0\right\}.$$

A normed space E is called *uniformly smooth* if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

It is well known (see *e.g.*, Chidume [14] p. 16, also Lindenstrauss and Tzafriri [29]) that ρ_E is nondecreasing. If there exist a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be q-*uniformly smooth*. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p\ (or\ l_p)\ or\ W_p^m \text{ is } \begin{cases} 2 - \text{uniformly smooth} & \text{if } \ 2 \leq p < \infty; \\ p - \text{uniformly smooth} & \text{if } \ 1 < p < 2. \end{cases}$$

A normed space E is said to be *strictly convex* if

$$\forall\, x, y \in E,\ \|x\| = \|y\| = 1, \quad x \neq y \implies \left\|\frac{x+y}{2}\right\| < 1.$$

A consequence of this is that, E is *strictly convex* if for any $R > 0$, we have

$$\forall\, x, y \in E,\ \|x\| = \|y\| = R, \quad x \neq y \implies \left\|\frac{x+y}{2}\right\| < R.$$

The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1;\ \epsilon = \|x-y\|\right\}.$$

The space E is *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. It is also well known (see *e.g.,* Chidume [14] p. 34, Lindenstrauss and Tzafriri [29]) that δ_E is nondecreasing. If there exist a constant $c > 0$ and a real number $p > 1$ such that $\delta_E(\epsilon) \geq c\epsilon^p$, then E is said to be p-*uniformly convex*. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p\ (or\ l_p)\ or\ W_p^m \text{ is } \begin{cases} p - \text{uniformly convex} & \text{if } \ 2 \leq p < \infty; \\ 2 - \text{uniformly convex} & \text{if } \ 1 < p < 2. \end{cases}$$

For $q > 1$, let J_q denote the *generalized duality map* from E to 2^{E^*} defined by:

$$J_q(x) := \left\{f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\right\},$$

where $\langle ., .\rangle$ denotes the generalized duality pairing. J_2 is called the *normalized duality map* and is denoted by J. It is well known that if E is smooth, then J_q is single-valued.

We now present the following definitions and lemmas which will be used in the sequel.

Definition 2.1. A map $T : E \to E^*$ is called J*-pseudo contractive* if

$$\langle Tx - Ty, x - y\rangle \leq \langle Jx - Jy, x - y\rangle \; \forall x, y \in E. \tag{2.1}$$

Definition 2.2. A map $T : E \to E^*$ is called J*-nonexpansive* if

$$\|Tx - Ty\|\|x - y\| \leq \langle Jx - Jy, x - y\rangle \; \forall \, x, y \in E. \tag{2.2}$$

Remark 2.1. We observe that if a map $T : E \to E^*$ is $J-$nonexpansive then T is $J-$pseudocontractive. For,

$$\langle Tx - Ty, x - y\rangle \leq \|Tx - Ty\|\|x - y\| \leq \langle Jx - Jy, x - y\rangle \; \forall \, x, y \in E,$$

and satisfies the inequality

$$\|Tx - Ty\| \leq \|Jx - Jy\| \; \forall x, y \in E. \tag{2.3}$$

Remark 2.2. If a map $T : E \to E^*$ is J-pseudocontractive, then the map $A := (J - T)$ is monotone. For

$$\langle Ax - Ay, x - y\rangle = \langle Jx - Jy, x - y\rangle - \langle Tx - Ty, x - y\rangle \geq 0 \; \forall \, x, y \in \; E.$$

In the sequel, we shall use the following important lemmas.

Lemma 2.1 (Cioranescu [20], corrollary 2.7 pg 156)**.** *Let $A : E \to E^*$ be monotone and semicontinuous with $D(A) = E$; then A is maximal monotone.*

Lemma 2.2 (Bruck, Jr., [11])**.** *Suppose E is strictly convex and $\{T_n\}$ is a sequence of nonexpansive maps $T_n : C \to E$, where C is a subset of E. Then, there exists a nonexpansive map $T : C \to E$ such that $F(T) = \cap_{n=1}^{\infty} F(T_n)$.*

3. MAIN RESULTS

Using the technique of Bruck [11], we prove the following lemma which will be central in the proof of our main theorem.

