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OPTIMIZATION AND APPROXIMATION FOR POLYHEDRA IN SEPARABLE HILBERT SPACES

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ABSTRACT. This paper studies infinite dimensional polyhedra, covering the case in which range spaces of operators defining inequality systems are not closed. A rangespace method of linear programming is generalized to infinite dimensions and finite dimensional methods of approximation are introduced.

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1. INTRODUCTION

This paper is a continuation of [4], in which the same author has presented a generalization to infinite dimension of the image space theory of polyhedra (as illustrated, in the finite dimensional case, in [5], [3], [1] and [2]).

In such generalization, polyhedra are countable intersections of closed semispaces in a real separable Hilbert space H. Since H is separable, the study is carried out in l_2. Both finite dimension and finite number of intersecting semispaces are replaced by their countably infinite counterparts (with finite dimension and / or finite intersection as a special case).

Therefore, as is well known, the unit ball (of the usual norm) in R^n becomes a polyhedron and the same is true for the unit ball in l_2. Separability allows to include in this setting any closed convex set, narrowing the gap between linear and convex programming.

We will keep here the assumption that the linear transformations G (or $\widehat{G}$), defining the inequality constraint (or the extended inequality constraint, for the case of optimization), be continuous. This assumption has been shown in [4] to be a rather mild one.

For a first simpler foray in the infinite dimensional range space territory, in [4], we also assumed that range $\mathcal{R}(G)$ (and $\mathcal{R}(\widehat{G})$) be closed (such ranges will be at times more simply denoted by F, leaving to the context to specify which of the two subspaces we refer to). This restriction has some interest in its own, and includes, for example, the case that G (or $\widehat{G}$) is a Fredholm operator. Naturally this special theory bears the maximum possible resemblance with the finite dimensional case. In particular, we showed that the classification of seven possible types of polyhedra goes through to infinite dimension.

There are two important set to consider in range space: the first is the slack set, which can be considered to be the polyhedron as seen from the range space viewpoint and has the form $(v+F)\cap P$ (where v is the bound vector and P is the positive cone, which, incidentally, has no interior in infinite dimensions). The second is the feasibility cone, which has the form $F+P$ and is the set of all bound vector that make the polyhedron non-void. Thus the motivation for the cited restriction was to have closed slack sets and to facilitate closedness of the feasibility cones.

Still the restriction is obviously a blanket hypothesis, which entails leaving out of the ensuing class of polyhedra, for example, such an important cases as the weakly compact convex bodies, as will be shown in the present paper.

In this respect however, intuition suggests that the radical dissymmetry between domain space (where feasible sets are always closed) and range space, where various issues look intractable when $\mathcal{R}(G)$ (or $\mathcal{R}(\widehat{G})$) are not closed (and hence the feasibility cones are surely not closed in turn) appear rather strange. Indeed one may suspect that, by some hidden and deep mechanisms, this dissymmetry should vanish in some way.

Actually one of the major goals of this paper is to show that this intuition is correct and to unveil the mechanisms, whereby the apparent inviability of range space techniques when the relevant operator ranges are not closed, does dissolve. In the crucial case of strict tangency (namely when $F^{-}\cap P=\{0\}$) non closedness turns out to be irrelevant, in the sense that we may solve optimization problems using the closure of $\mathcal{R}(\widehat{G})$ instead of $\mathcal{R}(\widehat{G})$ itself.

Thus range space techniques in infinite dimension are at no disadvantage with respect to domain space techniques. Moreover, we show here, not only that finite dimensional range space algorithms of optimization can be extended to infinite dimensions, but also that they can be complemented by various finite dimensional approximation techniques.

In the case where F meets the "quasi interior" (which is also called here intern - see the sequel for its definition) of the positive cone P, the finite dimensional result ($F+P$ is the whole space whoever is F) is weakened to a density result. This is not much of a concern in optimization because, as we will show, the case where $\mathcal{R}(\widehat{G})$ is intern is anyway of no practical use, just as happens in the finite dimensional case.

But this fact is, at first sight, of much concern in the weakly tangent case (namely the case where F is neither strictly tangent nor intern to P). In fact, in this case, there is an intern relaxation and the density property for such relaxation apparently jeopardizes the extension to infinite dimensions of the fundamental concept of strictly tangent relaxation (see e.g. [1] for the finite dimensional case).

Despite these adverse circumstances, we will be able to show that the existence of the strictly tangent relaxation goes through untouched in the infinite dimensional case, although this generates (as might be expected) a solution for the whole system, that satisfies the remaining block of constraints either exactly or to an arbitrarily small degree of approximation.

Further comments on this topics are given in the conclusion, along to a guide for a few emendations for the preceding paper [4] and a brief guide to the connections with the present paper. For the moment we only add that the results of Section 4 (which studies the feasibility cone) have been given minimizing the use of the relativized product topology in favor of standard Hilbert spaces techniques. However it is possible to give, for some of the Theorems of Section 4, alternative (and much shorter) proofs, based on the results regarding the product topology given in [4]. We did not include such alternative proofs for the sake of brevity.

Some of the optimization methods introduced in the papers cited at the beginning, were based on the computation of extreme rays of certain polyhedral cones, in order to derive both enumerative algorithms (which solve whole classes of problems) as well as on evolutive algorithms. In infinite dimensions the approach based on internal generation of cones requires, of course, the computation of infinite generating sets. This problem is left open here, as it seems initially more convenient, and more promising on the practical side, to try to generalize an optimization method based on tangency conditions rather than internal generation (introduced, for the finite dimensional case in [3]).

Such method was called primal external linear programming method, as it reaches the optimum, expressed as the condition of tangency of an affine space to the positive cone, starting from an initial condition of void intersection (so that, so to speak, the affine lands on the cone). This method enjoys exact finite convergence in finite dimension. Incidentally, the "dual" possibility of reaching the optimum from an initial condition in which the affine and the cone are intersecting (so that, so to speak, the affine emerges on the boundary of the cone) is precisely the evolutive method introduced in [2], which also enjoys exact finite convergence.

In the sections devoted to Optimization we will show that the primal external method goes through in infinite dimension, naturally in terms of asymptotic convergence. Moreover, we introduce various techniques that allow to find approximate solutions, solving suitable finite dimensional problems.

2. A SPECIAL CASE OF MINIMUM DISTANCE OF CLOSED CONVEX SETS

The minimum distance problem for two closed convex sets is an important ingredient of the optimization technique illustrated in the sequel. We need now to clarify a few facts that will be of use later on.

In particular, we will be interested to the special case where the two sets are the positive cone P of l_2 and a closed affine $v + F$, where F is a closed linear subspace and v is an arbitrary vector.

Consider two closed convex disjoint sets C and D in l_2. Assume that

$$d(C, D) = \inf\{\|z = w\|: z \in C, y \in D\} > 0$$

The minimum distance problem consists in verifying the existence of two points $x \in C$ and $y \in D$ such that:

$$\|x - y\| = \inf\{\|z = w\|:z \in C, y \in D\}$$

and, in the positive case determining at least one such pair or, possibly and better yet, the set of all the minimum distance pairs.

An immediate sufficient condition for existence of a solution is that $C - D$ (which is convex) be closed. In fact, if this is the case, the Projection Theorem insures that there exists a (unique) minimum norm vector in $C - D$, and this in turn implies that pairs of vectors that solve the minimum distance problem for C and D do exist. Notice that C and D are disjoint if and only if $0 \notin C - D$. Since $\{0\}$ is compact, if $C - D$ is closed, the Strong Separation Theorem ([6] Corollary 14.4) implies $d(\{0\}, C - D) > 0$, which is equivalent to $d(C, D) > 0$.

The following Lemma is an important tool in the development of the Optimization techniques introduced in the sequel. Assume that, for two closed convex sets C and D, the minimum distance problem has solution. Denote the projections on C and on D by P_C and P_D respectively. Then we can state the following:

Lemma 2.1. *Let $C$and D be closed, convex and disjoint sets with $d(C, D) > 0$. Then two points $x \in C$ and $y \in D$ (with $x \neq y$) are a solution pair for the minimum distance problem if and only if:*

$$P_D x = y \text{ and } P_C y = x$$

Proof. Necessity is obvious for, if x and y make a solution pair, the two conditions hold a fortiori by the very definition of projection. Sufficiency follows easily from the fact that, by virtue of the projection Theorem, vectors $y - x$ and $x - y$ are, respectively, normal to C at x and normal to D at y. ∎

3. CONVEX CONES AND LINEAR SUBSPACES

We gather in this Section some mathematical preliminaries on cones and subspaces, instrumental for the study of polyhedra and optimization over polyhedra. There is a lot more on this topic, and a few results are also included for completeness only.

We are mainly interested in l_2 and its positive cone P, but, at times, if it obvious that no additional effort is required, we make more general statements. However, we do not strive for maximum generality.

In the l_2 environment we will sometimes have to consider three topologies. The native (strong) topology, denoted by $\mathcal{S}$, the weak topology, denoted by $\mathcal{W}$ and the (relative) product topology, denoted by $\mathcal{X}$.

We start with a few useful Lemmata:

Lemma 3.1. *Suppose that F and G are closed subspaces of Hilbert space H, that $F \perp G$, and that, for two non-void subsets C and D of H, $C \subset F$ and $D \subset G$. Then $C + D$ is closed if and only if both C and D are closed. Moreover:*

$$y = P_F y + P_G y \in C + D \Longleftrightarrow P_F y \in C \text{ and } P_G y \in D$$

Proof. We can assume without restriction of generality that $G = F^\perp$ because, if this were not the case we can take in lieu of H the Hilbert space $H_1 = F + G$, which is in fact a closed subspace of H. Suppose that C and D be closed and consider a sequence $\{z_i\}$ in $C + D$ with $\{z_i\} \to z$. We can write in a unique way:

$$z_i = P_F z_i + P_G z_i$$

with $P_F z_i \in C$ and $P_G z_i \in D$. By continuity of projection and the assumption that C and D be closed, $\{P_F z_i\} \to P_F z \in C$ and $\{P_G z_i\} \to P_G z \in D$, but since $P_F z + P_G z = z$ it follows

$z \in C + D$ and we are done. Conversely suppose that, for example D is not closed so that there exists a sequence $\{d_i\}$ in D, that converges to $d \notin D$, but, of course, $d \in G$. Take a vector $c \in C$. The sequence $\{c + d_i\}$ converges to $c + d$. But $c = P_F(c + d)$ and $d = P_G(c + d)$, and therefore, by uniqueness of the decomposition, it is not possible to express $c + d$ as a sum of a vector in C plus a vector in D. It follows that $c + d \notin C + D$. The second statement follows immediately from uniqueness of decomposition of a vector y in the sum $P_F y + P_G y$, and we so are done. ∎

In the sequel we will often write $\forall y \in H$, $P_F y = y_F$ and $P_G y = y_G$.

Remark 3.1. This Lemma has an obvious extension for a finite family of mutually orthogonal closed subspace. we do not make a formal statement for the sake of brevity

In the following Lemma proper cone means "not equal to the whole space". Beware that Phelps ([7]) calls proper cone what is for us a pointed cone (see the definition below).

Lemma 3.2. *A closed cone in a Hilbert space H is proper if and only if it is contained in a closed half-space.*

Proof. Let C be a closed proper cone. Then there is a singleton $\{y\}$ disjoint from C. Singletons are convex and compact and therefore the Strong Separation Corollary 14.4 in [6] applies. The rest is immediate. ∎

The following Lemma on inclusion of translated cones is valid in a very general setting.

