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ROBUST ERROR ANALYSIS OF SOLUTIONS TO NONLINEAR VOLTERRA INTEGRAL EQUATION IN L^p SPACES

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ABSTRACT. In this paper, we propose a novel strategy for proving an important inequality for a contraction integral equations. The obtained inequality allows us to express our iterative algorithm using a "for loop" rather than a "while loop". The main tool used in this paper is the fixed point theorem in the Lebesgue space. Also, a numerical example shows the efficiency and the accuracy of the proposed scheme.

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1. INTRODUCTION

Solutions of integral equations play a major role in the many fields of science and engineering [14, 23]. Usually, physical events are modeled by a differential equation, an integral or an integro-differential equation, or combinations them [6, 10]. Since few of these equations can not be solved explicitly, it is often necessary to resort to numerical techniques [3, 18]. There are several numerical methods for solving integral equations, such as Galerkin's method [11], collocation method[7], Taylor series [19], Legendre wavelets [20, 29], Jacobi polynomials [16], homotopy perturbation [5, 13, 25], block-pulse functions [21], expansion [27, 28], and recently, Chebyshev polynomials [9]. On the other hand, investigations on existence theorems for diverse functional-integral equations have been presented in other references such as [1, 2, 5, 8, 12, 15, 17, 22, 24]. It seems that the method presented in this paper has a best stopping rule for iterative algorithm in integral equation comparison with other researches.

The paper is organized as follows. In Section 2, by using the weighted norm method, a contraction mapping is obtained. Thereafter in Section 3, by a simple technique, the stopping rule for our iterative algorithm has been introduced. Finally, we report numerical results and demonstrate the efficiency and the accuracy of the proposed scheme by considering a numerical example in Section 4.

In this paper, we intend to prove the existence and the uniqueness of solutions of the nonhomogeneous nonlinear Volterra integral equations of the form

$$x(t) = f(t) + \varphi\Big(\int_0^t F(t, s, x(s))ds\Big). \tag{1.1}$$

Here, we consider the following hypothesises:
(H.1) $f \in L^p(I, \mathbb{R})$ for $I := \{t \in \mathbb{R} : 0 \leq t \leq 1\}$ and $p > 1$,
(H.2) $F : T \times \mathbb{R} \to \mathbb{R}$ is measurable, where $T := \{(t, s) \in I \times I : t \leq s\}$.
We further assume that:
(H.3) the function $t \mapsto \int_0^t F(t, s, f(s))ds$ belongs to $L^p(I, \mathbb{R})$;
(H.4) $|F(t, s, x) - F(t, s, y)| \leq L(t, s)|x - y|$, $x, y \in \mathbb{R}, (t, s) \in T$, where L is a nonnegative and measurable function for which

$$M(t) := \Big(\int_0^t L^q(t, s)ds\Big)^{\frac{p}{q}}, t \in I, \frac{1}{p} + \frac{1}{q} = 1$$

exists and is integrable over I.
(H.5) φ is Lipschitz, that is, there exists $\alpha > 0$ such that for all $x, y \in \mathbb{R}$, $|\varphi(x) - \varphi(y)| \leq \alpha|x - y|$.

Remark 1.1. It is important to note that in the condition (H.5), the function φ is not necessary to be linear. For example $\varphi(x)$ can be chosen $\sin(x)$ or $\arctan(x)$.

There are many papers dealing with the existence and the uniqueness results for integral equations. However, these equations are usually discussed in the space of continuous functions [2, 12, 24]. In this note, we extend the Volterra integral equation and discuss its solutions in L^p spaces. To this end, we use the weighted norm method instead of the successive approximation method. In the reminder of this section, we recall some basic results which we will need in this paper.

