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EXAMPLES OF FRACTALS SATISFYING THE QUASIHYPERBOLIC BOUNDARY CONDITION

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ABSTRACT. In this paper we give explicit examples of bounded domains that satisfy the quasihyperbolic boundary condition and calculate the values for the constants. These domains are also John domains and we calculate John constants as well. The authors do not know any other paper where exact values of parameters has been estimated.

Key words and phrases: Quasihyperbolic boundary condition; John constant; Fractal.

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1. INTRODUCTION

Domains that satisfy the quasihyperbolic boundary condition with a constant $\beta \in (0,1]$ (see Definition 2.1) were introduced Gehring and Martio in [3] and after that they have been studied intensively. The constant β plays a crucial role in these studies and many properties have been proved in the terms of it. For example in [9] Koskela and Rohde showed that the Minkowski dimension of the boundary of the domain is at most $d - c\beta^{d-1}$, where d is the dimension of the boundary of the domain and the constant c depends only on the dimension d. Another example is the paper [5] by Hurri-Syrjänen, Marola and Vähäkangas, where the Poincaré inequality is stated in terms of β. However, there seems to be very few examples where the exact value for β is known. In fact the authors do not know any nontrivial example with exact constants.

John domains form a proper subclass of domains that satisfy the quasihyperbolic boundary condition [3, Lemma 3.11]. They were originally introduced in [6] but the more intensive studies started from the article [12] by Martio and Sarvas. John domains are recognized as a wide class of irregular domains where the classical results are known to hold, see for example the article [1] by Buckley and Koskela. Thus it is surprising that the value of the parameter is known only for trivial examples; all proofs seems to give only existence of the parameters. The aim of this paper is to give explicit examples of these domains.

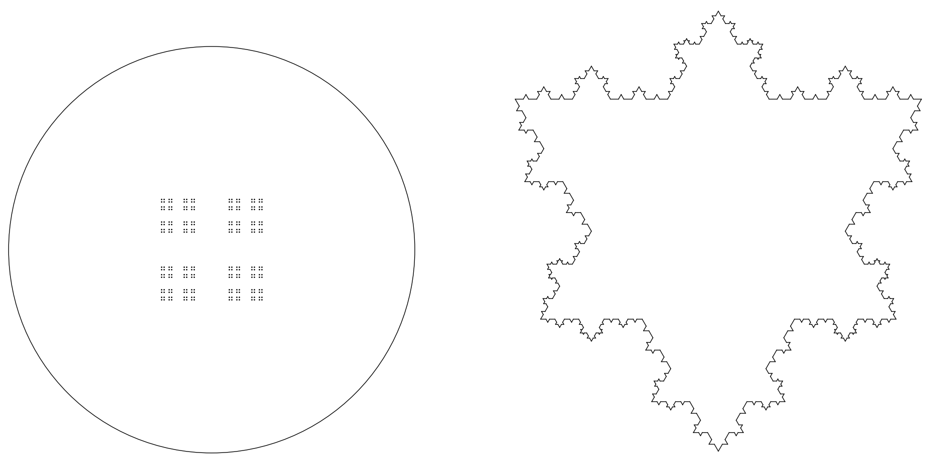

Figure 1: Left: a Cantor dust-type domain with $\alpha = 1/3$.Right: von Koch snowflake domain with $a = 1/4$.

We remove a Cantor dust-type fractal with a ratio $\alpha \in (0,1)$ from an open ball $B(0,2) \subset \mathbb{R}^2$, see Figure 1. Then we calculate two constants β_1 and β_2 depending only on α and show that our domain satisfies the quasihyperbolic boundary condition for $\beta \le \beta_1$ and it does not satisfies the quasihyperbolic boundary condition for $\beta \ge \beta_2$ (Theorem 3.1). Although $\beta_1 < \beta_2$, we see that $\beta_2 - \beta_1 < 0.04$. Similarly we analyze when this domain is a John domain and show that it is $4.37/\alpha$-John (Theorem 3.3).

We construct a von Koch snowflake in the plane by replacing the middle a-th portion, $a \in (0, \frac{1}{2}]$, of each line segment by the other two sides of an equilateral triangle, see Figure 1. We show that the von Koch snowflake domain satisfies the quasihyperbolic boundary condition for $\beta \le \beta_1'$ but not for $\beta \ge \beta_2'$ (Theorem 4.1), here β_1' and β_2' depend only on a. Finally we show that the von Koch snowflake domain is a John domain with a constant $\max\left\{2, \frac{4}{3(1-a)}\right\}$ (Theorem 4.2). So in particularly for $a \in (0, \frac{1}{3}]$ it is 2-John. In this range the result is sharp and surprisingly the constant does not depend on the parameter a since the worst case is the equilateral triangle inside the von Koch snowflake domain and every equilateral triangle is a John domain with a constant 2.

2. PRELIMINARY RESULTS

Let $D \subsetneq \mathbb{R}^d$ be a domain. The quasihyperbolic length of a rectifiable curve $\gamma \subset D$ is

$$\ell_k(\gamma) = \int_\gamma \frac{|dz|}{\operatorname{dist}(z, \partial D)},$$

where $\operatorname{dist}(z, \partial D)$ is the Euclidean distance between z and ∂D. The quasihyperbolic distance k_D is defined by

$$k_D(x,y) = \inf_\gamma \ell_k(\gamma), \quad x, y \in D,$$

where the infimum is taken over all rectifiable curves in D joining x and y. By the definition it is clear that the quasihyperbolic metric is monotone with respect to domains, which means that if $D \subsetneq \mathbb{R}^d$ and $D' \subset D$ are domains, and $x, y \in D'$, then $k_D(x,y) \le k_{D'}(x,y)$.