Lemma 3.3. *Given two closed cones C_1 and C_2 in a real linear topological space E and two vectors y_1 and y_2:*

$$y_1 + C_1 \subset y_2 + C_2 \Leftrightarrow y_1 - y_2 \in C_2 \text{ and } C_1 \subset C_2$$

$$y_1 + C_1 = y_2 + C_2 \Leftrightarrow y_1 - y_2 \in lin(C_2) \text{ and } C_1 = C_2$$

Proof. The second statement is an immediate consequence of the first one. As to the first statement, sufficiency is obvious. Next suppose that although

$$y_1 - y_2 + C_1 = y + C_1 \subset C_2$$

there is a vector z in C_1 that does not belong to C_2. Thus kz is in C_1 but not in C_2. By hypothesis $y + kz \in C_2$, for any positive integer k. Therefore $\{(y/k) + z\}$ is in C_2. But this sequence converges to z and so, being C_2 closed $z \in C_2$. This contradiction concludes the proof. ∎

Next we recall the following definition:

Definition 3.1. Let C be a (convex) cone in a linear space E. The *lineality space* of a cone C, denoted by $lin(C)$, is the linear subspace:

$$lin(C) = C \cap (-C)$$

The cone C is called *pointed* if $lin(C) = \{0\}$.

The lineality space of a cone is the maximal linear subspace contained in the cone itself. If E is a linear topological space then, if C is closed, $lin(C)$ is obviously closed too.

Lemma 3.3 has the following very intuitive consequence about lines and convex cones. A line is a one dimensional affine space, that is, a set of the form $\{y : y = w + \alpha z : \alpha \in R\}$ where $w, z \in E$.

Lemma 3.4. *No closed pointed cone in a linear topological space E can contain a whole line.*

Proof. Let C be the cone an $l = \{y : y = w + \alpha z : \alpha \in R\}$ be the line, which is, evidently, a translated closed cone. Thus:

$$l \subset C \Longleftrightarrow w \in C \,\&\{y : y = \alpha z : \alpha \in R\} \subset C$$

The second of these conditions contradicts that the cone C be pointed. Thus we are done. ∎

We note in passing that the image under a linear isomorphism of a cone is pointed if and only if the cone is pointed, and that the origin is the only extreme point of a cone C if and only if C is pointed. If C is not pointed it has no extreme points. For brevity, we omit the proofs of these statements.

In finite dimension, if a cone is not pointed, it can be decomposed in the sum of a pointed cone plus its lineality space [8]. This decomposition can be generalized to infinite dimension, under the assumption that the lineality space be closed. This fact was stated without proof in [4]. It is convenient to restate this result here and include a proof.

Theorem 3.5. *Consider a convex cone C in a Hilbert space H and assume that its lineality space be closed. Then*

$$(lin(C)^{\perp} \cap C) = P_{lin(C)^{\perp}} C$$

where the cone $lin(C)^{\perp} \cap C$ is pointed. Consequently, if C is closed the cone $P_{lin(C)^{\perp}} C$ is closed too. Moreover, the cone C can be expressed as:

$$C = lin(C) + (lin(C)^{\perp} \cap C) = lin(C) + P_{lin(C)^{\perp}} C$$

Proof. First we prove that

$$C = lin(C) + (lin(C)^{\perp} \cap C)$$

That the rhs is contained in the lhs is obvious. Consider any vector $x \in C$ and for brevity let $\Gamma = lin(C)$. Decompose $x \in C$:

$$x = x_{\Gamma} + x_{\Gamma^{\perp}} \tag{3.1}$$

where $x_{\Gamma} \in \Gamma$ and $x_{\Gamma^{\perp}} \in \Gamma^{\perp}$. Next $x_{\Gamma^{\perp}} = x - x_{\Gamma}$ as sum of two vectors in C is in C and hence in $\Gamma^{\perp} \cap C$. Thus we have proved that the lhs is contained in the rhs. Next we show that the cone $lin(C)^{\perp} \cap C$ is pointed. Suppose that both a vector $x \neq 0$ and its opposite $-x$ belong to $lin(C)^{\perp} \cap C$ and decompose x as above (3.1). Because $lin(C)^{\perp} \cap C \subset lin(C)^{\perp}$, $x_{\Gamma} = 0$, so that $x = x_{\Gamma^{\perp}} \neq 0$. Do the same for $-x$, to conclude that $x_{\Gamma^{\perp}}$ and $-x_{\Gamma^{\perp}}$ are in C (but obviously not in $lin(C)$). Because this is a contradiction, $lin(C)^{\perp} \cap C$ is pointed. Finally we prove that:

$$lin(C)^{\perp} \cap C = P_{lin(C)^{\perp}} C$$

In fact,

$$P_{\Gamma^{\perp}}(\Gamma^{\perp} \cap C) = \Gamma^{\perp} \cap C \subset P_{\Gamma^{\perp}}(C)$$

On the other hand if $z \in P_{\Gamma^{\perp}}(C)$, for some $w \in C$, $z = P_{\Gamma^{\perp}} w = w - P_{\Gamma} w$ so that $z \in C$. Hence $z \in \Gamma^{\perp} \cap C$. ∎

Next we dwell a little on cone polarity.

Definition 3.2. Given a cone in a real Hilbert space H, the *polar cone* of a cone C, denoted by C^p is given by:

$$C^p = \{n : (n, y) \leq 0, \forall y \in C\}$$

That C^p is a convex cone is immediate by direct computation. As usual, thanks to the continuity of the inner product, it is also immediate to show that the cone C^p is always closed.

One can readily prove that the following formulas hold:

$$(-C)^p = -C^p$$

$$(C^p)^p = C^-$$

$$C^{-p} = C^p$$

The cone $(C^p)^p$ will be briefly denoted by C^{pp}

Remark 3.2. The polar cone of a convex cone is the normal cone at the origin to the given convex cone. Also, the polar cone of a closed convex cone is the set of all points in the space, whose projection onto the cone, coincides with the origin.

Note that if the cone is a linear subspace F then:

$$F^p = F^\perp$$

Notice also that polarity is anti-monotone in a way similar to orthogonal complementation. In fact,

$$\text{if } C_1 \subset C_2 \text{ then } C_1^p \supset C_2^p$$

The analogy with orthogonal complementation goes on a long way. A first noteworthy example arises looking at the polar of a sum of cones:

Proposition 3.6. *Let C_1 and C_2 are two convex cones then:*

$$(C_1 + C_2)^p = C_1^p \cap C_2^p$$

Proof. Since C_1 and C_2 are convex cones, $C_1 + C_2 \supset C_1$ and $C_1 + C_2 \supset C_2$, so that $(C_1 + C_2)^p \subset C_1^p$ and $(C_1 + C_2)^p \subset C_2^p$ and hence $(C_1 + C_2)^p \subset C_1^p \cap C_2^p$. On the other hand if $z \in C_1^p \cap C_2^p$ then $\forall y \in C_1$, $w \in C_2$, $(z, y) \leq 0$ and $(z, w) \leq 0$ so that summing these two, $(z, y + w) \leq 0$ which shows that $(C_1 + C_2)^p \supset C_1^p \cap C_2^p$. This completes the proof. ∎

Theorem 3.7. *Let T be an operator $H \to H$ and C a cone in H. Then*

$$(TC)^p = T^{*-1}C^p$$

Proof. In fact

$$(TC)^p = \{n : (n, Tx) \leq 0 : \forall x \in C\}$$

$$= \{n : (T^* n, x) \leq 0 : \forall x \in C\} = T^{*-1}C^p$$

∎

It is now in order to deal with the crucial notion of *strict tangency* and with the positive cone P of l_2.

Definition 3.3. A non-dense (linear) subspace F is said to be strictly tangent to a pointed closed cone C if $F^- \cap C = \{0\}$.

Remark 3.3. It is important to notice that this property is hereditary under inclusion, in the sense that if F is strictly tangent to a pointed closed cone C then any linear subspace of F is also strictly tangent to P.

Definition 3.4. In $H = l_2$, we define the positive cone P by:

$$P = \{y : y_i \geq 0, \forall i\}$$

The cone P is closed, pointed and with void interior. Moreover:

$$P^p = -P$$

and:

$$P^{pp} = P$$

Let $\mathfrak{N}$ be the set of positive integers
For $y \in l_2$ we introduce the notation:

$$ip(y) = \{i : y_i > 0\}$$

$$iz(y) = \{i : y_i = 0\}$$

$$in(y) = \{i : y_i < 0\}$$

It will useful in the sequel to bear in mind that the projection a vector w on P is obtained from w zeroing all its negative components. The projection of a vector on $-P$ is instead obtained zeroing all its positive components. When convenient, we will use the notation:

$$P_P w = w^+ \text{ and } P_{-P} w = w^-$$

Note that

$$w = w^+ + w^- \text{ and } w^+ \perp w^-$$

It is possible to show that this decomposition in a vector in P plus a vector in $P^p = -P$ is unique, under the requirement that the decomposition be orthogonal.

Actually a similar, but much more general, result can be stated, which provides another important example of the analogy with orthogonal complementation. However, for the sake of brevity, and since we won't need this generality in the sequel, we omit a proof of the following:

Theorem 3.8. *Let H a real Hilbert space and C a proper convex cone in H. Then for any vector $x \in H$ there exists a unique decomposition of the form $x = x_C + x_{C^p}$ under the conditions that $x_C \in C$, $x_{C^p} \in C^p$ and $x_C \perp x_{C^p}$.*

From the above computation about the polar of a sum of cones, the important formula below, valid for an arbitrary subspace F, follows:

$$(F + P)^p = F^\perp \cap -P$$

Although P has a void interior, one can surrogate interior and boundary by means of the concepts of quasi-interior (more briefly intern) and quasi boundary (more briefly extern).

Definition 3.5. The set of all strictly positive vectors $\{y : y_i > 0, \forall i\}$ is denoted by $P^\vee$ and called the intern of P. The set $P \backslash P^\vee$ is denoted by $P^\wedge$ and called the extern of P.

Definition 3.6. We say that a linear subspace F is intern to P if

$$F \cap P^\vee \neq \phi$$

We say that a closed subspace F is weakly tangent or extern to P if it is neither strictly tangent nor intern.

4. THE CONE $F + P$

In this Section we study the cone $F + P$ in l_2, establishing various important results, in particular some main Theorems stating that, if F is closed and strictly tangent to P, then F is contained in an closed hyperplane strictly tangent to P, the lineality space of $F + P$ is F, and $F + P$ is closed.