Lemma 3.1. *Let K be a closed convex nonempty subset of a uniformly smooth real Banach space E, and $T_n : K \to E^*$, $n = 1, 2, ...$ be a family of $J-$nonexpansive maps such that $\cap_{n=1}^{\infty} F_J(T_n) \neq \emptyset$. Define a map $T : K \to E^*$ by $Tx := \sum_{n=1}^{\infty} \beta_n T_n x$, where $\{\beta_n\} \in (0, 1)$ and $\sum_{n=1}^{\infty} \beta_n = 1$. Then,*

(a) *T is $J-$nonexpansive;*
(b) *The set of $J-$fixed points of T is equal to the set of common $J-$fixed points of $\{T_n\}_{n=1}^{\infty}$ i.e.,*

$$F_J(T) = \cap_{n=1}^{\infty} F_J(T_n).$$

(c) *$(J - T)$ is maximal montone.*

Proof. The map T is well defined since

$$\|T_n v\| \leq \|T_n v - Ju_0\| + \|Ju_0\| \leq \|Jv - Ju_0\| + \|Ju_0\|;$$

yielding that $\sum_{n=1}^{\infty} \beta_n T_n v$ converges absolutely in K.

(a) Using the J-nonexpansiveness of the $T_n's$ we obtain that

$$\begin{aligned}\|Tv-Tu\|\|v-u\| &= \Big\|\sum_{n=1}^{\infty}\beta_nT_nv-\sum_{n=1}^{\infty}\beta_nT_nu\Big\|\|v-u\|\\ &= \Big\|\sum_{n=1}^{\infty}\beta_n(T_nv-T_nu)\Big\|\|v-u\|\\ &\leq \sum_{n=1}^{\infty}\beta_n\|T_nv-T_nu\|\|v-u\| \leq \sum_{n=1}^{\infty}\beta_n\langle Jv-Ju,v-u\rangle\\ &= \langle Jv-Ju,v-u\rangle.\end{aligned}$$

So T is J-nonexpansive.

(b) The inclusion $\cap_{n=1}^{\infty}F_J(T_n)\subset F_J(T)$ is obvious. We prove the reverse. Let $u_0\in\cap_{n=1}^{\infty}F_J(T_n)$ and $v\in F_J(T)$. Then,

$$\|Jv-Ju_0\| = \|Tv-Ju_0\| = \Big\|\sum_{n=1}^{\infty}\beta_nT_nv-Ju_0\Big\| \tag{3.1}$$

$$= \Big\|\sum_{n=1}^{\infty}\beta_n(T_nv-Ju_0)\Big\| \leq \sum_{n=1}^{\infty}\beta_n\|T_nv-Ju_0\|. \tag{3.2}$$

But $T_nu_0=Ju_0$ and T_n is $J-$nonexpansive. Thus by inequality (2.3), we have that $\|T_nv-T_nu_0\|\leq\|Jv-Ju_0\|$. Since $\sum_{n=1}^{\infty}\beta_n=1$, equations (3.1) and (3.2) imply that,

$$\Big\|\sum_{n=1}^{\infty}\beta_n(T_nv-Ju_0)\Big\| = \|Jv-Ju_0\|, \text{ and} \tag{3.3}$$

$$\|(T_nv-Ju_0)\| = \|Jv-Ju_0\|\ \forall\, n\geq\ 1. \tag{3.4}$$

Now, the fact that E^* is strictly convex, each $\beta_n>0$ and $\sum_{n=1}^{\infty}\beta_n=1$, equations (3.3) and (3.4) imply that $T_nv-Ju_0=T_kv-Ju_0,\ \forall\, n,k\geq1$, so that $T_nv=T_kv\ \forall\, n,k\geq1$. Hence,

$$Jv=Tv=\sum_{n=1}^{\infty}\beta_nT_nv=\sum_{n=1}^{\infty}\beta_nT_kv=T_kv\ \forall\, k\geq1.$$

This implies that $v\in\cap_nF_J(T_n)$.

(c) We observe that since T is J-nonexpansive, it follows from Remark 2.1 that it is J-pseudocontractive and hence, by Remark 2.2, $(J-T)$ is monotone. Clearly, $(J-T)$ is continuous and is defined on the whole of E. Therefore, by Lemma 2.1, $(J-T)$ is maximal monotone.

∎

We now prove the following Theorem.

Theorem 3.2. *Let E be a uniformly convex and uniformly smooth real Banach space and let E^* be its dual. Let $\{T_i\}_{i=1}^{\infty},\ T_i:E\to2^{E^*}$ be a family of J-nonexpansive maps. Suppose*

$\cap_{i=1}^{\infty} F_J(T_i) \neq \emptyset$. *For arbitrary* $x_1,\ u \in E$, *define a sequence* $\{x_n\}$ *iteratively by:*

$$x_{n+1} = J^{-1}\left[(1-\lambda_n)Jx_n + \lambda_n Tx_n - \lambda_n\theta_n(Jx_n - Ju)\right],\ n \geq 1, \tag{3.5}$$

where $T := \sum_{i=1}^{\infty} \beta_i T_i,\ \{\beta_i\} \in (0,1),\ \sum_{i=1}^{\infty} \beta_i = 1;\ \{\lambda_n\}$ *and* $\{\theta_n\}$ *are sequences in* $(0,1)$ *satisfying the same conditions as in theorem 1.3. Then, the sequence* $\{x_n\}$ *converges strongly to some* $x^* \in \cap_{n=1}^{\infty} F_J(T_i)$.