Let $\omega : I \longrightarrow \mathbb{R}_+, \mathbb{R}_+ = (0, +\infty)$, be a continuous function.
Put

$$\|u\|_{p,\omega} = \Big(\sup \Big\{ \omega^{-1}(x) \int_0^x |u(s)|^p ds; x \in I \Big\} \Big)^{\frac{1}{p}}. \tag{1.2}$$

Note that for $\omega \equiv 1$ we obtain the classical norm $\|u\|_p$ which makes $L^p(I, \mathbb{R})$ become a Banach space. In general, it is easy to see that (1.2) defines a norm for any ω. Indeed, multiplying the Minkowski inequality by $\Big(\omega(x)\Big)^{\frac{-1}{p}}$ we obtain

$$\Big(\omega^{-1}(x) \int_0^x |u(s) + v(s)|^p ds \Big)^{\frac{1}{p}} \leq \Big(\omega^{-1}(x) \int_0^x |u(s)|^p ds \Big)^{\frac{1}{p}} + \Big(\omega^{-1}(x) \int_0^x |v(s)|^p ds \Big)^{\frac{1}{p}}.$$

Now, taking the supremum with respect to $x \in I$, we obtain the triangle inequality

$$\|u + v\|_{p,\omega} \leq \|u\|_{p,\omega} + \|v\|_{p,\omega}.$$

It is clear that $\|.\|_{p,\omega}$ has the other norm properties. Moreover, the inequality

$$c_1.\|u\|_p \leq \|u\|_{p,\omega} \leq c_2.\|u\|_p$$

is true for

$$c_1 = \Big(\sup\{\omega(x) : x \in I\} \Big)^{\frac{-1}{p}}, c_2 = \Big(\inf\{\omega(x) : x \in I\} \Big)^{\frac{-1}{p}}.$$

This means that the norm $\|.\|_{p,\omega}$ is equivalent to $\|.\|_p$.

2. A CONTRACTION MAPPING FOR THE INTEGRAL EQUATIONS IN $L^p(I, \mathbb{R})$

In this section, we prove that $\mathcal{F}$ defined by the right hand side of the equation (1.1), is a contraction with respect to the special norm $\|.\|_{p,\omega_\lambda}$ that is defined in the following theorem.

Theorem 2.1. *Consider equation (1.1) satisfiying hypothesises $(H.1) - (H.5)$. Let $\mathcal{F}$ be an operatoe defined by the right hand side of equation (1.1) which is a contraction in $L^p(I, \mathbb{R})$ with respect to the norm $\|.\|_{p,\omega_\lambda}$, where λ is sufficiently large and ω_λ is defined by*

$$\omega_\lambda(x) = exp\Big(\lambda \int_0^x M(s)ds\Big), M(s) = \Big(\int_0^s L^q(s,t)dt \Big)^{\frac{p}{q}}, \lambda > 1. \tag{2.1}$$

The equation (1) has a unique solution $u^* \in L^p(I, \mathbb{R})$, which is the limit in $L^p(I, \mathbb{R})$ of the sequence of iteration $\{\mathcal{F}^n u_0\}$, for any u_0 in $L^p(I, \mathbb{R})$.

Proof. First of all, one can observe that $\mathcal{F}(L^p(I,\mathbb{R})) \subset L^p(I,\mathbb{R})$. Now, by using Hölder inequality we obtain

$$\begin{aligned}\Big|\mathcal{F}u(t)-\mathcal{F}v(t)\Big|^p &\leq \Big|\varphi\Big(\int_0^t F(t,s,u(s))ds\Big)-\varphi\Big(\int_0^t F(t,s,v(s))ds\Big)\Big|^p\\ &\leq \alpha^p\Big|\Big(\int_0^t F(t,s,u(s))ds-\int_0^t F(t,s,v(s))ds\Big)\Big|^p\\ &\leq \alpha^p\Big(\int_0^t L(t,s)|u(s)-v(s)|ds\Big)^p\\ &\leq \alpha^p\Big(\int_0^t L^q(t,s)ds\Big)^{\frac{p}{q}}.\int_0^t |u(s)-v(s)|^p ds\\ &\leq \alpha^p M(t)\int_0^t |u(s)-v(s)|^p ds.\end{aligned}$$