We start with the following:

Theorem 4.1. *A closed hyperplane L is strictly tangent to P if and only if $L^{\perp}$ is intern to P (or, equivalently, there is a normal n to L with $n \in P^{\vee}$). Moreover, if L is strictly tangent to P, $L + P$ is a proper cone, more precisely, it is a closed semispace, and:*

$$lin(L + P) = L$$

Proof. First statement: one direction is straightforward because if $L^{\perp}$ is intern to P then L must clearly be strictly tangent in order not to violate orthogonality. Conversely suppose L is strictly tangent to P and let $L^{\perp} = \mathcal{L}(\{n\})$ (with, of course $n \perp L$). Clearly n cannot have zero components, for otherwise the fact that L is strictly tangent would be easily falsified by a vector with a single positive component in correspondence of the index for which the component of w is zero. By a similar simple argument n cannot have pair of non-zero components of opposite sign. Thus either $n \in P^{\vee}$ or $-n \in P^{\vee}$ and so we are done with the first statement. Suppose, without restriction of generality, that $n \in P^{\vee}$. Then:

$$L + P \subset \mathcal{N}(n, .) + P \subset$$
$$\subset \{y : (n, y) \geq 0\} + P = \{y : (n, y) \geq 0\}$$

Hence we have proved that $L + P$ is a proper cone. Clearly $lin(L + P) \supset L$. If this inclusion were proper, we would have $lin(L + P) = H$, which is a contradiction. Thus $lin(L + P) = L$. Finally note that, if we write $y = y_L + y_{L^{\perp}}$, the condition $(n, y) \geq 0$ yields $y_{L^{\perp}} = \alpha n$ with $\alpha \geq 0$. Thus:

$$\{y : (n, y) \geq 0\} \subset L + P$$

Therefore $L + P = \{y : (n, y) \geq 0\}$ as stated. ∎

Next we cover the case in which a finite dimensional space is strictly tangent to P with the following:

Theorem 4.2. *If a finite dimensional) subspace F is strictly tangent to P, then F is contained in a closed hyperplane L strictly tangent to P (or, equivalently there exists a vector $n \perp F$ with $n \in P^{\vee}$). Consequently, $F + P$ is a proper cone. Moreover:*

$$lin(F + P) = F$$

Proof. Consider the sequence of coordinate subspaces $\Gamma_k = \mathcal{L}(\{e_i : i = 1, ..k\})$ for $k = 1, 2, ...$ For sufficiently large k, $F \subset \Gamma_k$ in order not to contradict that F is finite dimensional. Fix such a k and in Γ_k, which for simplicity we denote by Γ, apply the finite dimensional theory (see [1]) to conclude that in Γ our statement hold good, so that there is a vector μ in the interior of the positive cone of Γ with $\mu \perp F$. Next take any $e \in P^{\vee}$ and consider the vector $n = \mu + P_{\Gamma^{\perp}} e$. It is obvious by construction that such vector has the properties specified in the statement and so we are done for this part of the Theorem. Next Let $L = \{n\}^{\perp}$. Then for $y \in F + P \subset L + P$, write $y = w + z$ with $w \in F$ and $z \in P$ and then, with obvious meaning of symbols, we can also write $y = w + z_L + \alpha n$ with $\alpha \geq 0$. If we also want $-y \in F + P$ then $\alpha \leq 0$. Hence $\alpha = 0$. But then $y - w = z_L \in P$ implies $z_L = 0$ and so $y = w$. Thus $lin(F + P) = F$. This completes the proof. ∎

The following Theorem extends this result to the infinite dimensional case.

Theorem 4.3. *If a closed (infinite dimensional) subspace F is strictly tangent to P, then F is contained in a closed hyperplane L strictly tangent to P (or, equivalently, there exists a vector $n \perp F$ with $n \in P^{\vee}$). Consequently, $F + P$ is a proper cone. Moreover:*

$$lin(F + P) = F$$

Proof. The proof is based on two separation exercises, arguing on which we will deduce our thesis. The weakly compact convex set $\Upsilon = \mathcal{C}^{-}(\{e^i\}\} \subset P$ clearly contains the origin. Thus we consider a vector $\zeta \in P^{\vee}$ with, to fix ideas $\|\zeta\| < 1$, and the weakly compact convex set $\Psi = \mathcal{C}^{-}(\{e^i + \zeta\}) = \zeta + \Upsilon \subset P$. Clearly Ψ is disjoint from $\{0\}$. Next we can apply to Ψ and F the strong separation principle ([6]) and consequently affirm that there exists a weakly continuous (and hence strongly continuous) functional $(n, .)$ (with $n \neq 0$) that strongly separates Ψ and F. Clearly it must be $(n, F) = 0$ and $\inf\{(n, y) : y \in \Psi\} = (n, \zeta) = m > 0$. Moreover, $\forall i$,

$$(n, e^i + \zeta) = n_i + (n, \zeta) \geq m \Rightarrow n_i \geq 0$$

Thus $n \in P$, $n \neq 0$ (and so $(n, P) = [0, +\infty)$), $n \perp F$, and so $(n, .)$ weakly separates F and P. We don't know about $iz(n)$. It might well be $ip(n) = \mathfrak{N}$, in which case we were done, but also it might be $ip(n) \subset \mathfrak{N}$ properly, and so, to begin with, we go on with another separation exercise. For each positive integer k let $K = \{1, .., k\}$. Consider the set $\Pi = \mathcal{C}^{-}(\{e^i : i \in K, e^i + \zeta : i \notin K\})$. Define $\Theta = \mathcal{C}(\{e^i : i \in K\})$ and $\Lambda = \mathcal{C}^{-}(\{e^i + \zeta : i \notin K\})$ so that $\Pi = \mathcal{C}^{-}(\Theta \cup \Lambda) = (\cup[x : y] : x \in \Theta, y \in \Lambda)^{-}$. Then use the same separation principle arguing in a similar manner as before, but taking care of using, for vectors in Λ, the minimum of the functional on Λ itself. In this way we find a functional $(\nu, .)$, with $\nu \in P$, $\nu \neq 0$ (and so $(n, P) = [0, +\infty)$), $\nu \perp F$ strongly separating Π and F, Moreover this time we can claim that $ip(\nu) \supset K$. At this point consider the set $\Omega = \{w, \|w\| = 1, w \in P, w \perp F\}$. We have just shown that Ω is non-void, and in fact that it is an infinite set. In what follows bear in mind that making finite sums of elements of Ω (and renormalizing the sum), we obtain another element of Ω, with an index set of positive components containing those of each addend vector. Next order the elements of Ω with the order $\succ$ defined by:

$$w^1 \succ w^1 \text{ if } ip(w^1) \supset ip(w^2) \text{ properly}$$

Clearly there are towers with respect to this order in Ω. By the maximal principle there is a maximum tower $\mathcal{T}$ in Ω. Note that $\mathcal{T}$ is countable and so we can write $\mathcal{T} = \{n^1, n^2, ..\}$. Finally take a vector $f \in P^{\vee}$ and define:

$$n = \sum_{i=1}^{\infty} f_i n^i$$

It is readily verified that, by construction, we have thereby defined a non-zero vector in l_2, such that $n \in P^{\vee}$ and $n \perp F$ (and of course we might renormalize n without changing this properties). Next let $L = \{n\}^{\perp}$. Then for $y \in F + P \subset L + P$, write $y = w + z$ with $w \in F$ and $z \in P$ and then, with obvious meaning of symbols, we can also write $y = w + z_L + \alpha n$ with $\alpha \geq 0$. If we also want $-y \in F + P$ then $\alpha \leq 0$. Hence $\alpha = 0$. But then $y - w = z_L \in P$ implies $z_L = 0$ and so $y = w$. Thus $lin(F + P) = F$. This concludes the proof. ∎

It is now in order to cover the intern subspace case. Notice that we do not assume, in the first part of next Theorem, that the subspace to be closed.

Theorem 4.4. *Suppose that a subspace F is intern to P. Then:*

$$(F + P)^{-} = H$$

A closed subspace F is intern to P if and only if $F^{\perp}$ is strictly tangent to P. Obviously the role of F and $F^{\perp}$ can be interchanged. Consequently a closed subspace F is extern to P if and only if $F^{\perp}$ is also extern to P.

Proof. Consider a vector $z \in F \cap P^{\vee}$. For any integer $i > 0$, we can add to $-z \in F + P$ the vector $w \in P$, which has all the component equal to minus the corresponding component of $-z$ except the ith, which is equal to zero. The resulting vector $-z + w$ is such that $(-z + w)_i < 0$ and all the other components are zero. It follows from this that the cone $F + P$ contains both the base $\{e^i\}$ and the base $\{-e^i\}$. Hence $F + P$ contains the dense linear subspace $\mathcal{L}(\{e^i\})$ and therefore $F + P$ itself is dense. Now, if is $F + P$ is dense, then:

$$(F + P)^p = F^{\perp} \cap -P = \{0\}$$

and this means that $F^{\perp}$ is strictly tangent to P. If $F^{\perp}$ is strictly tangent to P, by the preceding Theorem, it is contained in a closed hyperplane L also strictly tangent to P, and with the unit normal $n \in P^{\vee}$. Thus:

$$F^{\perp} \subset L \Longrightarrow F^{\perp\perp} = F \supset \mathcal{L}(\{n\})$$

and, because $\mathcal{L}(\{n\})$ is intern to P, we are done. ∎

At this point we know that, if the closed subspace F is strictly tangent to P, then F is the lineality space of $F + P$ and that $F^{\perp}$ is intern to P. The next natural step is to apply to $F + P$ the decomposition Theorem 3.5. In view of such a Theorem, we get:

$$F + P = F + P_{F^{\perp}}(F + P) = F + P_{F^{\perp}}P$$

This formula motivates us to take a close look at the cone $P_{F^{\perp}}P$, under the assumption that $F^{\perp}$ is intern to P. In the next Theorem we will use a basic capping technique for pointed cones, compared to the theory in e.g. [7]. In fact our capping method uses continuous linear functionals, and the right topology for obtaining compactness of the capped cone is the $\mathcal{X}$ topology. Incidentally, this is also the right topology to introduce the the infinite dimensional counterpart of polytope (see [4]).

Regarding the following proof, we will work with sequential compactness. This is legal because the underlying topological space (l_2 with the relative product topology) is metrizable (see Theorem 6.10 in [6]), and a metrizable space is sequentially compact if and only if it is compact. We exploit this fact to simplify notations and exposition, although a proof using nets would be possible.

Theorem 4.5. *If a closed subspace F is intern P, then the (pointed) cone $P_F P$ is closed.*

Proof. Consider a unit vector $f \in F \cap P^{\vee}$. We use the functional $(f, .)$ to cap the cone P. Let for $\alpha > 0$:

$$\Sigma_{\alpha} = \{y : (f, y) \leq \alpha\}$$

Then we represent the cone as the union of what we call caps of integer level $\alpha = r$:

$$P = \cup\{\Sigma_r \cap P : r = 1, 2, ...\}$$

To simplify notations we use a short symbol for the cap of level r:

$$\Omega_r = \Sigma_r \cap P$$

Thus

$$P = \cup\{\Omega_r\} \text{ and } P_F P = \cup\{P_F \Omega_r\}$$

(we omit $r = 1, 2, ...$ for brevity). Now we know from [4] that each cap Ω_r is (convex and) $\mathcal{X}$ - compact. Because $f \in F \cap P$ it follows that $P_F f = f$. We now claim that:

$$y \in P_F \Omega_r \Leftrightarrow y \in P_F P \text{ and } (f, y) \leq r$$

which will be used in the equivalent form:

$$y \in P_F P \backslash P_F \Omega_r \Leftrightarrow y \in P_F P \text{ and } (f, y) > r$$

