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# SOME INTERESTING PROPERTIES OF FINITE CONTINUOUS CESÀRO OPERATORS

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ABSTRACT. A complex scalar $\lambda$ is called an extended eigenvalue of a bounded linear operator $T$ on a complex Banach space if there is a nonzero operator $X$ such that $TX = \lambda XT$, the operator $X$ is called extended eigenoperator of $T$ corresponding to the extended eigenvalue $\lambda$.
In this paper we prove some properties of extended eigenvalue and extended eigenoperator for $C_1$ on $L^p([0,1])$, where $C_1$ is the Cesàro operator defined on the complex Banach spaces $L^p([0,1])$ for $1 < p < \infty$ by the expression

$$(C_1 f)(x) = \frac{1}{x}\int_0^x f(t)dt.$$

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## 1. INTRODUCTION

We represent by $B(E)$ the algebra of all bounded linear operators on a complex Banch space $E$. A complex scalar $\lambda$ is called an extended eigenvalue of an operator $T \in B(E)$ provided that there is a nonzero operator $X \in B(E)$ such that $TX = \lambda XT$, $X$ called an extended eigenoperator of $T$ corresponding to the extended eigenvalue $\lambda$. We represent by $\{T\}^{'}$ the commutant of the operator $T$, i.e. the set of operators that commute with $T$, or in other words, the family of all the extended eigenoperators for $T$ corresponding to the extended eigenvalue $\lambda = 1$.
Recently, the study of the extended eigenvalues for some classes of operators has received a considerable amount of attention[1, 2, 3]. The finite continuous Cesàro operator $C_1$ is defined on the complex Banach spaces $L^p([0,1])$ for $1 < p < \infty$ by the expression

$$(C_1 f)(x) = \frac{1}{x}\int_0^x f(t)dt \tag{1.1}$$

Brown, Halmos and Shields [4] proved in the Hilbertian case that $C_1$ is indeed a bounded linear operator, and they also proved that $I - C_1^*$ is unitarily equivalent to a unilateral shift of multiplicity one. Consider a unilateral shift of multiplicity one $S \in B\left(L^2\left[0,1\right]\right)$ and a unitary operator $U \in B\left(L^2\left[0,1\right]\right)$ such that $I - C_1^* = U^*SU$. We have $C_1 = U^*\left(I - S^*\right)U$, it follows that the extended eigenvalues of $C_1$ are precisely the extended eigenvalues of $I - S^*$, and the extended eigenoperators of $C_1$ are in one to one correspondence with the extended eigenoperators of $I - S^*$ under conjugation with $U$.
In [6], the authors proved $X_0$ is an extended eigenoperator for $C_1$ on $L^p([0,1])$ where $X_0$ is given by

$$(X_0 f)(x) = x^{(1-\lambda)/\lambda} f(x^{1/\lambda}), \quad 0 < \lambda \leq 1 \tag{1.2}$$

In this paper we prove some properties of extended eigenvalue and extended eigenoperator for $C_1$ on $L^p([0,1])$.

## 2. PRELIMINARIES

**Theorem 2.1.** [6] *if $0 < \lambda \leq 1$ then $\lambda$ is an extended eigenvalue for the Cesàro operator $C_1$ on $L^p([0,1])$ for $1 < p < \infty$ and a corresponding extended eigenoperator is the weighted composition operator $X_0 \in B(L^p[0,1])$ defined by* (1.2).

*Proof.* First of all, let us show that $X_0$ is a bounded linear operator.
We have for every $f \in L^p([0,1])$ for $1 < p < \infty$

$$\begin{aligned}\int_0^1 \left|(X_0 f)(x)\right|^p dx &= \int_0^1 x^{p(1-\lambda)/\lambda}\left|f(x^{1/\lambda})\right|^p dx \\ &= \lambda\int_0^1 y^{(p-1)(1-\lambda)}\left|f(y)\right|^p dy \leq \lambda\int_0^1 \left|f(y)\right|^p dy.\end{aligned}$$

Therefore, $X_0$ is bounded on $L^p([0,1])$ with $\|X_0\| \leq \lambda^{1/p}$.
Now let us show that $X_0$ is an extended eigenoperator of $C_1$ associated with the extended eigenvalue $\lambda$.
Let $n \in \mathbb{N}$ and notice that $X_0 x^n = x^{(n+1-\lambda)/\lambda}$, so that

$$\begin{aligned} C_1 X_0 x^n &= C_1 x^{(n+1-\lambda)/\lambda} \\ &= \frac{\lambda}{n+1} x^{(n+1-\lambda)/\lambda} \\ &= \frac{\lambda}{n+1} X_0 x^n \\ &= \lambda X_0 C_1 x^n \end{aligned}$$

and since the linear subspace span $\{x^n : n \in \mathbb{N}\}$ is a dense subset of $L^p([0,1])$, it follows that $C_1 X_0 = \lambda X_0 C_1$, that is $X_0$ is an extended eigenoperator of $C_1$ associated with the extended eigenvalue $\lambda$. ∎