Then, by integrating with respect to t we have

$$\begin{aligned}&\int_0^x \Big|\mathcal{F}u(t)-\mathcal{F}v(t)\Big|^p dt\\ &\leq \int_0^x \Big(\alpha^p M(t)\int_0^t |u(s)-v(s)|^p ds\Big)dt\\ &=\int_0^x \Big[\alpha^p M(t)\exp\Big(\lambda\int_0^t M(s)ds\Big)\exp\Big(-\lambda\int_0^t M(s)ds\Big)\int_0^t |u(s)-v(s)|^p ds\Big]dt\\ &\leq \alpha^p\|u-v\|^p_{p,\omega_\lambda}.\int_0^x M(t)\exp\Big(\lambda\int_0^t M(s)ds\Big)dt\\ &\leq \frac{\alpha^p}{\lambda}\|u-v\|^p_{p,\omega_\lambda}\exp\Big(\lambda\int_0^x M(s)ds\Big).\end{aligned}$$

Last inequality implies that

$$\exp\Big(-\lambda\int_0^x M(s)ds\Big).\int_0^x \Big|\mathcal{F}u(t)-\mathcal{F}v(t)\Big|^p dt \leq \frac{\alpha^p}{\lambda}\|u-v\|^p_{p,\omega_\lambda},$$

which means that

$$\|\mathcal{F}u-\mathcal{F}v\|^p_{p,\omega_\lambda} \leq \frac{\alpha^p}{\lambda}\|u-v\|^p_{p,\omega_\lambda},$$

and

$$\|\mathcal{F}u-\mathcal{F}v\|_{p,\omega_\lambda} \leq K_\lambda.\|u-v\|_{p,\omega_\lambda}, \tag{2.2}$$

with $K_\lambda = \sqrt[p]{\frac{\alpha^p}{\lambda}}$. It is clear that the operator $\mathcal{F}$ is a contraction in $L^p(I,\mathbb{R})$ if λ is sufficiently large. The final assertion of the theorem is an obvious consequence of Banach contraction mapping principle. ∎

3. MAIN RESULT

Let $p > 1$ be arbitrary. Suppose λ is the smallest positive integer number for which $\mathcal{F}$ is a contraction with respect to $\|.\|_{p,\omega_\lambda}$.

By using (2.2) for m times, $m \geq 1$ we obtain

$$\|\mathcal{F}^m u_1-\mathcal{F}^m u_2\|_{p,\omega_\lambda} \leq K_\lambda^m.\|u_1-u_2\|_{p,\omega_\lambda}.$$

Triangle inequality, yields

$$\|u_1-u_2\|_{p,\omega_\lambda} \leq \|u_1-\mathcal{F}u_1\|_{p,\omega_\lambda}+\|\mathcal{F}u_1-\mathcal{F}u_2\|_{p,\omega_\lambda}+\|\mathcal{F}u_2-u_2\|_{p,\omega_\lambda}.$$

So, we have

$$\|u_1 - u_2\|_{p,\omega_\lambda} \leq \frac{1}{1-K_\lambda}\Big(\|u_1 - \mathcal{F}u_1\|_{p,\omega_\lambda} + \|u_2 - \mathcal{F}u_2\|_{p,\omega_\lambda}\Big). \tag{3.1}$$

In particular, if u_1 and u_2 are the fixed points of $\mathcal{F}$, we get $\|u_1 - u_2\|_{p,\omega_\lambda} = 0$. This shows that the contraction mapping $\mathcal{F}$ has at most one fixed point. For any $u \in (L^p(I,\mathbb{R}), \|.\|_{p,\omega_\lambda})$, by letting $u_1 = \mathcal{F}^n u$ and $u_2 = \mathcal{F}^m u$ in (3.1) we find that

$$\begin{aligned}\|\mathcal{F}^n u - \mathcal{F}^m u\|_{p,\omega_\lambda} &\leq \frac{1}{1-K_\lambda}\Big(\|\mathcal{F}^n u - \mathcal{F}^n(\mathcal{F}u)\|_{p,\omega_\lambda} + \|\mathcal{F}^m u - \mathcal{F}^m(\mathcal{F}u)\|_{p,\omega_\lambda}\Big)\\ &\leq \frac{K_\lambda^n + K_\lambda^m}{1-K_\lambda}\|\mathcal{F}u - u\|_{p,\omega_\lambda}.\end{aligned}$$