In fact suppose that $y \in P_F \Omega_r$ so that $y = P_F w$ with $w \in \Omega_r$. Then

$$(f, y) = (f, P_F w) = (P_F f, w) = (f, w) \leq r$$

Conversely suppose that for $y \in P_F P$ it is true that $(f, y) \leq r$. Then $y = P_F w$ with $w \in P$ and

$$(f, y) = (f, P_F w) = (P_F f, w) = (f, w) \leq r$$

so that $y \in P_F \Omega_r$. Now consider a (strongly) convergent sequence $\{y^i\} \to y$ in $P_F P$. We claim that for sufficiently large r, $\{y^i\}$ is eventually in $P_F \Omega_r$ (For simplicity we will denote $P_F \Omega_r$ by Ψ_r). For if this were not true, so that $\forall r$, $\{y^i\}$ would be repeatedly in $P_F P \backslash \Psi_r$, then, in view of what we have just proved, the sequence $\{(f, y^i)\}$ could not converge. But this is a contradiction, because $(f, .)$ is continuous. It follows that we can assume (and neither we do change symbols) that the sequence $\{y^i\}$ is in Ψ_r, and writing $\forall i$, $y^i = P_F z^i$, we have that $\{z^i\}$ is in Ω_r. Because Ω_r is (sequentially) $\mathcal{X}$ - compact, there is a subsequence of $\{z^i\}$, which we will denote with the same symbol $\{z^i\}$, which converges, in the $\mathcal{X}$ -topology to a (unique) limit $z \in \Omega_r$. Using the properties of the $\mathcal{X}$ -topology as well as basic Hilbert space facts, if X_k is the coordinate subspace of l_2 generated by $\{e^1, .., e^k\}$, then we can write:

$$\forall k, \ \lim_i \{P_{X_k} z^i\} \to_s P_{X_k} z$$

$$\forall i, \ \lim_k \{P_{X_k} z^i\} \to_s z^i$$

$$\lim_k \{P_{X_k} z\} \to_s z$$

where $\to_s$means: converges strongly. Therefore,using the last, for any $\varepsilon > 0$, there exist $\mathbf{k}(\varepsilon)$ such that:

$$\forall k \geq \mathbf{k}, \ \|P_{X_k} z - z\| \leq \frac{\varepsilon}{2}$$

and,using the first, for any $k, \varepsilon > 0$,there exists an $\mathbf{i}(k)$ such that:

$$\forall i \geq \mathbf{i}(k), \ \|P_{X_k} z^i - P_{X_k} z\| \leq \frac{\varepsilon}{2}$$

Notice, incidentally, that $\mathbf{i}(k)$ can be taken to be an increasing function and thus it is invertible. From the last two inequalities, we have that $\forall k \geq \mathbf{k}$, $\forall i \geq \mathbf{i}(k)$:

$$\|P_{X_k} z^i - z\| \leq \|P_{X_k} z^i - P_{X_k} z\| + \|P_{X_k} z - z\| \leq \varepsilon$$

Now we can extract from the double sequence a diagonalized sequence. Take $\varepsilon = 1/3j$, $\mathbf{k}(1/3j)$, $\mathbf{i}(\mathbf{k}(1/3j))$ and define $\{w^j\} = \{P_{X_{\mathbf{k}(1/3j)}} z^{\mathbf{i}(\mathbf{k}(1/3j))}\}$, which is evidently in Ω_r and converges strongly to $z \in \Omega_r$. We slight modify the subsequence taking a function $k(j) \geq \mathbf{k}(1/3j)$ in such a way that $k(.)$ is strictly increasing with j. We substitute the function $\mathbf{i}(\mathbf{k}(.))$ accordingly and end up with a subsequence (denoted by the same symbol) $\{w^j\} = \{P_{X_{k(j)}} z^{i(k)}\}$. Using the second limit we can say:

$$\forall i, j, \exists h(i, j) \text{ s.t. } \forall h \geq h(i, j), \ \|P_{X_h} z^i - z^i\| \leq 1/3j$$

Next we extract a subsequence from $\{z^i\}$ and a corresponding subsequence from $\{w^j\}$ according to the following procedure. For $r = 1$ take $z^r = z^1$ and take $w^1 = w^{h(1)}$ in such a way that $h = \min\{j \text{ s.t. } k(j) \geq h(r, 1)\}$. The subsequent steps are similar. Given $z^r = z^\gamma$, z^{r+1} is chosen in $\{z^{\gamma+1}, z^{\gamma+2}, ..\}$ in such a way as to minimize $h(i, r)$, and w^{r+1} is chosen in $\{w^{h(r)+1}, w^{h(r)+2}, ..\}$ taking the minimum index δ for which $k(\delta) \geq h(r, 1/3r)$. By construction we can write:

$$\forall r, \forall \rho \geq r \ \|z^\rho - w^\rho\| \leq \|w^\rho - P_{X_\rho} z\| + \|P_{X_\rho} z^\rho - P_{X_\rho} z\| + \|P_{X_\rho} z^\rho - z^\rho\| \leq 1/r$$

Therefore both $\{w^r\}$ and $\{z^r\}$ converges strongly to the same limit z. But $\{P_F z^r\}$ is a subsequence of $\{y^i\}$ and hence converge strongly to y, as well as, by continuity of P_F, to $P_F z$. By uniqueness of limits $y = P_F z$. Thus $y \in \Psi_r = P_F \Omega_r \subset P_F P$. It follows that $P_F P$ is closed and the proof is finished. ∎

We are now in a position to quickly prove the following main

Theorem 4.6. *If a linear subspace F of l_2 is closed and strictly tangent to P, then $F + P$ is closed. Consequently, if F is not closed:*

$$(F + P)^- = F^- + P$$

Proof. As to the first statement suppose F is closed and strictly tangent to P. Then, as we have proved:

$$F + P = F + P_{F^\perp} P$$

where the cone pointed $P_{F^\perp} P \subset F^\perp$ is closed. Thus by an application of Lemma 3.1 the first statement is proved. Next, by what we have just proved:

$$(F + P)^- \subset F^- + P$$

On the other hand

$$F^- + P \subset (F + P)^-$$

comes from an elementary topological computation, and hence the proof is finished. ∎

The generalization to non-closed subspaces of this result goes through only half the way. Nevertheless, in Section 8, we will manage to achieve a full generalization, under strict tangency, of Optimization Theory for non closed range operators.

But for now here is our result for strictly tangent non closed subspaces.

Theorem 4.7. *Suppose the subspace F is non-closed and strictly tangent to P. Then in $F + P$, and a fortiori in $lin(F + P)$, there are no vectors of $B = F^- \backslash F$. Moreover:*

$$lin(F + P) = F$$

Proof. We start observing that if for two vectors $\gamma \in F^- \backslash F$ and $\delta \in F$ then $\eta = (\gamma - \delta) \notin F$. In fact if it were $\eta \in F$ it would also be $\gamma = \eta + \delta \in F$, which is a contradiction. Let $z \in F + P$ so that $z = y + w$ with $y \in F$ and $w \in P$. Suppose that $z \in F^- \backslash F$. Then $w = z - y \in F^- \backslash F$. Since $F^- \cap P = \{0\}$, $w = 0$. But then $z = y \in F$, which is a contradiction. Hence in $F + P$, and a fortiori in $lin(F + P)$, there are no vectors of $F^- \backslash F$. At this point we obviously know:

$$F \subset lin(F + P) \subset lin(F^- + P) = F^-$$

where we have applied Theorem 4.3. Combining this with the first part of the present Theorem, which insures:

$$lin(F + P) \cap (F^- \backslash F) = \phi$$

the reverse inclusion follows:

$$lin(F + P) \subset F$$

and hence:

$$lin(F + P) = F$$

as we wanted to show. ∎

5. POLYHEDRA

We first provide an overview of the setting for infinite dimensional polyhedra in l_2. Then we will connect weakly compact convex bodies, which are always polyhedra, to non-closed range operators and strict tangency.

We define polyhedra as follows:

Definition 5.1. A polyhedron $\mathcal{G}$ (in l_2) is a countable intersection of closed semispaces:

$$\mathcal{G} = \cap \{x : (g^i, x) \leq v_i, i = 1, 2, ...\}$$

where $g^i \in l_2$ and $g^i \neq 0$, $\forall i$. If $v_i = 0$, $\forall i$ then $\mathcal{G}$ is a cone and is called polyhedral cone.

Without restriction of generality, we can divide each inequality by $\|g^i\|$, and hence we can assume, whenever convenient, that $\|g^i\| = 1$.

We may also express a polyhedron by:

$$\mathcal{G} = \mathcal{G}(G, v) = \{x : Gx \leq v\}$$

where G is an infinite matrix whose rows are the vectors g^i, the order $\leq$ denotes the product vector ordering in $R^{\mathfrak{N}}$, and $v \in R^{\mathfrak{N}}$ (called the bound vector) has components v_i. Note that G is continuous $(l_2, \mathcal{S}) \to (R^{\mathfrak{N}}, \mathcal{X})$. However the matrix G does not represent in general an operator $(l_2, \mathcal{S}) \to (l_2, \mathcal{S})$.

Once polyhedra are defined in this way, well known properties of separable Hilbert spaces allow us to state that any closed convex set is a polyhedron (see e.g. [4]):

Theorem 5.1. *Consider a non-void strongly closed convex subset $C \neq H$ in a separable Hilbert space H and let D be a countable dense subset of H. Then, $\forall \zeta \in \mathcal{B}(C)$ (the boundary of C), there exists a sequence of support points $\{z_i\}$ of C, such that $\{z_i\} \to \zeta$ strongly, where $\{z_i\}$ is in the countable set $P_C(D \backslash C)$. Thus there exists a countable set of support points dense in $\mathcal{B}(C)$. Moreover, the countable intersection of supporting semispaces defined by the points of the countable set $P_C(D \backslash C)$ and the corresponding normals (i.e. if $y \in D \backslash C$ the normal is $y - P_C y$) coincides with C. Hence any non-void closed convex set C is a polyhedron and any non-void closed cone is a polyhedral cone.*

Regarding translations of polyhedra we can state:

Lemma 5.2. *The translate of a polyhedron is a polyhedron. In particular, if we translate a polyhedron $\mathcal{G}$ by the opposite one of its points $-t$ (so that $0 \in \mathcal{G} - t$) then, in the representation of $\mathcal{G} - t$, the bound vector is non-negative.*

Proof. Just note that

$$-t + \mathcal{G}(G, v) = \mathcal{G}(G, v - Gt)$$

∎

We are especially interested to the case where G is an $(l_2, \mathcal{S}) \to (l_2, \mathcal{S}))$ operator and the bound vector is in l_2. This restriction is milder than one may expect.

In fact we can state (after [4]):

Theorem 5.3. *Consider a polyhedron $\mathcal{G}(G, v)$ and assume that the bound vector $v \in l_\infty$, then the same polyhedron is representable as*

$$\mathcal{G} = \{x : \Gamma x \leq \gamma\}$$

where $\gamma \in l_2$ and Γ is an operator (i.e. continuous linear transformation) $(l_2, \mathcal{S}) \to (l_2, \mathcal{S})$. All polyhedral cones (and all closed subspaces), all bounded polyhedra and all convex and weakly compact set can be (and will be) represented in this way. In the case of cones we can always *assume that $\gamma = 0$. Moreover, by the preceding Lemma, this class of polyhedra is closed under translation.*

Remark 5.1. As to convex and weakly compact sets, there might well be in their definition some redundant constraints that prevent the bound to stay in l_∞. However, because the set is contained in a closed sphere, it is obvious that they may be replaced by new constraints in such way that the bound stay in l_∞. We underline that the method of giving and taking redundant constraints could be used to achieve more favorable features in various cases. An example in point is the Theorem for weakly compact convex bodies below.

Throughout the rest of the paper the assumption that G is an operator and $\gamma \in l_2$ will be in force.

Note that the positive cone P of l_2 is a polyhedral cone with $G = -I$.

The feasibility problem for a polyhedron $\mathcal{G}(G, v)$ is that of determining whether or not $\mathcal{G}(G, v) \neq \phi$.