**Theorem 2.2.** *Space $C_c^\infty(\Omega)$ of infinitely differentiable functions with compact support is dense in $L_p(\Omega)$ for $1 < p < \infty$.*

## 3. MAIN RESULTS

**Theorem 3.1.** *If $0 < \lambda \leq 1$ then $\lambda$ is an extended eigenvalue for the Cesàro operator $C_1$ on $L^p([0,1])$ for $1 < p < \infty$ and a corresponding extended eigenoperator is the weighted composition operator $X_k \in B(L^p[0,1]), k \in \mathbb{N}$ defined by*

$$(X_k f)(x) = x^{(\frac{1}{\lambda}+k-1)} \frac{d^k f(x^{\frac{1}{\lambda}})}{dx^k}, \tag{3.1}$$

*where $f$ is infinitely differentiable functions with compact support.*

*Proof.* First of all, let us show that $X_k$ is indeed a bounded linear operator. We have for every $f \in L^p([0,1])$ for $1 < p < \infty$

$$\begin{aligned} \int_0^1 |(X_k f)(x)|^p \, dx &= \int_0^1 \left| x^{(\frac{1}{\lambda}+k-1)} \frac{d^k f(x^{\frac{1}{\lambda}})}{dx^k} \right|^p dx \\ &= \int_0^1 x^{p(\frac{1}{\lambda}+k-1)} \left| \frac{d^k f(x^{\frac{1}{\lambda}})}{dx^k} \right|^p dx \\ &= \lambda \int_0^1 y^{(pk\lambda+(p-1)(1-\lambda))} \left| \frac{d^k f(y)}{dy^k} \right|^p dy \\ &\leq \lambda \int_0^1 \left| \frac{d^k f(y)}{dy^k} \right|^p dy \end{aligned}$$

and this shows that $X_k$ is bounded on $L^p([0,1])$ with $\|X_k\| \leq \lambda^{1/p}$.
Now let us show that $X_k$ is an extended eigenoperator of $C_1$ associated with the extended eigenvalue $\lambda$.
Let $n \in \mathbb{N}$ and notice that

$$\begin{aligned} X_k x^n &= x^{(\frac{1}{\lambda}+k-1)} \frac{d^k x^{\frac{n}{\lambda}}}{dx^k} \\ &= x^{(\frac{1}{\lambda}+k-1)} (\frac{n}{\lambda})(\frac{n}{\lambda}-1)....(\frac{n}{\lambda}-k+1) x^{\frac{n}{\lambda}-k} \\ &= (\frac{n}{\lambda})(\frac{n}{\lambda}-1)....(\frac{n}{\lambda}-k+1) x^{\frac{n+1-\lambda}{\lambda}}, \end{aligned}$$

so that

$$\begin{aligned} C_1X_kx^n &= C_1\left((\frac{n}{\lambda})(\frac{n}{\lambda}-1)....(\frac{n}{\lambda}-k+1)x^{\frac{n+1-\lambda}{\lambda}}\right) \\ &= \left((\frac{n}{\lambda})(\frac{n}{\lambda}-1)....(\frac{n}{\lambda}-k+1)\right)C_1x^{\frac{n+1-\lambda}{\lambda}} \\ &= \frac{\lambda}{n+1}\left((\frac{n}{\lambda})(\frac{n}{\lambda}-1)....(\frac{n}{\lambda}-k+1)\right)x^{\frac{n+1-\lambda}{\lambda}} \\ &= \frac{\lambda}{n+1}X_kx^n \\ &= \lambda X_k\frac{x^n}{n+1} \\ &= \lambda X_kC_1x^n, \end{aligned}$$

and since the linear subspace span $\{x^n : n \in \mathbb{N}\}$ is a dense subset of $L^p([0,1])$, it follows that $C_1X_k = \lambda X_kC_1$, that is $X_k, k \in \mathbb{N}$ is an extended eigenoperator of $C_1$ associated with the extended eigenvalue $\lambda$. ∎

**Example 3.1.** *Let us suppose that an operator $X_k$ defined by* (3.1)

(1) *If $k = 0$ then $(X_0f)(x) = x^{(1-\lambda)/\lambda}f(x^{1/\lambda})$, by definition.*
(2) *If $k = 1$ then $(X_1f)(x) = x^{1/\lambda}\frac{df(x^{1/\lambda})}{dx}$.*