We recall next the definitions of the quasihyperbolic boundary condition and the class of John domains.

Definition 2.1. [3] A domain $D \subsetneq \mathbb{R}^d$ satisfies a *quasihyperbolic boundary condition* with constants $\beta \in (0,1]$ and $c > 0$, or shortly D satisfies β-QHBC, if there exists a distinguished point $x_0 \in D$ such that

$$k_D(x_0, x) \le \frac{1}{\beta} \log \frac{1}{\operatorname{dist}(x, \partial D)} + c \tag{2.1}$$

for all $x \in D$.

Note that if $D' \subset D$ and $x, y \in D'$, then $k_D(x,y) \le k_{D'}(x,y)$. We use this property when we obtain lower estimates for the quasihyperbolic distance.

Definition 2.2. [12] A domain D is a c-*John domain*, $c \ge 1$, if there is a distinguished point $x_0 \in D$ such that any $x \in D$ can be connected to x_0 by a rectifiable curve $\gamma : [0, l] \to D$, which is parametrized by arclength and with $\gamma(0) = x$, $\gamma(l) = x_0$ and

$$\operatorname{dist}(\gamma(t), \partial D) \ge \frac{1}{c} t$$

for every $0 \le t \le l$. The distinguished point x_0 is called the John center.

Punctured space $\mathbb{R}^d \setminus \{0\}$ is one of the very few domains where the explicit formula for the quasihyperbolic distance is known. Martin and Osgood proved the following result in 1986 [11, p. 38].

Proposition 2.1. *Let $G = \mathbb{R}^d \setminus \{0\}$ and $x, y \in G$. Then*

$$k_G(x,y) = \sqrt{\theta^2 + \log^2 \frac{|x|}{|y|}},$$

where $\theta = \measuredangle(x, 0, y)$.

Finally, we give a formula for the quasihyperbolic length of a Euclidean line segment in twice-punctured space.

Lemma 2.2. [8, Remark 4.26] *Let $G = \mathbb{R}^d \setminus \{a, b\}$ for $a \ne b$, $c = (a+b)/2$, the line l be the perpendicular bisector of $[a, b]$ and $x \in l$. Then*

$$\ell_k([x,c]) = \log\left(2\left(|x-c| + \sqrt{|a-b|^2/4 + |x-c|^2}\right)\right) - \log|a-b|.$$

3. CANTOR DUST-TYPE FRACTAL

Let $\alpha \in (0,1)$. Let $Q_0 \subset \mathbb{R}^2$ be the closed square in the plane which side length is 1 and which is centered at the origin. We make a Cantor construction in Q_0. We remove from Q_0 strips $\{-\frac{\alpha}{2} < x < \frac{\alpha}{2}\}$ and $\{-\frac{\alpha}{2} < y < \frac{\alpha}{2}\}$. We get four closed squares Q_1^j, $j = 1, \ldots, 2^2$. We continue the process by removing from each Q_1^j vertical and horizontal strips of width $\alpha(Q_i^j)$. We set

$$C_\alpha = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{2^{2i}} Q_i^j \cap Q_0.$$

Thus C_α consists of the corner points of all squares Q_i^j. The set C_α is self-similar and thus its Hausdorff dimension is equal to its Minkowski dimension [10, Lemma 3.1, p. 488]. By [2, Theorem 9.3, p. 118] we can calculate

$$\dim_{\mathcal{H}}(C_\alpha) = \dim_{\mathcal{M}}(C_\alpha) = \frac{\log 4}{\log \frac{2}{1-\alpha}}.$$

Thus $\alpha \mapsto \dim_{\mathcal{H}}(C_\alpha) = \dim_{\mathcal{M}}(C_\alpha)$ is a strictly decreasing bijective mapping from $(0,1)$ to $(0,2)$. Note that in the range $\alpha \in (0, \frac{1}{2}]$ we have $\dim_{\mathcal{H}}(C_\alpha) = \dim_{\mathcal{M}}(C_\alpha) < 1$.

We set

$$\Omega_\alpha = B(0,2) \setminus C_\alpha \subset \mathbb{R}^2.$$

Then Ω_α is a bounded domain with $\dim_{\mathcal{H}}(\partial\Omega_\alpha) = \dim_{\mathcal{M}}(\partial\Omega_\alpha) = \max\{1, \dim_{\mathcal{M}}(C_\alpha)\}$ and for every $\lambda \in [1,2)$ there exists a unique $\alpha \in [0, \frac{1}{2}]$ such that $\lambda = \dim_{\mathcal{H}}(\partial\Omega_\alpha) = \dim_{\mathcal{M}}(\partial\Omega_\alpha)$. For the domain Ω_α see Figure 1 or 3.