Because $0 < K_\lambda < 1$, we have $K_\lambda^n \to 0$ as n tends to infinity. Hence $\|\mathcal{F}^n u - \mathcal{F}^m u\|_{p,\omega_\lambda} \to 0$ as n and m tend to infinity, that is $\mathcal{F}^n u$ is a cauchy sequence. Since $(L^p(I,\mathbb{R}), \|.\|_{p,\omega_\lambda})$ is a Banach space, there exists $u^* \in L^p(I,\mathbb{R})$ for which $\{\mathcal{F}^n u\}$ converges to u^*, which completes the proof.
The stopping rule : Now, by letting m tend to infinity in the abovementioned inequality, the following important inequality is obtained

$$\|\mathcal{F}^n u - u^*\|_{p,\omega_\lambda} \leq \frac{K_\lambda^n}{1-K_\lambda}\|\mathcal{F}u - u\|_{p,\omega_\lambda}. \tag{3.2}$$

Let us explain the importance of the inequality (3.2). Suppose we are willing to accept an error of ϵ, i.e., instead of the actual fixed point u^* of $\mathcal{F}$, we will be satisfied with a point $\mathcal{F}^n u$ satisfying $\|\mathcal{F}^n u - u^*\|_{p,\omega_\lambda} < \epsilon$, and suppose also that we start our iteration at some point u_0 in $L^p(I,\mathbb{R})$. Since we need $\|\mathcal{F}^n u_0 - u^*\|_{p,\omega_\lambda} < \epsilon$, we just must take N_λ so large that $\frac{K_\lambda^{N_\lambda}}{1-K_\lambda}\|\mathcal{F}u_0 - u_0\|_{p,\omega_\lambda} < \epsilon$. Now, the quantity $\|\mathcal{F}u_0 - u_0\|_{p,\omega_\lambda}$ is something that we can compute after the first iteration and we can then compute how large N_λ has to be by taking the log of the above inequality and solving for N_λ (note that $\log(K_\lambda)$ is negative). The result is

$$N_\lambda > \frac{\log(\epsilon) + \log(1-K_\lambda) - \log(\beta_\lambda)}{\log(K_\lambda)},$$

where $\beta_\lambda := \|\mathcal{F}u_0 - u_0\|_{p,\omega_\lambda}$. From a practical programming point of view, this inequality allows us to express our iterative algorithm with a "for loop" rather than a "while loop", but it has another interesting interpretation. Suppose we take $\epsilon = 10^{-m}$ in our stopping rule inequality. What we see is that the growth of N_λ with m is a constant plus $\frac{m}{|\log(K_\lambda)|}$, or in other words, to get one more decimal digit of precision we have to do (approximately) $\frac{1}{|\log(K_\lambda)|}$ more iteration steps. Stated differently, if we need N_λ iterative steps to get m decimal digits of precision, then we need another N_λ to double the precision to $2m$ digits.

4. A NUMERICAL EXAMPLE

In this section, we present an example of classical integral and functional equations which are particular cases of (1.1), and subsequently, for some initial guesses, the values of the parameters have been calculated.

Example 4.1. *(* [26]*) Consider the following linear Volterra integral equation*

$$u(t) = f(t) - \int_0^t \sin(2(t-s))u(s)ds, \quad t \in I. \tag{4.1}$$

The exact solution is

$$u(t) = f(t) - \frac{2}{\sqrt{6}} \int_0^t \sin(\sqrt{6}(t-s)) f(s) ds.$$

In particular, for $f(t) = \cos(t)$, this solution becomes $u^*(t) = 0.6\cos(t) + 0.4\cos(\sqrt{6}t)$. Now, by taking $\epsilon = 10^{-m}$, we guess that after N_λ iterative steps, m decimal digits of precision must be obtained. In Table 4.1, for some initial guesses u_0, the values of the parameters are calculated.

U_0	ϵ	p	q	λ	K_λ	β_λ	N_λ	$\|U^* - U_{N_\lambda}\|_{p,\omega_\lambda}$
$\cos(t)$	10^{-6}	2	2	50	0.1414	0.0050	5	$2.5003e-010$
t	10^{-4}	2	2	40	0.1581	0.0739	4	$3.3858e-006$
1	10^{-8}	2	2	30	0.1825	0.0939	10	$6.4212e-017$

Table 1: Numerical results for Example 4.1.

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