*Let us show that $X_1$, is an extended eigenoperator of $C_1$ associated with the extended eigenvalue $\lambda$.*
*Let $n \in \mathbb{N}$ and notice that $X_1x^n = \frac{n}{\lambda}x^{\frac{n+1-\lambda}{\lambda}}$ so that*

$$\begin{aligned} C_1X_1x^n &= C_1\left(\frac{n}{\lambda}x^{\frac{n+1-\lambda}{\lambda}}\right) \\ &= \frac{\lambda}{n+1}\frac{n}{\lambda}x^{\frac{n+1-\lambda}{\lambda}} \\ &= \frac{\lambda}{n+1}X_1x^n \\ &= \lambda X_1C_1x^n \end{aligned}$$

*$X_1$ is an extended eigenoperator of $C_1$ associated with the extended eigenvalue $\lambda$.*

**Theorem 3.2.** *Let $Q$ be the square root of $X_0$ defined by* (1.2)*. If $0 < \lambda \leq 1$ then $\sqrt{\lambda}$ is an extended eigenvalue for the Cesàro operator $C_1$ on $L^p([0,1])$ for $1 < p < \infty$ and a corresponding extended eigenoperator is the weighted composition operator $Q \in B(L^p[0,1])$ defined by*

$$(Qf)(x) = x^{\frac{1-\lambda}{\lambda+\sqrt{\lambda}}}f(x^{\frac{1}{\sqrt{\lambda}}}). \tag{3.2}$$

*Proof.* First of all, let us show that $X_0 = Q^2$.
Let $n \in \mathbb{N}$ and notice that $X_0x^n = x^{(n+1-\lambda)/\lambda}$, so that

$$\begin{aligned} Qx^n = x^{\frac{1-\lambda}{\lambda+\sqrt{\lambda}}}x^{\frac{n}{\sqrt{\lambda}}} &= x^{\frac{(1-\lambda)\sqrt{\lambda}+n(\lambda+\sqrt{\lambda})}{\sqrt{\lambda}(\lambda+\sqrt{\lambda})}} \\ &= x^{\frac{(1-\sqrt{\lambda})(1+\sqrt{\lambda})\sqrt{\lambda}+n(\lambda+\sqrt{\lambda})}{\sqrt{\lambda}(\lambda+\sqrt{\lambda})}} \\ &= x^{\frac{n+1-\sqrt{\lambda}}{\sqrt{\lambda}}} \end{aligned}$$

And it

$$Qx^{\frac{n+1-\sqrt{\lambda}}{\sqrt{\lambda}}} = x^{\frac{1-\lambda}{\lambda+\sqrt{\lambda}}} x^{\frac{n+1-\sqrt{\lambda}}{\lambda}} = x^{\frac{(1-\lambda)\lambda+(n+1-\sqrt{\lambda})(\lambda+\sqrt{\lambda})}{\lambda(\lambda+\sqrt{\lambda})}}$$
$$= x^{\frac{(1-\sqrt{\lambda})(1+\sqrt{\lambda})\lambda+(n+1-\sqrt{\lambda})(\lambda+\sqrt{\lambda})}{\lambda(\lambda+\sqrt{\lambda})}}$$
$$= x^{\frac{(1-\sqrt{\lambda})(\lambda+\sqrt{\lambda})\sqrt{\lambda}+(n+1-\sqrt{\lambda})(\lambda+\sqrt{\lambda})}{\lambda(\lambda+\sqrt{\lambda})}}$$
$$= x^{(n+1-\lambda)/\lambda}.$$

And this shows $X_0 = Q^2$.
Also $Q$ is bounded on $L^p([0,1])$ with $\|Q\| \leq (\sqrt{\lambda})^{\frac{1}{p}}$.

Now we will demonstrate $Q$ is an extended eigenoperator of $C_1$ associated with the extended eigenvalue $\sqrt{\lambda}$.

We have $Qx^n = x^{\frac{n+1-\sqrt{\lambda}}{\sqrt{\lambda}}}$, so that

$$C_1Qx^n = C_1x^{\frac{n+1-\sqrt{\lambda}}{\sqrt{\lambda}}} = \frac{1}{x}\frac{1}{\frac{n+1-\sqrt{\lambda}}{\sqrt{\lambda}}+1}x^{\frac{n+1-\sqrt{\lambda}}{\sqrt{\lambda}}+1}$$
$$= \frac{\sqrt{\lambda}}{n+1}x^{\frac{n+1-\sqrt{\lambda}}{\sqrt{\lambda}}}$$
$$= \frac{\sqrt{\lambda}}{n+1}Qx^n$$
$$= \sqrt{\lambda}QC_1x^n.$$

The linear subspace span $\{x^n : n \in \mathbb{N}\}$ is a dense subset of $L^p([0,1])$, it follows that $C_1Q = \sqrt{\lambda}QC_1$, that is, $Q$ is an extended eigenoperator of $C_1$ associated with the extended eigenvalue $\sqrt{\lambda}$. ∎