Theorem 3.1. *The domain $\Omega_\alpha \subset \mathbb{R}^2$ (defined above) satisfies the β-QHBC for*

$$\beta \le \beta_1 = \frac{\log \frac{2}{1-\alpha}}{\log \frac{2+\sqrt{4+(1-\alpha)^2}}{1-\alpha} + \frac{3}{2\alpha} + \frac{\pi}{2} - \frac{3}{2}} \tag{3.1}$$

and it does not satisfy β-QHBC for

$$\beta \ge \beta_2 = \frac{\log \frac{2}{1-\alpha}}{\log \frac{2+\sqrt{4+(1-\alpha)^2}}{1-\alpha} + \frac{1-\alpha}{\alpha} + \frac{\pi}{2}}. \tag{3.2}$$

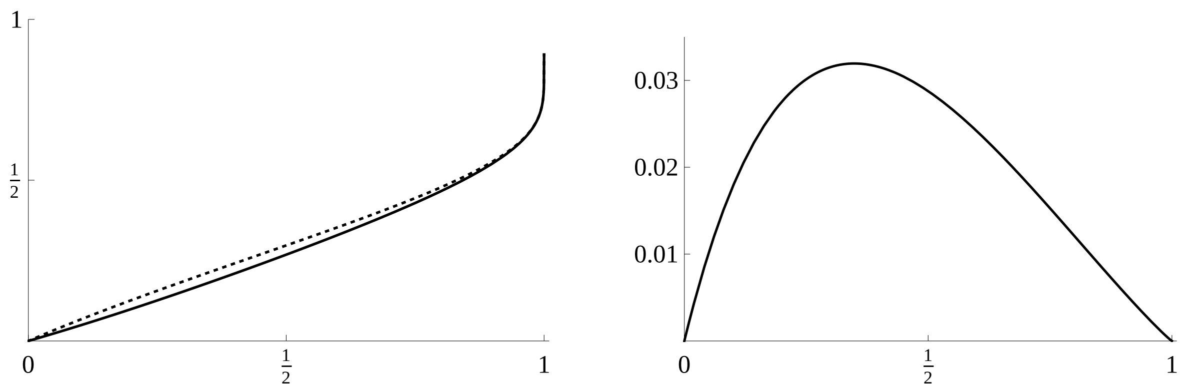

Figure 2: Left: bounds β_1 (solid line) and β_2 (dashed line) of Theorem 3.1 plotted as functions of α. Right: $\beta_2 - \beta_1$ plotted as a function of α.

Note that although $\beta_1 < \beta_2$ we have $\beta_2 - \beta_1 < 0.04$, see Figure 2.

Proof. Let x_0 be a center of Q_0 and let x_n be a center of Q_n^j in an upper right corner, see Figure 3. We want to give upper and lower estimates for the quasihyperbolic distance $k_{\Omega_\alpha}(x_0, x_n)$. Then by the geometry of the domain we can connect by a line segment any $x \in Q_0 \cap \Omega_\alpha$ to a suitable center point. Thus if the center points satisfy the β-QHBC then, by increasing the constant c in (2.1), all $x \in Q_0 \cap \Omega_\alpha$ satisfy β-QHBC for the same β. We start with the upper estimate. We connect x_0 and x_n as in the Figure 3, where we use line segments and circle arcs near the points $x_1, \ldots, x_{n-1}$. Let us denote $l \in \{1, 2, \ldots, n\}$. We first estimate the dotted part of the path denoted by p_l. Let y_l and u_l be as in Figure 3. By Lemma 2.2 we obtain

$$\begin{aligned} k(p_l) &= k(y_l, u_l) \le k([y_l, u_l]) \\ &= \log\Big(\alpha\big(\tfrac{1-\alpha}{2}\big)^{l-1}\Big(1 + \sqrt{1 + \tfrac{1}{4}(1-\alpha)^2}\Big)\Big) - \log\big(\alpha\big(\tfrac{1-\alpha}{2}\big)^{l}\big) \\ &= \log\frac{2 + \sqrt{4 + (1-\alpha)^2}}{1-\alpha}. \end{aligned}$$

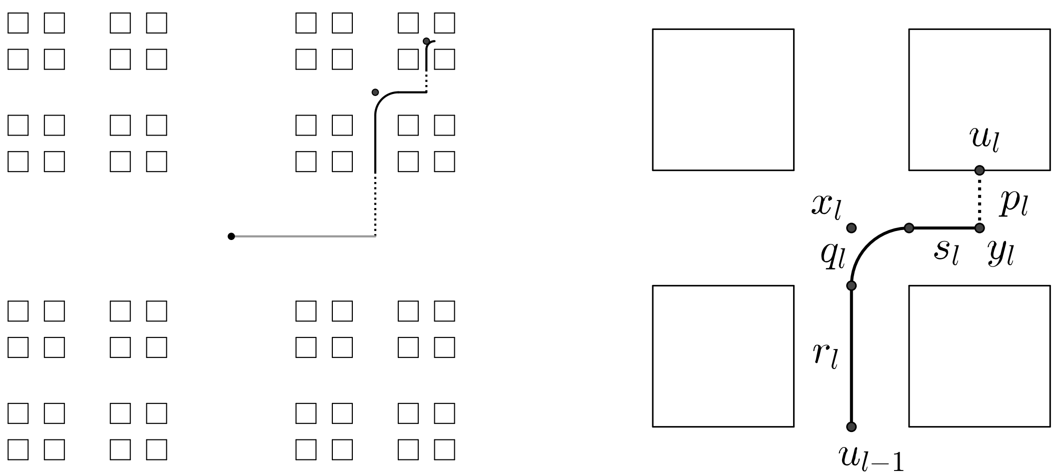

Figure 3: The path used in the proof of Theorem 3.1.