We now state some general necessary and sufficient range space conditions for the polyhedron $\mathcal{G}(G, v)$ to be non-void, which are formally identical to their finite dimensional counterpart (e.g. [1]), and are readily proved much in the same way. They are given by:

$$\mathcal{G}(G, v) \neq \phi \Leftrightarrow v \in \mathcal{R}(G) + P \Leftrightarrow$$
$$(v + \mathcal{R}(G)) \cap P \neq \phi \Leftrightarrow (v - P) \cap \mathcal{R}(G) \neq \phi$$

The set

$$S = (v + \mathcal{R}(G)) \cap P$$

is called the *slack set* (essentially the slack set can be considered as the polyhedron, as viewed from the Range Space side). The slack set is of course closed, if $\mathcal{R}(G)$ is closed.

Another important set, which determines a polyhedron by a suitable inverse image is

$$\Sigma = (v - P) \cap \mathcal{R}(G) = v - S$$

In fact we can write:

$$\mathcal{G}(G, v) = G^{-1}(\Sigma)$$

Using a suitable form of the Induced Map Theorem (as in [4]) we can write

$$G = G|_{\mathcal{N}(G)^\perp} P_{\mathcal{N}(G)^\perp}$$

where both map are continuous, $G|_{\mathcal{N}(G)^\perp}$ is one to one from $\mathcal{N}(G)^\perp$ onto $\mathcal{R}(G)$ and, also, it is a topological isomorphism if and only if $\mathcal{R}(G)$ is closed. From this it follows:

$$\mathcal{G}(G, v) = (G|_{\mathcal{N}(G)^\perp})^{-1}\Sigma + \mathcal{N}(G)$$

Note that if $\mathcal{N}(G) = \{0\}$ and $\mathcal{R}(G)$ is closed, this formula implies that $\mathcal{G}^i = \phi$ (because P has void interior).

The next Theorem characterizes polyhedra that are weakly compact convex bodies.

Theorem 5.4. *Assume that $\mathcal{G}$ is bounded and has non-void interior. Then $\mathcal{G}$ admits a representation where $\mathcal{R}(G)$ is strictly tangent to P. Moreover, $\mathcal{R}(G)$ cannot be closed and G must be one to one.*

Proof. Without restriction of generality we can assume that $0 \in \mathcal{G}^i$ (for otherwise we can translate $\mathcal{G}$ by $-t$ where $t \in \mathcal{G}^i$). Let D be a countable dense subset of l_2. Applying the usual argument based on projecting the set $D \cap \overline{\mathcal{G}}$ on $\mathcal{G}$ and exploiting the fact that the projection is continuous and contractive, we can affirm that in the boundary $\mathcal{B}(\mathcal{G})$ of $\mathcal{G}$ there is a countable dense subset of support points. Notice that in view of radiality at 0 the origin itself cannot be a support point. Let $\{\zeta^j\}$ be the set of such support points and $\{n^j\}$ the set of their (normalized) normals. If not already present (but from Theorem 5.1, this is exactly how the inequality system may be built), add to the inequalities describing the polyhedron, each inequality:

$$(n^j, x) \le (n^j, \zeta^j)$$

as well as the constraint:

$$(-n^j, x) \le -\min\{(n^j, x) : x \in \mathcal{G}\}$$

where we have of course exploited the fact that $\mathcal{G}$ is weakly compact. The polyhedron stays the same in the new representation. In what follows recall that in the new representation the bound vector is non-negative. Because $\mathcal{G}$ is radial at 0, it suffices to prove that for each x, $x \neq 0$, $x \in \mathcal{G}$ there are two functionals, in the system of inequalities, that assume opposite sign on x. Notice that $(n^j, \zeta^j) > 0$, because we know that $\zeta_j \neq 0$ and if it were $(n^j, \zeta^j) = 0$, given that the above constraint can be read as:

$$(n^j, x) \le (n^j, \zeta^j) = \max\{(n^j, x) : x \in \mathcal{G}\}$$

we would qualify the origin as a support point, which is a contradiction. It follows that the functional $(-n^j, .)$ assumes a negative value at ζ^j and so we are done for any vector proportional to some ζ^j. Next consider any other $x \neq 0$, $x \in \mathcal{G}$, and notice that, clearly, there must be an $\alpha \ge 1$ such that $\alpha x \in \mathcal{B}(\mathcal{G})$. It suffices to argue on the vector αx which, for simplicity, we call again x. At this point note that there is a closed sphere S_r around the origin of radius $r > 0$ such that $S_r \subset \mathcal{G}$. By the way we expressed the constraints defined by the functionals $(n_j, .)$ and this inclusion relations we can write $\forall j$:

$$(n^j, x) \le (n^j, \zeta^j) = \max\{(n^j, x) : x \in \mathcal{G}\} \ge r$$

Now we know that there exists a sequence $\{\zeta^k\}$ in $\mathcal{B}(\mathcal{G})$ converging to x strongly. Take an integer $\mu >> 1$ and $\overline{k}$ such that:

$$\|x - \zeta^k\| \le r/\mu$$

and write:

$$(n^k, x) = (n^k, x - \zeta^k) + (n^k, \zeta^k)$$

and notice that $(n^k, \zeta^k) \ge r$ and (by CBS inequality):

$$|(n^k, x - \zeta^k)| \le r/\mu$$

which evidently implies $(n^k, x) > 0$. It follows, as before that the functional $(-n^k, .)$ assumes a negative value at x, and so we are also done with any other point of $\mathcal{B}(\mathcal{G})$. It remains one last observation to complete the proof of the first statement. We have now, in our inequality systems, new rows, so that the ensuing extended matrix, that we still call G, might not represent an operator anymore. However the new bound vector, which we still call v, is, by construction, in l_∞. Therefore, if we finally transform our polyhedron according to the proof of Theorem 5.3, we obtain a new matrix, still called G, (along with a new bound vector still called $v \in l_2$), which now represents an operator, while clearly the property of strict tangency remains invariant. Finally we turn to the last statement and observe that in the expression:

$$\mathcal{G}(G, v) = (G|_{\mathcal{N}(G)^\perp})^{-1}\Sigma + \mathcal{N}(G)$$

it must be $\mathcal{N}(G) = \{0\}$ in order to not contradict boundedness. Therefore G must be one to one (in other words a linear isomorphism). Next if $\mathcal{R}(G)$ were closed, $G|_{\mathcal{N}(G)^\perp}$ would be a topological isomorphism. But then, since Σ is a subset of P and therefore cannot have interior, it would follow that $\mathcal{G}(G, v)$ has no interior. But this is a contradiction and, consequently, $\mathcal{R}(G)$ cannot be closed. This concludes the proof. ∎

Remark 5.2. This Theorem could be generalized using the space $\mathcal{L}^{-}(\mathcal{G}(G,v))$ in place of H. But we do not pursue this here.

6. OPTIMIZATION, THE STRICTLY TANGENT AND THE INTERNAL CASE

The linear optimization problem (LP) consists in determining whether a (non-zero) continuous linear functional f has a maximum on a (non void) polyhedron $\mathcal{G}(G,v)$ and, in the positive case, in determining both the maximum and the subset of $\mathcal{G}$ (or, at least, one point of $\mathcal{G}$) where the maximum is attained. We indicate this problem writing:

$$\max(f,x) : x \in \ \mathcal{G}(G,v) = \{x : Gx \leq v\}$$

The problem is *feasible* if $\mathcal{G}(G,v) \neq \phi$. Assuming feasibility, we look for, if it exists at all, the maximum of the set of reals $f(\mathcal{G})$, which is convex and hence is a non-void interval. If and only if the interval has finite right extremum and is right closed, the problem admits (exact) solutions.

In [4] three cases were considered, the last of which is divided in two subcases. Here we prefer to classify directly four possible cases.

First, the polyhedron is void (*unfeasible problem*). Of course in this case the problem vanishes.

Second, the polyhedron is non-void, but $\sup(f(\mathcal{G})) = \infty$ (*feasible unbounded problem*).

Third, the polyhedron is non-void, and $\sup(f(\mathcal{G})) < \infty$, but $f(\mathcal{G})$ is right open. In this case we can compute the supremum m of the functional on the polyhedron $\mathcal{G}$, but this supremum is not attained in any point $x \in \mathcal{G}$ (*feasible bounded indefinite problem*). However, one should consider in this case approximate solutions, that is points in the domain space, on which the functional takes on the value $m - \varepsilon$, for some arbitrary $\varepsilon > 0$.

Fourth, the polyhedron is non-void, $\sup(f(\mathcal{G})) < \infty$ and $f(\mathcal{G})$ is right closed, so that the supremum of the functional is attained on some point x of $\mathcal{G}$ (*feasible bounded definite problem*). In this case we define $ma = \max\{f(x) : x \in \mathcal{G}(G,v)\}$.

An important special case in which the problem is feasible bounded and definite, whatever f might be, is when $\mathcal{G}$ is bounded and hence a weakly compact set. In this case, since f is weakly continuous, $f(\mathcal{G})$ must be compact too and hence it must be a closed bounded interval.

Next, we define:

$$\widehat{G} = \begin{pmatrix} -f \\ G \end{pmatrix}; \widehat{v}(h) = \begin{pmatrix} -h \\ v \end{pmatrix}$$

where clearly $-f$ is disposed as a row and h is a real parameter. The matrix $\widehat{G}$ represents again an operator $H \to \widehat{H}$, where $\widehat{H}$ is an extended Hilbert space of which H is a closed subspace of codimension 1. Note that, in the case of infinite dimensions, spaces H and $\widehat{H}$ are isomorphic. Also we leave to the context to distinguish between the positive cones P and $\widehat{P}$, using for both the same symbol P.

It seems reasonable to exclude that $\mathcal{R}(\widehat{G})$ be dense in H. Throughout the rest of the paper this assumption will be in force. It will appear justified on formal ground in the sequel.

In this setting problem LP is recasted in the form:

$$\max\{h : \widehat{\mathcal{G}}(\widehat{G}, \widehat{v}(h)) \neq \phi\}$$

(notationwise, we should write more correctly $\mathcal{G}(\widehat{G}, \widehat{v}(h))$, however $\widehat{\mathcal{G}}(\widehat{G}, \widehat{v}(h))$ is more suggestive, especially if, for brevity, we drop the arguments).

To determine if a maximum exists, we have to look for an $\overline{h}$ (that will be the maximum value of the functional on $\mathcal{G}(G,v)$) such that, if $h \leq \overline{h}$, then $\widehat{\mathcal{G}}(\widehat{G}, \widehat{v}(h))$ is non-void, whereas, for all $h > \overline{h},$ $\widehat{\mathcal{G}}(\widehat{G}, \widehat{v}(h))$ is void, We will use two of the previously recalled feasibility conditions, to state two equivalent optimality conditions. The first is formulated saying that it must exist an $\overline{h}$ (which, in the positive case will be the maximum of the problem) such that

$$(\widehat{v}(h) + \mathcal{R}(\widehat{G})) \cap P \neq \phi \text{ for } h \leq \overline{h}$$

and

$$(\widehat{v}(h) + \mathcal{R}(\widehat{G})) \cap P = \phi \text{ for } h > \overline{h}$$

In this way the optimality condition has the form of a tangency condition.

In the second formulation, $\overline{h}$ must be such that:

$$\widehat{v}(h) \in \mathcal{R}(\widehat{G})) + P \text{ for } h \leq \overline{h}$$

and

$$\widehat{v}(h) \notin \mathcal{R}(\widehat{G})) + P \text{ for } h > \overline{h}$$

In presence of strict tangency, feasibility may or may not occur. For the finite dimensional case this is stated in [1]. Lets make an infinite dimensional example. Suppose that $\mathcal{R}(\widehat{G})$ is closed. Then we know from [4] that $\mathcal{R}(G)$ is closed too. Next suppose $\mathcal{R}(G)$ is strictly tangent to P. Then $\mathcal{R}(\widehat{G})$ is strictly tangent too. Now we know from Theorems 4.3 and 4.6 that the feasibility cone $\mathcal{R}(G) + P$ is a proper closed cone. Thus the bound vector v may or may not be in such a cone and thus the problem may or may not be feasible.