Proof. From Lemma 3.1, T is J-nonexpansive and is hence bounded. Furthermore, from Remark (2.1), T is J-pseudocontractive. Moreover, from (c) of Lemma 3.1, $(J-T)$ is maximal monotone. Therefore, it follows from Theorem 1.3 that $\{x_n\}$ converges strongly to some $x^* \in F_J(T)$. But we know from condition (b) of Lemma 3.1 that $F_J(T) = \cap_{n=1}^{\infty} F_J(T_i)$, completing the proof. ∎

Corollary 3.3. *Let* H *be a real Hilbert space. Let* $\{T_i\}_{i=1}^{\infty},\ T_i : H \to 2^H$ *be a family of nonexpansive maps. Suppose* $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. *For arbitrary* $x_1,\ u \in H$, *define a sequence* $\{x_n\}$ *iteratively by:*

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n Tx_n - \lambda_n\theta_n(x_n - u),\ n \geq 1, \tag{3.6}$$

where $T := \sum_{i=1}^{\infty} \beta_i T_i,\ \{\beta_i\} \in (0,1),\ \sum_{i=1}^{\infty} \beta_i = 1;\ \{\lambda_n\}$ *and* $\{\theta_n\}$ *are sequences in* $(0,1)$ *satisfying the same conditions as in theorem 1.3. Then, the sequence* $\{x_n\}$ *converges strongly to some* $x^* \in \cap_{n=1}^{\infty} F(T_i)$.

Remark 3.1. We compare Corollary 3.3 with Theorem BSK 1, Theorem OPH 1, Theorem OPH 2 and Theorem S 1.

(i) In Theorem BSK 1, the recursion formular (1.5) will certainly require less computing time than the recursion formular (3.6) of Corollary 3.3. However, Theorem BSK 1 is proved for a finite family $\{T_i\}_{i=1}^{r}$ of nonexpansive maps and also under the condition that the family $\{T_i\}_{i=1}^{r}$ satisfies the following additional condition.

$$\cap_{i=1}^{r} Fix(T_i) = Fix(T_r T_{r-1} ... T_1) = Fix(T_1 T_r ... T_2) = ... = F(T_{r-1}T_{r-2}...T_1T_r).$$

(ii) In Theorem OPH 1, an infinite family $\{T_i\}_{i=1}^{\infty}$ of nonexpansive maps is studied. While the recursion formular (1.6) may require less computation time than the recursion formular (3.6) of Corollary 3.3, the theorem is proved under the additional condition that $V_1, V_2, ..., V_n : K \to K$ are nonexpansive maps with the property: for all $k = 1, 2, ..., N$ and for any bounded subset C of K, the following condition holds $\lim_{n\to\infty} \sup_{x\in C} \|T_n x - V_k(T_n x)\| = 0$.

(iii) In Theorem OPH 2, while the recursion formular (1.7) studied may require less computation time than the recursion formular (3.6) of Corollary 3.3, the theorem is proved for a finite family of nonexpansive maps, $\{T_i\}_{i=1}^{N}$, and under the additional assumption that $\cap_{n=1}^{N} Fix(T_i) = Fix(T_N \cdots T_1) = Fix(T_1 T_N \cdots T_2) = \cdots = Fix(T_{N-1}T_{N-2}\cdots T_N)$.

(iv) In Theorem S 1, an infinite family of nonexpansive maps is studied. The recursion formular (1.8) studied may require more computation time than the recursion formular (3.6) of Corollary 3.3. Furthermore, even though the theorem is proved in an arbitrary Banach space, the domain of the maps $T_i, i = 1, 2, ...$ is required to be *compact* and convex, and the family $\{T_i\}_{i=1}^{\infty}$ is also *commuting*.

Finally, given the fact that the parameters, λ_n and θ_n in Corollary 3.3 can easily be chosen as in (1.11), it is obvious that Corollary 3.3 is a welcome complement to Theorems BSK 1, OPH 1, OPH 2 and S 1 for providing algorithms for approximating common fixed points of families of nonexpansive maps defined on real Hilbert space.

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