For the circle arc the radius is $\alpha\frac{1}{2}\big(\frac{1-\alpha}{2}\big)^l$ and hence the quasihyperbolic length of the circle arc is

$$k(q_l) = \frac{\frac{\pi}{2}\alpha\frac{1}{2}\big(\frac{1-\alpha}{2}\big)^l}{\alpha\frac{1}{2}\big(\frac{1-\alpha}{2}\big)^l} = \frac{\pi}{2}.$$

There are two line segments inside the square Q_{l-1}^j. The longer has length $\ell(Q_l^j) = \big(\frac{1-\alpha}{2}\big)^l$ and the shorter $\frac{1}{2}\ell(Q_l^j)$. In both parts the distance to the boundary is equal to or greater than $\frac{1}{2}\alpha\big(\frac{1-\alpha}{2}\big)^{l-1}$. For the line segments we obtain an upper bound for the quasihyperbolic length

$$k(r_l) + k(s_l) = \frac{\frac{3}{2}\big(\frac{1-\alpha}{2}\big)^l}{\frac{1}{2}\alpha\big(\frac{1-\alpha}{2}\big)^{l-1}} = \frac{3}{\alpha}\Big(\frac{1-\alpha}{2}\Big).$$

Putting these three estimates together and adding the first and last parts of the path, we have

$$\begin{aligned} k_{\Omega_\alpha}(x_0,x_n) &\leq \frac{\frac{1}{2}\alpha+\frac{1}{2}\left(\frac{1-\alpha}{2}\right)}{\frac{1}{2}\alpha}+n\log\frac{2+\sqrt{4+(1-\alpha)^2}}{1-\alpha} \\ &\quad +(n-1)\left(\frac{3}{\alpha}\left(\frac{1-\alpha}{2}\right)+\frac{\pi}{2}\right)+\frac{\frac{1}{2}\left(\frac{1-\alpha}{2}\right)^n}{\frac{1}{2}\alpha\left(\frac{1-\alpha}{2}\right)^n} \\ &= 2-\frac{\pi}{2}+n\left(\log\frac{2+\sqrt{4+(1-\alpha)^2}}{1-\alpha}+\frac{3}{2\alpha}+\frac{\pi}{2}-\frac{3}{2}\right). \end{aligned} \tag{3.3}$$

Next we calculate a lower bound for the quasihyperbolic distance. We do not need to know where exactly quasihyperbolic geodesic is located. But if a geodesic connects x_0 and x_n in the upper right corner, then the geodesic should go from the boundary of Q_l^j to the boundary of Q_{l+1}^j. Thus we can give a lower estimate to the quasihyperbolic distance $k(u_{l-1},u_l)$. First we estimate the path from the boundary of Q_l^j to the 'middle square' of Q_l^j. Here the shortest route is in the middle of the strip and in the same time the distance to C_α is the greatest. Thus we obtain

$$k(r_l)\geq\frac{\left(\frac{1-\alpha}{2}\right)^{l+1}}{\frac{1}{2}\alpha\left(\frac{1-\alpha}{2}\right)^l}=\frac{2}{\alpha}\left(\frac{1-\alpha}{2}\right).$$

Then we estimate the path across the 'middle square' to the boundary of Q_{l+1}^j. We use a circular arc to estimate the path through the 'middle square', see Figure 3, and obtain $k(q_l)\geq\frac{\pi}{2}$. Finally we estimate the path from the 'middle square' to the boundary of Q_{l+1}^j. In the boundary of Q_{l+1}^j the distance to C_α is at most $\frac{1}{2}\alpha\ell(Q_{l+1}^j)$. Thus we get a lower estimate for the later half by approaching to the middle of the strip perpendicular to the boundary of Q_{l+1}^j as we did in the dotted part of the upper bound. We get the term $\log\frac{2+\sqrt{4+(1-\alpha)^2}}{1-\alpha}$. Collecting the terms together we obtain

$$\begin{aligned} k_{\Omega_\alpha}(x_0,x_n) &\geq n\tfrac{2}{\alpha}\left(\tfrac{1-\alpha}{2}\right)+(n-1)\tfrac{\pi}{2}+n\log\tfrac{2+\sqrt{4+(1-\alpha)^2}}{1-\alpha} \\ &= -\tfrac{\pi}{2}+n\left(\log\tfrac{2+\sqrt{4+(1-\alpha)^2}}{1-\alpha}+\tfrac{1-\alpha}{\alpha}+\tfrac{\pi}{2}\right). \end{aligned} \tag{3.4}$$

In the definition of the quasihyperbolic boundary condition we choose $x_0=0$ and let $x=x_n$ be a center of Q_n^j. Now

$$\operatorname{dist}(x_n,\partial C_\alpha)=\sqrt{2}\frac{\alpha}{2}\left(\frac{1-\alpha}{2}\right)^n$$

and thus

$$\log\frac{1}{\operatorname{dist}(x_n,\partial C_\alpha)}=\log\frac{\sqrt{2}}{\alpha}+n\log\frac{2}{1-\alpha}. \tag{3.5}$$

Combining (3.3) and (3.5) and letting $n\to\infty$ we deduce that Ω_α satisfies (2.1) in the QHBC for

$$\beta\leq\frac{\log\frac{2}{1-\alpha}}{\log\frac{2+\sqrt{4+(1-\alpha)^2}}{1-\alpha}+\frac{3}{2\alpha}+\frac{\pi}{2}-\frac{3}{2}}.$$

Similarly combining (3.4) and (3.5) and letting $n\to\infty$ we see that Ω_α does not satisfies (2.1) in the definition of the QHBC for

$$\beta\geq\frac{\log\frac{2}{1-\alpha}}{\log\frac{2+\sqrt{4+(1-\alpha)^2}}{1-\alpha}+\frac{1-\alpha}{\alpha}+\frac{\pi}{2}}. \qquad \blacksquare$$