In the next Theorem we assess the situation for strict tangency (for the moment assuming $\mathcal{R}(\widehat{G})$ is closed) and for the intern case, both in the case of closed $\mathcal{R}(\widehat{G})$ and in that of non-closed $\mathcal{R}(\widehat{G})$.

Theorem 6.1. *If $\mathcal{R}(\widehat{G})$ is closed and strictly tangent to P then feasibility of the optimization problem implies that the same problem is also bounded and definite. If $\mathcal{R}(\widehat{G})$ (closed or not) is intern to P and the optimization problem is feasible, then it is also unbounded.*

Proof. We know from the previous analysis that $\mathcal{R}(\widehat{G}) + P$ is a proper closed cone. And also that :

$$\mathcal{R}(\widehat{G}) + P \subset L + P$$

where L is a closed hyperplane strictly tangent to P and finally that that $L + P$ is a closed semispace. Let n be the external normal to the closed semispace $L + P$, so that:

$$L + P = \{y : (n, y) \leq 0\}$$

We also know that $n \in -P^{\vee}$. Next we claim that the line

$$l = \{\widehat{v}(h) : h \in R\}$$

cannot be contained in L. In fact, if it were so, take a difference of two vectors in this line to obtain a vector of the form:

$$z = \begin{pmatrix} \eta \\ 0 \end{pmatrix}$$

with $\eta > 0$. Because $z \in L$, it follows that L cannot be strictly tangent. This is a contradiction and therefore l is not contained in L. On the other hand if we take an arbitrary vector $y \in l$ and denote the first coordinate axis $\mathcal{L}(\{e^1\})$ by X_1 we can write:

$$l = y + X_1$$

But then $(n, l) = R$ and this implies that a whole halfline is in $\overline{L + P}$ (corresponding to $h > \widetilde{h}$ for a certain real $\widetilde{h}$) and all the more such a halfline is outside the closed cone $\mathcal{R}(\widehat{G}) + P$. Thus, because the problem is feasible, $l \cap (\mathcal{R}(\widehat{G}) + P)$ is a closed half line. This means that the LP problem is bounded and definite. As to the intern case, we know from Theorem 4.4 that, whether or not $\mathcal{R}(\widehat{G})$ be closed, $\mathcal{R}(\widehat{G})+P$ is dense. Since we assume feasibility, there is at least a point $w \in l \cap (\mathcal{R}(\widehat{G}) + P)$. We have seen in the proof of Theorem 4.4 that $X_1 \subset \mathcal{R}(\widehat{G}) + P$. But then, since $l = w + X_1$ and $\mathcal{R}(\widehat{G}) + P$ is a cone, it follows:

$$l \subset \mathcal{R}(\widehat{G}) + P$$

and this means that the LP problem is unbounded. ∎

The line $l = \{\widehat{v}(h) : h \in R\}$, mentioned in the proof, will be called feasibility line, for obvious reasons connected with the previously recalled feasibility conditions.

7. OPTIMIZATION, THE WEAKLY TANGENT CASE

We now study the extern case. We assume that $F = \mathcal{R}(\widehat{G})$ be closed.

We will show that the system can be partitioned in two blocks in such a way that the first "subsystem" is internal, whereas the second is strictly tangent. If the functional falls in the first block, then the whole system, if feasible, is also unbounded. If the functional falls in the second subsystem then if the whole system is feasible, the strictly tangent relaxation is obviously also feasible and therefore, as we know, it is bounded definite.

Recall from [4] that there exists a maximal face M of P (corresponding to an index set denoted by Υ) whose relative intern (the term is self-explanatory) is met by F. Reasoning in the same way as the intern case, we may affirm that $F + P \supset \mathcal{L}(M)$. It might well happen that $F + P \supset \mathcal{L}(M)^-$, in which case the argument in [4] allow us to define a strictly tangent relaxation.

But, when $F + P \supset \mathcal{L}(M)^-$ is false, the argument in [4] cannot be applied. However, we prove here that, surprisingly enough, the definition of a strictly tangent relaxation goes through formally identical to that of the finite dimensional case, even in this general case.

Naturally, the technique of proof is quite different. It is based on the idea of altering the intern relaxation and leaving unchanged the rest of the system. This will allows us to exploit a finite dimensional argument to reach the desired conclusion.

Once we show that we are able to define a strictly tangent relaxation we use the same argument as in Theorem 8.1.1 in [1] to infer from an optimal the solution of the relaxation, an optimal solution of the whole problem (a process that was called "backtracking" in such reference). However, we face the hurdle that the intern block has a feasibility cone that is only dense. For this reason we can derive from the solution of the strictly tangent relaxation a solution for the whole system, which, at least in general, satisfies the constraint given by the internal subsystem to an arbitrary degree of approximation.

Theorem 7.1. *Assume that $\mathcal{R}(\widehat{G})$ is closed and extern to P and that the problem is feasible. Deleting the rows of $\widehat{G}$ in Υ (and doing the same on the components of $\widehat{v}$) we obtain a block G_2 out of $\widehat{G}$. This corresponds to rewriting the extended inequality system as (a reshuffling of the order of inequalities might be required):*

$$Gx \leq v \rightleftharpoons \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} x \leq \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

where, according to the cases, the first row of either block 1 or block 2 corresponds to the functional. Recall that, as proved in [4] $\mathcal{R}(G_1)$ and $\mathcal{R}(G_2)$ are closed. The second block makes up a strictly tangent (feasible) system, (called the strictly tangent relaxation) no matter what block the row corresponding to functional happen to belong. If the inequality corresponding to the functional is in the first block then the LP problem is unbounded whatever may be the constant components (from the second component on) of the bound vector block v_1. If the *inequality corresponding to the functional is in the second block, then the second block defines a strictly tangent, and hence bounded and definite, LP problem with maximum ma. In this latter case either of the following two cases are possible. First case: the whole LP problem is bounded definite, its maximum is equal to that of the strictly tangent relaxation $G_2x \leq v_2$ and is independent of v_1. Second case: for any $\varepsilon > 0$ it is possible to find an vector χ such that it is an optimum solution for the strictly tangent relaxation and $\|G_1\chi - v_1\| \leq \varepsilon$.*

Proof. We first of all prove that the system $G_2x \leq v_2$ is strictly tangent. Indeed note that this property (or the lack thereof) is intrinsic to the second block, the first one being used only as an mean to reach the desired conclusion. At this point, without changing symbols, we relax the first system leaving only a finite number of inequalities. The new system is intern as well, the difference being that now the maximal face whose relative intern is met by $\mathcal{R}(\widehat{G})$ is finite dimensional. At this point the proof that system 2 is strictly tangent is identical to that that was given in the finite dimensional case (see [1]), where $\mathcal{L}(M)$ is obviously closed. Next suppose that the functional happen to be in the first block. Since $\mathcal{R}(\widehat{G}) + P \supset \mathcal{L}(M)$, the feasibility line, which has the form:

$$l = \begin{pmatrix} -h \\ \widetilde{v}_1 \\ v_2 \end{pmatrix}$$

is contained in $\mathcal{R}(\widehat{G}) + P$ whatever is $\widetilde{v}_1$ and so the problem is unbounded whatever is $\widetilde{v}_1$. Finally, assuming that the functional is in the second block (placed in the first position of the second block, in this case), the feasibility line has as parallel linear subspace the first axis of the second block, so that the first block of inequalities do not influence feasibility dependence on h, whatever be the block vector v_1 may be. And solving the strictly tangent relaxation, that we know know to be feasible and hence also bounded and definite, yields the maximum of the relaxed LP problem. Let x be the optimal solution of the strictly tangent relaxation, ma its maximum, and consider then a vector w such that Gw belongs to the relative intern of M. Then, by construction, it is clear that, according to the cases, either there exists an $\alpha > 0$ such that $x + \alpha w$ is the optimal solution of the whole system, or for any $\varepsilon > 0$ there exits and $\alpha(\varepsilon)$ such that $x + \alpha(\varepsilon)w$ is an optimum solution of the strictly tangent relaxation with $h = ma$, and $\|G_1(x + \alpha(\varepsilon)w) - v_1\| \leq \varepsilon$. ∎

For the sake of brevity we do not pursue any generalization to the case of non closed ranges of the theory of the strictly tangent relaxation.

8. Removing the Hypothesis That $\mathcal{R}(\widehat{G})$ Be Closed Under Strict Tangency

In this Section we remove, under strict the tangency hypothesis, the restriction that $\mathcal{R}(\widehat{G})$ be closed In practice the lack of closedness of $\mathcal{R}(\widehat{G})$ turns out to be inconsequential, as precisely stated in the next two Theorems.

Theorem 8.1. *Suppose that the LP problem be feasible and that $\mathcal{R}(\widehat{G})$ be not closed but strictly tangent to P. Then the LP problem is bounded.*

Proof. By definition, $\mathcal{R}(\widehat{G})^-$ is strictly tangent to P. From the assumption of feasibility we can assert that there exists a point in the feasibility line $y \in l$ such that

$$y \in \mathcal{R}(\widehat{G}) + P$$

But from the theory we have developed so far:

$$\mathcal{R}(\widehat{G}) + P \subset \mathcal{R}(\widehat{G})^- + P = (\mathcal{R}(\widehat{G}) + P)^-$$

Now from the previous proof we know that $l \cap \mathcal{R}(\widehat{G})^- + P$ is a closed half-line. Hence $l \cap \mathcal{R}(\widehat{G}) + P$ is an halfline (closed or not we don't know yet) and this means that the LP problem is bounded. ∎

Naturally the next question is: is the problem definite? And, if it is definite, what is the maximum?

The following Theorem answers these questions.