**Theorem 3.3.** *Let $Q_k$ operator where $Q_k = X_0^{\frac{1}{k}}, k \in \mathbb{N}^*, X_0$ defined by* (1.2). *If $0 < \lambda \leq 1$ then $\lambda^{\frac{1}{k}}$ is an extended eigenvalue for the Cesàro operator $C_1$ on $L^p([0,1])$ for $1 < p < \infty$ and a corresponding extended eigenoperator is the weighted composition operator $Q_k \in B(L^p[0,1])$ defined by*

$$(Q_kf)(x) = (X_0^{\frac{1}{k}}f)(x) = x^{\frac{1-\lambda}{\lambda^{\frac{1}{k}}+\lambda^{\frac{2}{k}}+\dots+\lambda^{\frac{k}{k}}}} f(x^{\frac{1}{\lambda^{1/k}}}). \tag{3.3}$$

*Proof.* We have $Q_k$ bounded on $L^p([0,1])$ with $\|Q_k\| \leq (\lambda)^{\frac{1}{pk}}$.
In order to show $Q_k$ is an extended eigenoperator of $C_1$ associated with the extended eigenvalue $\lambda^{\frac{1}{k}}$, we have

$$Q_kx^n = x^{\frac{1-\lambda}{\lambda^{\frac{1}{k}}+\lambda^{\frac{2}{k}}+\dots+\lambda^{\frac{k}{k}}}} x^{\frac{n}{\lambda^{\frac{1}{k}}}} = x^{\frac{(1-\lambda)\lambda^{1/k}+n(\lambda^{1/k}+\lambda^{2/k}+\dots+\lambda)}{\lambda^{1/k}(\lambda^{1/k}+\lambda^{2/k}+\dots+\lambda)}}$$
$$= x^{\frac{(1-\lambda)\lambda^{1/k}+n\lambda^{1/k}\frac{1-\lambda}{1-\lambda^{1/k}}}{\lambda^{1/k}\lambda^{1/k}\frac{1-\lambda}{1-\lambda^{1/k}}}}$$
$$= x^{\frac{(1-\lambda^{1/k})(1-\lambda)\lambda^{1/k}+n(1-\lambda)\lambda^{1/k}}{\lambda^{1/k}(1-\lambda)\lambda^{1/k}}}$$
$$= x^{\frac{n+1-\lambda^{1/k}}{\lambda^{1/k}}},$$

such that

$$\begin{aligned} C_1Q_kx^n = C_1x^{\frac{n+1-\lambda^{1/k}}{\lambda^{1/k}}} &= \frac{1}{x}\frac{1}{\frac{n+1-\lambda^{1/k}}{\lambda^{1/k}}+1}x^{\frac{n+1-\lambda^{1/k}}{\lambda^{1/k}}+1} \\ &= \frac{\lambda^{1/k}}{n+1}x^{\frac{n+1-\lambda^{1/k}}{\lambda^{1/k}}} \\ &= \frac{\lambda^{1/k}}{n+1}Q_kx^n \\ &= \lambda^{1/k}Q_kC_1x^n. \end{aligned}$$

The linear subspace span $\{x^n : n \in \mathbb{N}\}$ is a dense subset of $L^p([0,1])$, it follows that $C_1Q_k = \lambda^{1/k}Q_kC_1$, that is, $Q_k$ is an extended eigenoperator of $C_1$ associated with the extended eigenvalue $\lambda^{1/k}$. ∎

**Theorem 3.4.** *Let $Q_{qk}$ operator where $Q_{qk} = X_0^{\frac{q}{k}}, q \in \mathbb{N}, k \in \mathbb{N}^*, X_0$ defined by* (1.2). *If $0 < \lambda \leq 1$ then $\lambda^{\frac{q}{k}}$ is an extended eigenvalue for the Cesàro operator $C_1$ on $L^p([0,1])$ for $1 < p < \infty$ and a corresponding extended eigenoperator is the weighted composition operator $Q_{qk} \in B(L^p[0,1])$ defined by*

$$(Q_{qk}f)(x) = (X_0^{\frac{q}{k}}f)(x) = (Q_k^qf)(x), \tag{3.4}$$

*where $Q_k$ is defined by* (3.3).

*Proof.* We have $Q_k$ is an extended eigenoperator of $C_1$ associated with the extended eigenvalue $\lambda^{1/k}$. Therefore,

$$\begin{aligned} C_1Q_k = \lambda^{1/k}Q_kC_1 &\Rightarrow C_1Q_k^q = \lambda^{q/k}Q_k^qC_1 \\ &\Rightarrow C_1Q_{qk} = \lambda^{q/k}Q_{qk}C_1. \end{aligned}$$

Thus, $Q_{qk}$ is an extended eigenoperator of $C_1$ associated with the extended eigenvalue $\lambda^{q/k}$. ∎

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