Proposition 3.2. *Let 0 be the John center. Then the domain Ω_α is c-John for $c \geq 4.37/\alpha$, and it is not c-John for $c \leq 4/\alpha$.*

Proof. We consider first the case that $x \in \Omega_\alpha \cap Q_0$. Let x_n be a center of Q_n^j in an upper right corner. We choose the curve $\gamma_{n,0}$ joining x_n and x_0 consisting of horizontal and vertical line segments as in Figure 4. We denote $u_k = \gamma_{n,0} \cap \partial Q_k^j$ and $y_k, z_k \in \gamma_{n,0}$ as in Figure 4. Now

$$\ell(\gamma_{n,0}) = 1 - \left(\frac{1-\alpha}{2}\right)^n$$

and

$$\ell(\gamma_{n,k}) = \left(\frac{1-\alpha}{2}\right)^k - \left(\frac{1-\alpha}{2}\right)^n,$$

where $\gamma_{n,k}$ is the subcurve of $\gamma_{n,0}$ connecting x_n to x_k with $k < n$. Let $x \in Q_n^j \setminus \cup Q_{n+1}^j$ and $\gamma_y = [x, x_n] \cup \gamma_{n,y}$ for $y \in \gamma_{n,0}$, where $\gamma_{n,y}$ is the subcurve of $\gamma_{n,0}$ connecting x_n to y. Now $|x - x_n| < ((1-\alpha)/2)^n \sqrt{1+\alpha^2}/2$ implying

$$\begin{aligned}\frac{\operatorname{dist}\left(\gamma_{z_k}(\ell(\gamma_{z_k}), \partial\Omega_\alpha)\right)}{\ell(\gamma_{z_k})} &\geq \frac{\frac{\alpha}{2}\left(\frac{1-\alpha}{2}\right)^k}{\left(\frac{1-\alpha}{2}\right)^n \frac{\sqrt{1+\alpha^2}}{2} + \left(\frac{1-\alpha}{2}\right)^k - \left(\frac{1-\alpha}{2}\right)^n + \frac{\alpha}{2}\left(\frac{1-\alpha}{2}\right)^k} \\ &= \frac{\alpha}{\left(\frac{1-\alpha}{2}\right)^{n-k}(\sqrt{1+\alpha^2}-2) + 2 - \alpha} > \frac{\alpha}{3}\end{aligned}$$

and

$$\begin{aligned}\frac{\operatorname{dist}\left(\gamma_{u_k}(\ell(\gamma_{u_k}), \partial\Omega_\alpha)\right)}{\ell(\gamma_{u_k})} &\geq \frac{\frac{\alpha}{2}\left(\frac{1-\alpha}{2}\right)^{k+1}}{\left(\frac{1-\alpha}{2}\right)^n \frac{\sqrt{1+\alpha^2}}{2} + \left(\frac{1-\alpha}{2}\right)^{k+1} - \left(\frac{1-\alpha}{2}\right)^n + \frac{1}{2}\left(\frac{1-\alpha}{2}\right)^{k+1}} \\ &= \frac{\alpha}{\left(\frac{1-\alpha}{2}\right)^{n-k-1}(\sqrt{1+\alpha^2}-2) + 3} > \frac{\alpha}{3}.\end{aligned}$$

Hence the definition holds if $\frac{1}{c} \leq \frac{\alpha}{3}$ i.e. if $c \geq 3/\alpha$.

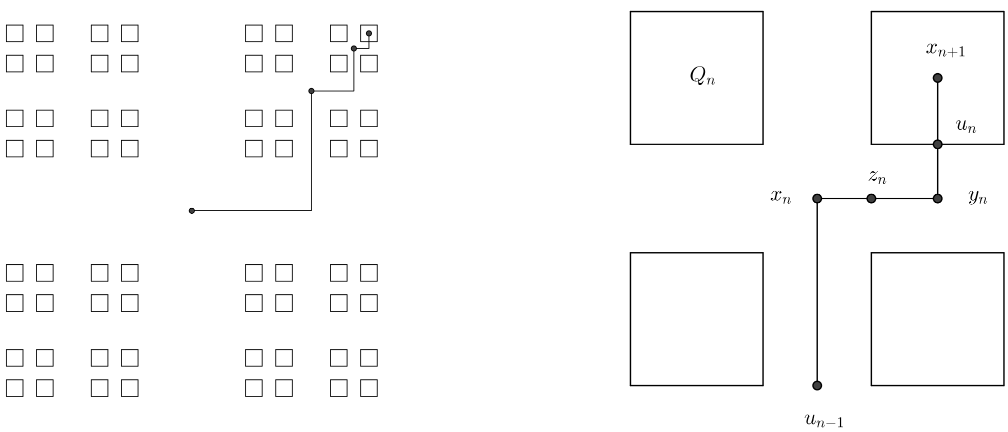

Figure 4: The curve $\gamma_{n,0}$ and points y_n, z_n and u_n used in the proof of Proposition 3.2.

Let us then consider $x_n \in \Omega_\alpha \setminus Q_0 = B^2(2) \setminus Q_0$. Let $x_n = i(2-1/n)$ and γ_n is the line segment joining x_n to x_0. Now

$$\frac{\mathrm{dist}(\gamma_n(t), \partial\Omega_\alpha)}{\ell(\gamma_n(t))} \le \frac{\frac{\alpha}{2}}{2-1/n} = \frac{\alpha}{4-2/n} < \frac{\alpha}{4}$$

and hence the definition does not hold if $\frac{1}{c} \ge \frac{\alpha}{4}$ i.e. if $c \le 4/\alpha$.