Theorem 8.2. *Suppose that the LP problem be feasible and that $\mathcal{R}(\widehat{G})$ be not closed but strictly tangent to P. Then the LP problem is bounded and definite so that there exists the maximum:*

$$ma = \max\{h : \widehat{v}(h) \in \mathcal{R}(\widehat{G}) + P\}$$

which is the maximum of f on $\mathcal{G}$. Moreover:

$$\max\{h : \widehat{v}(h) \in \mathcal{R}(\widehat{G})^- + P\} = \max\{h : \widehat{v}(h) \in \mathcal{R}(\widehat{G}) + P\}$$

Proof. For brevity we will use, whenever convenient, the notation $\mathcal{R}(\widehat{G}) = F$. We look at the intersection of the feasibility line with the feasibility cone: $l \cap\, (\mathcal{R}(\widehat{G}) + P)$. We have proved that since the problem is feasible and:

$$\mathcal{R}(\widehat{G}) + P \subset \mathcal{R}(\widehat{G})^- + P = (\mathcal{R}(\widehat{G}) + P)^-$$

this intersection must be and half line. We will show that such halfline is closed. For the moment we may assume that the interval of feasibility in terms of the parameter h has the form $\mathcal{I} = (-\infty, m\rangle$ where $\rangle$ may either be $)$ or $]$, we still don't know. So $\widehat{v}(h) \in \mathcal{R}(\widehat{G}) + P$ for $h \in \mathcal{I}$. For $h \in \mathcal{I}$ we can write:

$$\widehat{v}(h) = \widehat{v}_F(h) + \widehat{v}_P(h)$$

with $\widehat{v}_F(h) \in F$ and $\widehat{v}_P(h) \in P$. These decomposition may well be non-unique, but we will fix this later. For two distinct values $h_1, h_2 \in \mathcal{I}$ with $h_2 > h_1$we can write:

$$\widehat{v}(h_1) = \widehat{v}_F(h_1) + \widehat{v}_P(h_1)$$

and

$$\widehat{v}(h_2) = \widehat{v}_F(h_2) + \widehat{v}_P(h_2)$$

By convexity we can write:

$$[\widehat{v}_F(h_2) : \widehat{v}_F(h_1)] \subset F$$

and:

$$[\widehat{v}_P(h_2) : \widehat{v}_P(h_1)] \subset P$$

Clearly for any $h \in [h_1, h_2]$ we have:

$$\widehat{v}(h) = \widehat{v}_F(h) + \widehat{v}_P(h)$$

Consider the line l_F generated by the segment $[\widehat{v}_F(h_2) : \widehat{v}_F(h_1)]$. Clearly

$$l_F = \{\widehat{v}_F(h) : h \in R\} \subset F$$

For $h \in [h_1, h_2]$

$$\widehat{v}_P(h) = \widehat{v}(h) - \widehat{v}_F(h)$$

But for any $h \in R$ this defines another line l_P generated by the segment $[\widehat{v}_P(h_2) : \widehat{v}_P(h_1)]$, and this latter formula holds good for any h. In other words:

$$l = l_F + l_P$$

Note that the symbol of the residual line l_P does not mean that all of it be contained in P. Instead the line l_F has a segment inside F and thus is entirely contained in F. Now we consider the orthogonal decomposition of the space into the parallel (and closed) subspace to the line l_F, and the closed subspace $l_F^{\perp}$, and project l (and hence both l_F and l_P) on these two orthogonal subspaces. We now have an univocal decomposition of l as:

$$l = \lambda_F + l_{\lambda_F^{\perp}} = \lambda_F + P_{\lambda_F^{\perp}} l_P$$

with the line $\lambda_F \subset F$. Now,

$$\lambda_F + P = \lambda_F + P_{\lambda_F^{\perp}} P \subset F + P$$

where we have applied Theorem 4.6 bearing in mind that λ_F is strictly tangent and closed and thus the cone $P_{\lambda_F^{\perp}} P$ is pointed and closed. Thus we have clearly reached the conclusion that $l(h)$ is in $F + P$ if and only if $\lambda_F + l_{\lambda_F^{\perp}}$ is in $\lambda_F + P_{\lambda_F^{\perp}} P$, and this is in turn true if and only if $l_{\lambda_F^{\perp}}(h)$ is in the pointed closed cone $P_{\lambda_F^{\perp}} P$. But just because this latter cone is closed, we have in this way proved that the maximum exists, that is, that the problem is (feasible) bounded and definite. Next we introduce the notation:

$$\mu = \max\{h : \widehat{v}(h) \in \mathcal{R}(\widehat{G})^{-} + P\}$$

thus $\widehat{v}(\mu) + \mathcal{R}(\widehat{G})^{-}$ is tangent to P. Let $y \in (\ \widehat{v}(\mu) + \mathcal{R}(\widehat{G})^{-}) \cap P$. Since $(\widehat{v}(\mu) + \mathcal{R}(\widehat{G}))^{-} = \widehat{v}(\mu) + \mathcal{R}(\widehat{G})^{-}$, there is a sequence $\{y_i\} \rightarrow_s y$ in $\widehat{v}(\mu) + \mathcal{R}(\widehat{G})$. Therefore $d(\widehat{v}(\mu) + \mathcal{R}(\widehat{G}), P) < \varepsilon$ for any $\varepsilon > 0$, But we know that the maximum exists and thus this can only happen for $\mu = ma$ and so the proof is finished. ∎

9. OPTIMIZATION METHOD

In this Section we present a generalization to infinite dimension of the algorithm for solving LP problems, introduced in [3]. Of course the optimization problem in question must be feasible (this can be ascertained beforehand using the feasibility conditions discussed in Section 5, using an obvious variant of the optimization method discussed below).

On the base of the developments presented so far, we may assume, without much harm for generality, that $\mathcal{R}(\widehat{G})$ be closed and strictly tangent to P (and thus also bounded definite, as we know).

We initiate drawing some consequences, for certain minimum distance problems, of the results given in Section 5. We have shown that if a linear subspace F is closed and strictly tangent to P, then $F + P$ is closed and, if instead F is strictly tangent but not closed, then $(F + P)^{-} = F^{-} + P$.

Consider now the set $(v + F) + P$. We can write:

$$(v + F) + P = v + (F + P)$$

Therefore, if F is closed, then $(v + F) + P$ is closed too. It also follows that the difference $P - (v + F)$, which has the same form, is, again, closed. We register now some consequences of what we established so far, in connection with the minimum distance problem.

Proposition 9.1. *If the linear subspace F is closed and strictly tangent to P, then $\forall v$, the sets $(v + F) + P$ and $P - (v + F)$ are closed. Thus in particular if $(v + F)$ and P are disjoint, their distance is positive, and there exist pairs of points $x \in (v + F)$ and $y \in P$, that solve the minimum distance problem.*

The method is based on the repeated solution of the minimum distance problem for the sets $\widehat{v}(h) + \mathcal{R}(\widehat{G})$ (also denoted by $\mathcal{A}(h)$) and P, for a sequence of values of h, which will converge to the maximum of the functional.

Let the maximum of the LP problem be ma. Fix $h = h_0 > ma$, so that the two sets $\widehat{v}(h_0) + \mathcal{R}(\widehat{G})$ and P are disjoint. Repeat the following two steps for $i = 0, 1, ...$

STEP A: find the minimum distance problem solution for $\widehat{v}(h_i) + \mathcal{R}(\widehat{G}) = \mathcal{A}(h_i)$ and P. Call the solution points $x^i \in \widehat{v}(h_i) + \mathcal{R}(\widehat{G})$ and $y^i \in P$. If $x^i = y^i$, then stop and put $h_i = ma$. Otherwise go to STEP B.

STEP B: Adjourn h_i as follows:

$$h_{i+1} = h_i - \frac{\|y^i - x^i\|^2}{(y^i - x^i)_1} = h_i - \Delta h_i$$

We put $n^i = x^i - y^i$ and $\delta_i = \|x^i - y^i\|$.

We will prove that the method is consistent (in particular in the above formula the denominator is never zero) and that it enjoys asymptotic convergence in the sense that $\{h_i\} \to ma$ and $\{y^i - x^i\} \to 0$.

Albeit P has no interior, the above quasi-topological concepts of intern and extern play the expected role:

Lemma 9.2. *If $w \notin P$ then $w^+ \in P^\wedge$ and $n = w - w^+ = w^- \in -P^\wedge$.*

Proof. Immediate bearing in mind that projection zeroes the negative part of w. ∎

Lemma 9.3. *All points z of the set:*

$$Cp = (\widehat{v}(ma) + \mathcal{R}(\widehat{G})) \cap P$$

have the first component $P_1 z = z_1 = 0$ and the residual vector $(I - P_1)z \geq 0$.

Proof. In fact all such points of Cp must get out of P when their first component is decreased. ∎

Lemma 9.4. *Consider two arbitrary $h_b > h_a > ma$, two corresponding pairs of minimum distance points $x^b \in \mathcal{A}(h_b)$, $y^b \in P$ and $x^a \in \mathcal{A}(h_a)$, $y^a \in P$ and let $n^b = x^b - y^b$, $\delta_b = \|x^b - y^b\|$ and $n^a = x^a - y^a$, $\delta_a = \|x^a - y^a\|$. Then the following is true: n^b and n^a (which are both in $-P^\wedge$) have a negative first component and $\delta_b \geq \delta_a$. In other words, the function $\delta(h)$ is monotone decreasing.*

Proof. First we show that for any $h > ma$ (so that $(\widehat{v}(h) + \mathcal{R}(\widehat{G})) \cap P = \phi$) if $x \in \mathcal{A}(h)$, $y \in P$ are a pair of minimum distance points it cannot happen $(y - x)|_1 = 0$. For, otherwise, $\forall \delta > 0$ it would be

$$\widehat{v}(h - \delta) - \widehat{v}(h) \perp ((y - x) = n$$

and this would imply that the affine $(\widehat{v}(h - \eta) + \mathcal{R}(\widehat{G}))$ would not intersect P for any $\eta > 0$, contrary to the fact that the maximum of the optimization problem exists. It follows by the preceding Lemmas that $(y - x)_1 > 0$. Next notice that, in passing from h_b to h_a, all vectors in $\mathcal{A}(h_b)$ have their first component incremented by a positive number and hence their distance from P either decreases or stays the same. It follows that

$$d(\mathcal{A}(h_b), P) = \delta_b \geq d(\mathcal{A}(h_a), P) = \delta_a$$

as we wanted to show. ∎

Now we can state the following main:

Theorem 9.5. *The sequence $\{h_i\}$ converges to ma and the sequence $\{y^i - x^i\}$ converges to zero.*

Proof. As long as $h_i > ma$, $(\widehat{v}(h_i) + \mathcal{R}(\widehat{G})) \cap P = \phi$ and $(y^i - x^i)_1 > 0$, by virtue of the last Lemma. If, instead, $h_i = ma$ the sequences of cycles is arrested. Thus not only the formula to adjourn h_i is consistent, but also $\Delta h_i > 0$. This means that the sequence $\{h_i\}$ is strictly decreasing. Next we show that, unless $x_i = y_i$, $h_i > ma$ implies $h_{i+1} \geq ma$. In fact when we change h_i to h_{i+1}, the affine space translates by a vector $\Delta h_i e^1$. Projecting this translation on the subspace generated by $y_i - x_i$ we obtain exactly $y_i - x_i$. The projection on the orthogonal complement tell us that the affine space translates within the support hyperplane to $y_i + (y_i - x_i)^{\perp}$. The first translation makes the two disjoint sets set intersect. The second either leaves the intersection still non-void, and hence the two sets at step $i+1$ are in tangency position and the procedure is arrested, or it detaches the affine $\mathcal{A}(h_{i+1})$ from the cone P. In the first case we have $h_{i+1} = ma$ and the algorithm stops because the minimum distance is zero. In the second case $h_{i+1} > ma$. Thus we can say that, unless the algorithm stops in a finite number of cycles, $\{h_i\}$ converges to some $h_t \geq ma$ and $\{\Delta h_i\}$ converges to zero. Notice that because projection is a contraction we have:

$$\Delta h_i = \|\Delta h_i e^1\| \geq \|y^i - x^i\| \geq (y^i - x^i)_1 > 0$$

Hence $\{\delta_i = \|y^i - x^i\|\}$ and $\{(y^i - x^i)|_1\}$ all converge to zero. Suppose $h_t > ma$. Then, by the last Lemma $\delta_i \geq \delta(h_t) > 0$. But this is a contradiction because we know that $\{\delta_i\} \to 0$. Hence $h_t = ma$ and the proof is finished. ∎

Once one has computed the maximum, the next step to complete the computations is to determine the feasible slack vector. For sufficiently high i the vector y^i approximates, to any degree of accuracy a feasible slack vector. Finally, via a pullback to the solution space ([4]) one can find a solution of the LP problem.

10. FINITE DIMENSIONAL APPROXIMATIONS

To move closer to numericall applications, we now investigate, in the first place, finite dimensional approximations for feasible bounded and definite LP problems.