Let $x_n = \sqrt{2}(1+i)(2-1/n)$ and $\gamma_n = [x_0, i/2] \cup \delta \cup [\alpha/2 + i(1+\alpha)/2, x_n]$, where δ is the circular arc joining $i/2$ and $\alpha/2 + i(1+\alpha)/2$ with center at $\alpha/2 + i/2$. Now

$$\begin{aligned}
\frac{\mathrm{dist}(\gamma_n(t), \partial\Omega_\alpha)}{\ell(\gamma_n(t))} &\ge \lim_{n\to\infty} \frac{\mathrm{dist}(\gamma_n(t))}{\ell(\gamma_n(t))} \\
&= \frac{\frac{\alpha}{2}}{\frac{1-\alpha}{2} + \frac{\pi\alpha}{4} + \sqrt{(\sqrt{2} - \frac{1}{2} - \frac{\alpha}{2})^2 + (\sqrt{2} - \frac{1}{2} + \frac{1-\alpha}{2})^2}} \\
&= \frac{\alpha}{1 - \alpha + \frac{\pi\alpha}{2} + \sqrt{17 - 4\sqrt{2} + 2\alpha(1 - 4\sqrt{2} + \alpha^2)}} \\
&> \frac{\alpha}{1 + \sqrt{17 - 4\sqrt{2}}} > \frac{\alpha}{4.37}.
\end{aligned}$$

Hence the definition holds if $\frac{1}{c} \le \frac{\alpha}{4.37}$ i.e. if $c \ge 4.37/\alpha$.

By the geometry it is clear that the assertion follows. ∎

When the parameter α is small then the origin is no longer a good choice for the John center. In the next theorem we use $5i/4$ instead and get a slightly better result. Most probably the optimal John center should depend on α and thus have the form $c(\alpha)i$.

Theorem 3.3. *The domain Ω_α is $4.37/\alpha$-John for $\alpha \in [1/3, 1)$ and $3/\alpha$-John for $\alpha \in (0, 1/3)$.*

Proof. By Proposition 3.2 the domain Ω_α is $4.37/\alpha$-John and thus we need to show that for $\alpha < 1/3$ it is $3/\alpha$-John.

Let $\alpha < 1/3$ and choose $x_0 = 5i/4$ to be the John center. By the proof of Proposition 3.2 it is clear that for all $y \in \Omega_\alpha \cap Q_0$ we have

$$\frac{\mathrm{dist}(\gamma(t), \partial\Omega_\alpha)}{\ell(\gamma(t))} > \frac{\alpha}{3},$$

where $\gamma = \gamma' \cup [0, x_0]$ is the curve joining y to x_0 and γ is as in Figure 4.

Let us now assume that $y \in B(0,2) \setminus Q_0$. We consider the curve γ from y to x_0, which consists of the line segment $[y, 5y/(4|y|)]$ and the shortest circular arc from $5y/(4|y|)$ to x_0 with center at 0. By the selection of γ we obtain

$$\frac{\mathrm{dist}(\gamma(t), \partial\Omega_\alpha)}{\ell(\gamma(t))} > \frac{\frac{5}{4} - \frac{\sqrt{2}}{2}}{\pi\frac{5}{4} + \frac{3}{4}} = \frac{5 - 2\sqrt{2}}{5\pi + 3} > \frac{1}{9} > \frac{\alpha}{3},$$

where the last inequality follows from the fact that $\alpha < 1/3$. Now the assertion follows as Ω_α is $\frac{3}{\alpha}$-John. ∎

4. VON KOCH SNOWFLAKE DOMAIN

We construct a von Koch snowflake. Let $a \in (0, 1/2]$. We start with an equilateral triangle with side length 1. We replace the middle a-th portion of each line segment by the other two sides of an equilateral triangle. We continue inductively and obtain a von Koch snowflake. We denote by S_a the bounded domain bordered by the von Koch snowflake. Then ∂S_a is self-similar and thus its Hausdorff dimension is equal to its Minkowski dimension [10, Lemma

3.1, p. 488]. Note that for $a \in (0, 1/2)$, ∂S_a is not self-intersecting [7, Theorem 3.1]. The Minkowski dimension of ∂S_a is the solution of $2a^s + 2\big(\frac{1}{2}(1-a)\big)^s = 1$ for $a \in (0, 1/2)$, [2, Example 9.5, p. 120].

Theorem 4.1. *The domain $S_a \subset \mathbb{R}^2$ satisfies the β-QHBC for*

$$\beta \le \beta_1 = \frac{\log \frac{1}{a}}{\log \frac{1+\sqrt{1+3a^2}}{a\sqrt{3}} + \log\big(3+2\sqrt{3}\big)}$$

and it does not satisfy β-QHBC for

$$\beta \ge \beta_2 = \frac{\log \frac{1}{a}}{\sqrt{\arcsin^2 \frac{\sqrt{3}}{\sqrt{2(1+a)(3+2a)}} + \log^2 \frac{\sqrt{(1+a)(3+2a)}}{a\sqrt{2}}}}.$$