We need in the first place to establish some notations. We are interested to certain finite dimensional relaxations of the original optimization problem.

Definition 10.1. The problem of maximizing the functional $(f, .)$ under the constraints system given by the first n inequalities is called the $n-th$ (finite) relaxation of the given problem.

In this case the range space is R^{n+1}, but we keep using the same symbol P for all positive cones. We call $\digamma_n$ the feasible region (a polyhedron of course) in the domain space, necessarily non-void, of the $n-th$ relaxation and $\digamma$ the feasible region of the original problem. Obviously $\digamma_n \supset \digamma$ and $\digamma_n \supset \digamma_{n+1}$. We call m_n the maximum of the $n-th$ relaxation, where $+\infty$ maxima are used to indicate that the relaxation is unbounded.

Now we can state the following:

Theorem 10.1. *Suppose that the optimization problem is feasible bounded and definite with maximum ma. Then there exists an integer k such that, for $n > k$, the nth relaxation is bounded. Moreover, the sequence of maxima $\{m_n\}$ of the finite dimensional relaxations converges to ma.*

Proof. We claim $\cap\{\digamma_n\} = \digamma$. In fact $\cap\{\digamma_n\} \supset \digamma$ is obvious. On the other hand if $x \in \cap\{\digamma_n\}$ then $\forall i$, $(g^i, x) \leq v_i$ and so $x \in \digamma$. Next suppose that $\forall n$, $m_n = \infty$. This would mean that if we take $\mu > ma$

$$\digamma_n \cap \{x : (f, x) \geq \mu\} \neq \phi$$

which implies, since $\{\digamma_n\}$ is decreasing and $\cap\{\digamma_n\} = \digamma$,

$$\digamma \cap \{x : (f, x) \geq \mu\} \neq \phi$$

which is a contradiction. Thus there exists an integer k such that, for $n > k$, the nth relaxation is bounded. At this point we can say that the non increasing sequence of maxima of finite relaxations $\{m_k\}$ converges to some m with $\infty > m \geq ma$. If $m = ma$ we are done. Suppose instead that $m > ma$. Denote by x_{ma} a solution of the problem and note that x_{ma} is a support point for $\digamma$ with supporting functional $(f, .)$. Consider η with $m > \eta > ma$ and the hyperplane:

$$H_\eta = \{x : (f, x) = \eta\}$$

Now notice that:

$$H_\eta \cap \digamma_k \neq \phi, \forall k \Leftrightarrow H_\eta \cap (\cap\{\digamma_k\}) \neq \phi \Leftrightarrow$$
$$H_\eta \cap \digamma \neq \phi$$

Because this latter is false, it follows that $\exists j$ such that for $k > j$, $H_\eta \cap \digamma_k = \phi$. But this implies $\lim\{m_k\} < \eta < m$. Since this is a contradiction we can conclude that $m = ma$ and the proof is finished. ∎

In the final part of this Section we investigate finite dimensional approximations for the minimum distance problem. We consider $v \in l_2$, a closed linear subspace F and assume that $(v + F) \cap P = \phi$. We also consider the subcones of P:

$$\Pi_i = \mathcal{C}o(\{e_i : i = 1, .., n + 1\})$$

The linear spaces $F_i = P_F(\mathcal{L}(\Pi_i))$ are a finite dimensional subspaces of F (and thus they are closed). Clearly $(v + F_i) \cap \Pi_i = \phi$, and minimum distance pairs for this two sets surely exists just in view of their finite dimensionality.

Now we can state the following:

Lemma 10.2. *Assume $(v + F) \cap \Pi_i = \phi, \forall i$. For any i, $(v + F) - \Pi_i$ is closed, and thus the problem of minimum distance between the sets $v + F$ and Π_i admits solution. Moreover, a pair of points x^i and y^i solve the minimum distance problem for $(v + F)$ and Π_i if and only if they solve the minimum distance problem for $v + F_i$ and Π_i.*

Proof. First we prove that $F + \Pi_i$ is closed. For this purpose the index i is irrelevant and so we fix an arbitrary i and omit the corresponding subfix. It is easy to verify that:

$$F + \Pi = F + P_{F^\perp}\Pi$$

Now notice that $P_{F^\perp}\Pi$ is a finite dimensional polyhedron and therefore is closed. Thus, applying the Lemma 3.1, it follows that $F + \Pi$ is closed and so the first statement is proved. Next note that the projection x on $v + F$ of a point $y \in \Pi_i$ must stay in $v + F_i$ and is a fortiori the unique point of F_i, that has minimum distance from y. Thus the projection of y on F and F_i are the same. Now an application of Lemma 2.1 leads to the desired conclusion. ∎

Our last Theorem give the finite dimensional approximations for the minimum distance problem relative to the sets $v + F$ and P.

Theorem 10.3. *Given a closed subspace F, strictly tangent to P, and $v \in l_2$, consider the sets $(v + F)$ and P, assuming $(v + F) \cap P = \phi$. Let $\delta = \|w - y\|$, where w and y are a pair of points solving the minimum distance problem relative to the sets $(v + F)$ and P. Let $\delta_i = \|w^i - y^i\|$ where w^i and y^i solve the minimum distance problems for the sets $(v + F)$ and Π_i (or, equivalently, $(v + F_i)$ and Π_i in view of the preceding Lemma). Then $\{\delta_i\} \to \delta$.*

Proof. Clearly $\delta_i \geq \delta$ and the sequence $\{\delta_i\}$ is non-increasing since the sequence of sets $\{\Pi_i\}$ is increasing. Thus $\{\delta_i\} \to \delta_f \geq \delta$. Now let $\gamma^i = P_{\Pi_i}(w)$, and notice that, as it is readily verified, $\{\gamma^i\} \to y$. On the other hand it is evident that:

$$\|w - \gamma^i\| \geq \delta_i \geq \delta_f$$

Passing to the limit $\{\|w-\gamma^i\|\} \to \delta$ and therefore:

$$\delta \geq \delta_f$$

It follows that $\delta = \delta_f$ and we are done. ∎

Let' give an example of how these finite dimensional approximation results might be applied (under the same assumptions made at the beginning of the preceding Section). One can first approximate the maximum ma to an arbitrary degree of precision using Theorem 10.1. In fact we proved that the sequence of maxima of finite dimensional relaxations is monotone and converges to ma. Thus we can define a stopping criterion requiring that the increment become sufficiently small. Let μ be such an approximation. Once we are done with this first phase, we can exploit Theorem 10.3 to estimate the distance

$$\Delta = d((\widehat{v}(\mu) + \mathcal{R}(\widehat{G})), P)$$

In this case too we have a monotone non increasing sequence $\{\delta_i\}$ converging to Δ and we can proceed in a similar fashion. Note that the estimate of Δ can give us a better idea of how good was the estimate μ of ma (since of course if it were $\mu = ma$ then it would follow $\Delta = 0$). If we are unsatisfied because we regard Δ as being too large, we might go back to the first phase and improve the approximation of ma. Once we are done with both phases, we will have at hand also two vectors, say w^l and y^l coming from the final step of the second phase. Now we can take $w^l \in (\widehat{v}(\mu) + \mathcal{R}(\widehat{G})$ as the approximate slack vector solution (in the range space) for the LP problem. A pull back to domain space will yield the corresponding approximate solution in the domain space.

11. PREVIOUS RESULTS AND THE PRESENT GENERALIZATIONS

The preceding paper [4], was devoted to the theory of polyhedra generated by operators with closed range.

The bulk of this paper deals with the study of the slack set $(v + \mathcal{R}(G)) \cap P$. Because the two intersected sets are closed, the slack set is closed too, and no technical problem about non-closed sets arises.

In a few places we did use the feasibility cone $\mathcal{R}(G) + P$. In this respect there was an error: the feasibility cone was stated to be closed when $\mathcal{R}(G)$ is closed, but this is not true in general. The correct result is given in the present paper: the feasibility cone is closed when $\mathcal{R}(G)$ is closed *and* strictly tangent to P. Thus a few additional corrections must be made, in the places where the erroneous result was applied. Corrections which, luckily, lead only to either slight variants or no modification at all according to the cases.

In the intern case, we have proved here that the feasibility cone is dense and hence it is not the whole space in general. However, this changes very little: instead of feasibility we have an "almost feasibility" property (that is, a non feasible problem can be made feasible by an infinitesimal perturbation of bounds) and, furthermore, when feasibility stands, it gives rise to an unbounded problem, just as it was stated in the previous paper. Therefore it is confirmed that the intern case has no practical relevance for optimization purposes.

As to the weakly tangent case, we have shown in the present paper that, despite the fact that the intern relaxation enjoys only a weaker density property, the existence of the strictly tangent relaxation goes through in infinite dimension. However, the extension of the optimal solution of the strictly tangent relaxation to the original system is weakened, in general, to approximate solution. The correct proofs are given here only, but the existence of the strictly tangent relaxation was also stated in the previous paper, and all its consequences drawn therein are anyway correct.

It is also useful to relate to the previous paper the generalizations obtained in the present paper. Regarding the relative position of a subspace and the positive cone, one has to bear in mind that, to allow a more general setting, where ranges may well be non-closed, the definition of strict tangency and internality have been changed accordingly (in Section 3 here).

We also stress that among the three cases (strict tangency, internality and weak tangency) strict tangency is by far the most important case. This is because the case of intern range has no practical relevance, and the technique for dealing with weakly tangent cases is to report the problem to the strictly tangent case, via the strictly tangent relaxation.

We give here not only a new proof that if F is closed and strictly tangent to P, then $F + P$ is closed, but also provide generalizations regarding the strictly tangent case with non closed subspace F in Section 3 and Section 7. The results in Section 7 here are of particular importance as explained in the Introduction. Indeed they demonstrate that the range space approach to LP as powerful in infinite dimension as it is in finite dimensions, despite the technical hurdle of non-closed ranges.

In the intern case too we have generalized here our analysis to both closed and non-closed ranges. The results turn out to be practically the similar in both cases.

Finally we stress that in [4] there were, outside of the main expository line, a few glimpses on possible generalizations for the case of non-closed ranges. These were base on a erroneously cited elementary computation of general topology, and were therefore wrong. The only generalizations of this kind are those given here, and so reference for this specific issue must be made to the present paper exclusively.

REFERENCES

[1] P. D'ALESSANDRO, *A Conical Approach to Linear Programming, Scalar and Vector Optimization Problems*, Gordon and Breach Science Publishers, Amsterdam, 1997.

[2] P. D'ALESSANDRO, A new conical internal evolutive LP algorithm, *J. Optim. Theory Appl. (JOTA)*, **Vol. 11**, No. 2, pp. 195-207.

[3] P. D'ALESSANDRO, A primal conical linear programming algorithm, *Optimization*, **25** 1992, pp. 197-207.

[4] P. D'ALESSANDRO, Generalizing polyhedra to infinite dimension, *AJMAA*, **Vol. 7**, Issue 2, 2010.

[5] P. D'ALESSANDRO, M. DALLA MORA and E. DE SANTIS, Techniques of linear programming based on the theory of convex cones, *Optimization*, **20** 1989, 6, pp. 761 - 777.

[6] J. L. KELLEY and I. NAMIOCA, *Linear Topological Spaces*, Springer, New York, 1963

[7] R. R. PHELPS, *Lectures on Choquet Theorem*, Springer, New York, 2001

[8] J. STOER, C. WITZGALL, *Convexity and Optimization in Finite Dimensions I*, Springer-Verlag Berlin-Heidelberg-New York 1970.