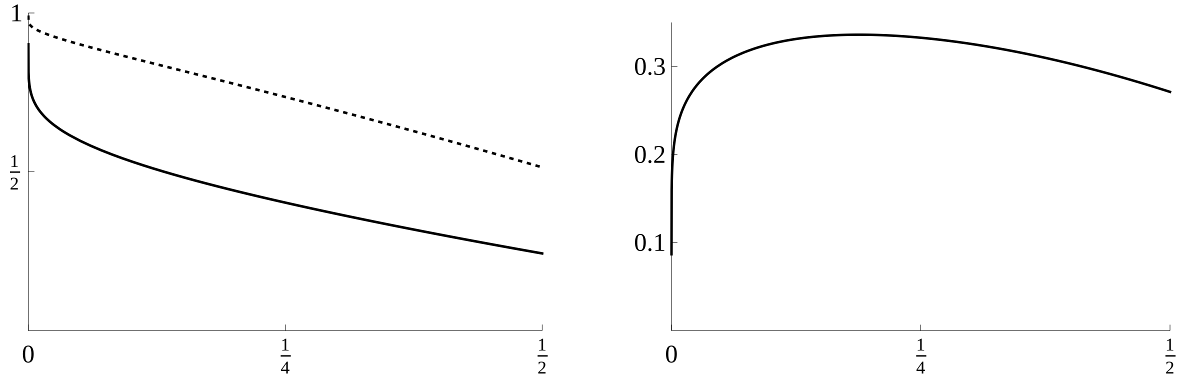

Figure 5: Left: bounds β_1 (solid line) and β_2 (dashed line) of Theorem 4.1 plotted as functions of α. Right:$\beta_2-\beta_1$ plotted as a function of α.

We have that $\beta_2 - \beta_1 < 0.4$, see Figure 5.

Proof. We calculate first the upper bound β_1. We concentrate on the worst situation, see Figure 6, where we first go up and then always to the left to the center of a triangle. Note that other points in the same triangle can be easily connect to the center point and thus they do not effect to the value of β. Let us denote by x_0 the center of S_a and by x_n the center of the triangle constructed on the n-th iteration as in Figure 6. We estimate $k_{S_a}(x_0, x_n)$ by using the curve $\gamma_n = \cup_{i=1}^n [x_{i-1}, x_i]$ and denote points y_n, z_n as in Figure 6. We estimate $k_{S_a}(x_n, y_n)$ by the quasihyperbolic length of line segments $[x_n, y_n]$ in the domain $\mathbb{R}^2 \setminus \{y_{n+1}\}$. By Lemma 2.2 we obtain

$$\begin{aligned} k_{S_a}(x_n, y_n) \;&\le\; \log\left(2\left(\frac{\sqrt{3}a^n}{4} + \sqrt{\left(\frac{a^n}{4}\right)^2 + \left(\frac{\sqrt{3}a^n}{4}\right)^2}\right)\right) - \log\left(\frac{a^n}{2}\right) \\ &\quad + \log\left(2\left(\frac{a^n}{4\sqrt{3}} + \sqrt{\left(\frac{a^n}{4}\right)^2 + \left(\frac{a^n}{4\sqrt{3}}\right)^2}\right)\right) - \log\left(\frac{a^n}{2}\right) \\ &= \log\big(2+\sqrt{3}\big) + \log\sqrt{3} = \log\big(3+2\sqrt{3}\big). \end{aligned}$$

Similarly $k_{S_a}(x_n, y_{n+1})$ is estimated by the quasihyperbolic length of line segments $[x_n, y_{n+1}]$ in $\mathbb{R}^2 \setminus \{z_n\}$and thus by Lemma 2.2 we obtain

$$\begin{aligned} k_{S_a}(x_n, y_{n+1}) &\le \log\left(2\left(\frac{a^n}{2\sqrt{3}} + \sqrt{\left(\frac{a^{n+1}}{2}\right)^2 + \left(\frac{a^n}{2\sqrt{3}}\right)^2}\right)\right) - \log a^{n+1} \\ &= \log \frac{1+\sqrt{3a^2+1}}{\sqrt{3}a}. \end{aligned}$$

Therefore we have

$$k_{S_a}(x_0, x_n) \le k_{S_a}(x_0, x_1) + (n-1)\Big(\log\big(3+2\sqrt{3}\big) + \log \tfrac{1+\sqrt{3a^2+1}}{\sqrt{3}a}\Big). \tag{4.1}$$

We easily obtain

$$\operatorname{dist}(x_n, \partial S_a) = \sqrt{\left(\frac{a^{n+1}}{2}\right)^2 + \left(\frac{a^n}{2\sqrt{3}}\right)^2} = \frac{a^n}{2}\sqrt{a+1/3}$$

and thus

$$\log \frac{1}{\operatorname{dist}(x_n, \partial S_a)} = \log \frac{2}{\sqrt{a+1/3}} + n \log \frac{1}{a}. \tag{4.2}$$

Combining (4.1) with (4.2) we obtain that S_a satisfies the β-QHBC for $\beta \le \beta_1$.

We prove next the lower bound β_2. We estimate $k(y_n, y_{n+1})$ by the quasihyperbolic distance between y_n and y_{n+1} in the domain $\mathbb{R}^2 \setminus \{z_n\}$. We deduce that $|y_{n+1} - z_n| = a^{n+1}/2$,

$$\begin{aligned} |y_n - z_n| &= \sqrt{\left(\frac{a^n}{2}\right)^2 + \left(\frac{a^n - a^{n+1}}{2}\right)^2 - \frac{a^n}{2}\frac{a^n - a^{n+1}}{2}\cos\frac{\pi}{3}} \\ &= \frac{a^n\sqrt{(1+a)(3+2a)}}{2\sqrt{2}} \end{aligned}$$

and by sine rule

$$\sin \measuredangle(y_n, z_n, y_{n+1}) = \frac{\sqrt{3}}{\sqrt{2(1+a)(3+2a)}}.$$

Therefore, by Proposition 2.1

$$k(y_n, y_{n+1}) \ge \sqrt{\arcsin^2 \frac{\sqrt{3}}{\sqrt{2(1+a)(3+2a)}} + \log^2 \frac{\sqrt{(1+a)(3+2a)}}{a\sqrt{2}}}.$$

and

$$k(x_0, x_n) \ge (n-1)\sqrt{\arcsin^2 \frac{\sqrt{3}}{\sqrt{2(1+a)(3+2a)}} + \log^2 \frac{\sqrt{(1+a)(3+2a)}}{a\sqrt{2}}}. \tag{4.3}$$

Combining (4.3) with (4.2) we obtain that S_a does not satisfy β-QHBC for $\beta \ge \beta_2$. ∎

Theorem 4.2. *Let $a \in (0, \frac{1}{2}]$. The set S_a is c-John with $c = \max\left\{2, \frac{4}{3(1-a)}\right\}$ and it is not c'-John for any $c' < 2$.*

Note that the result is sharp in the range $a \in (0, \frac{1}{3}]$.

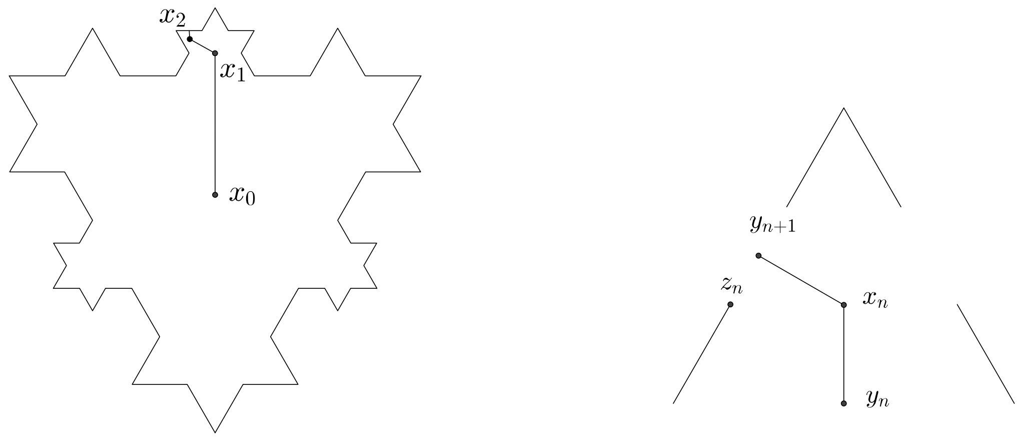

Figure 6: Points x_n, y_n and z_n as in the proof of Theorem 4.1.

Proof. Let us denote by T_0 the open equilateral triangle, which has sidelength 1 and is contained in S_a. We choose the John center x_0 to be the center of T_0 and let x be any point in S_a.

If $x \in T_0$, then we choose γ to be the line segment joining x to x_0. It is clear that

$$\frac{\operatorname{dist}(\gamma(t), \partial S_a)}{\ell(\gamma(t))} \geq \frac{1}{2}, \tag{4.4}$$

(and hence every open equilateral triangle is 2-John).

If $x \notin T_0$ then $x \in T_n$, where T_n is a maximal equilateral triangle in $S_a \setminus T_0$ with sidelength a^n. Let s be the side of T_n with $s \cap S_a = \emptyset$ and y_n the midpoint of s (see Figure 6). We denote $\gamma = [x, y_n] \cup [y_n, x_n] \cup [x_n, y_{n-1}] \cup \cdots \cup [y_1, x_0]$, where x_n is the center of T_n as in Figure 6. We easily obtain that $|y_{n+1} - x_n| = |x_n - y_n| = a^n/(2\sqrt{3})$ and thus $|y_{n+1} - x_n| + |x_n - y_n| = a^n/\sqrt{3}$. This yields for every $k = 0, \dots n$ that

$$\ell(\gamma_{y_k}) = \frac{1}{\sqrt{3}}(a^n + \ldots + a^k) = \frac{a^k}{\sqrt{3}} \frac{1 - a^{n-k+1}}{1 - a},$$

where γ_{y_k} is the subpath of γ that joins x to y_k. Since $\operatorname{dist}(y_k, \partial S_a) = \frac{\sqrt{3}}{4} a^k$ we obtain for every a, n and k that

$$\frac{\operatorname{dist}(\gamma(t), \partial S_a)}{\ell(\gamma(t))} = \frac{\frac{\sqrt{3}}{4} a^k}{\frac{a^k}{\sqrt{3}} \frac{1 - a^{n-k+1}}{1 - a}} = \frac{3}{4} \frac{1 - a}{1 - a^{n-k+1}} \geq \frac{3}{4}(1 - a), \tag{4.5}$$

where $\gamma(t) = y_k$. Note that the last inequality is sharp when k is fixed and $n \to \infty$. When $a \in [0, \frac{1}{3}]$, we have $\frac{3}{4}(1 - a) \geq \frac{1}{2}$; the inequality is sharp when $a = \frac{1}{3}$. By (4.4) and (4.5) the set S_a is $\max\left\{2, \frac{4}{3(1-a)}\right\}$-John.

Next we show that S_a is not c-John for any $c < 2$. Let us denote by y one of the corners of T_0 and consider $\gamma = [z, x_0]$ for $z \in [x_0, y]$. We obtain that S_a is not c-John for $c < c_z = 3|x_0 - z|$. As $z \to y$ we have $|x_0 - y| \to 2/3$ and thus $c_z \to 2$ implying the assertion. ∎

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