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ON SOME CONSTRUCTIVE METHOD OF RATIONAL APPROXIMATION

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ABSTRACT. We study a constructive method of rational approximation of analytic functions based on ideas of the theory of Hankel operators. Some properties of the corresponding Hankel operator are investigated. We also consider questions related to the convergence of rational approximants. Analogues of Montessus de Ballore's and Gonchars's theorems on the convergence of rows of Padé approximants are proved.

Key words and phrases: Rational approximation, Padé approximation, Montessus de Ballore's theorem, Hankel operator, Singular value.

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1. INTRODUCTION

1.1. **Rational approximation of a power series. Padé approximation.** Let f be a function analytic at the point $z = 0$. We represent the function f in some neighborhood of the point $z = 0$ by a convergent power series

$$f(z) = \sum_{k=0}^{\infty} f_k z^k. \tag{1.1}$$

Denote by R_0 the radius of convergence of the power series (1.1). Let $D_0 = \{z : |z| < R_0\}$ be the disk of convergence. Here and in what follows we assume that $R_0 > 1$. One of the classical constructive methods of approximation of analytic functions given by a power series is the Padé approximation (see the monograph [3] and the references therein, and also [4], [8], [9], [16], [17], [18]). This method is a method of rational approximation with free poles (there is no restriction on poles). Padé approximants are locally the best rational approximants to a given power series and constructed in terms of its coefficients. These approximants localize the singular points of a function determined by the power series and enable us to obtain, under certain conditions, an efficient analytic continuation of the power series beyond its circle of convergence.

For any nonnegative integer n denote by $\mathbf{P}_n$ the class of all polynomials of degree at most n. Let n and m be nonnegative integers. The Padé approximant $[n/m]$ of type (n, m) of the function f given by power series (1.1) is the unique rational function $[n/m] = p/q$, $p \in \mathbf{P}_n,\ q \in \mathbf{P}_m,\ q \not\equiv 0$, satisfying the following relation:

$$(qf - p)(z) = Az^{n+m+1} + \dots. \tag{1.2}$$

It is easy to see that polynomials

$$q(z) = \det(\begin{bmatrix} f_{n-m+1} & f_{n-m+2} & \cdots & f_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ f_n & f_{n+1} & \cdots & f_{n+m} \\ z^m & z^{m-1} & \cdots & 1 \end{bmatrix})$$

and

$$p(z) = \sum_{k=0}^{n} (qf)_k z^k,$$

where $(qf)_k$ is the k-th coefficient of the power series of the function qf, satisfy formula (1.2). The table $\{[n/m]\}_{n,m=0}^{\infty}$ is called the Padé table of the function f. The sequence $\{[n/m]\}_{n=0}^{\infty}$, where a nonnegative integer m is fixed, is called the m-th row of the Padé table.

In the present article we consider a constructive method of approximation of analytic functions given by a power series (1.1). The corresponding method is based on ideas of the theory of Hankel operators. As Padé approximants, these approximants are rational functions and constructed in terms of the coefficients f_k of the power series (1.1). Fix nonnegative integers n and m. There are $m + 1$ rational approximants $P/Q,\ P \in \mathbf{P}_n,\ Q \in \mathbf{P}_m,\ Q \not\equiv 0$, of the function f given by power series (1.1), satisfying the following relation on the unit circle $\Gamma = \{z : |z| = 1\}$:

$$(Qf - P)(z) = s\overline{Q(z)}z^{n+m+1} + Az^{n+m+2} + \dots,$$

where constants s are singular numbers of the matrix

$$\begin{bmatrix} f_{n+m+1} & f_{n+m} & \cdots & f_{n+1} \\ f_{n+m} & f_{n+m-1} & \cdots & f_n \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+1} & f_n & \cdots & f_{n-m+1} \end{bmatrix}.$$

In this paper we define the Hankel operator $D_{f,m,n}$ on the class $\mathbf{P}_m$ of all polynomials of degree at most m and use singular numbers and eigenfunctions of $D_{f,m,n}$ to obtain rational approximants P/Q. The main results include the proof of an analogue of the AAK theorem [1], [2] and an investigation of asymptotics of singular numbers of the Hankel operator $D_{f,m,n}$, when m is fixed and $n \to \infty$. Moreover, we study convergence of the corresponding rational approximants to f under the same conditions (m is fixed and $n \to \infty$). The corresponding results are analogues of classical theorems of Montessus de Ballore and Gonchar related to convergence of rows of Padé approximants.

1.2. **Convergence of rows of Padé approximants and Hadamard's theorem.** Let m be a positive integer. Denote by D_m the maximal open disk with center at $z = 0$ in which the function f is meromorphic and has at most m poles (counting multiplicities). Let R_m be the radius of D_m. It easy to see that $R_0 \le R_1 \le R_2 \dots$, and $D_0 \subseteq D_1 \subseteq D_2 \subseteq \dots$.

The classical Montessus de Ballore's theorem [13], [3] solves the problem of meromorphic recovery of a function f given by the power series (1.1) in the case when the function f has exactly $m, m \ge 1$, poles in the open disk D_m.

Montessus de Ballore's theorem

Let $m \ge 1$. Suppose that a function f has exactly m poles $\alpha_1, \dots, \alpha_m$ in the disk D_m. Then the sequence $\{[n/m]\}_{n=0}^{\infty}$ converges uniformly to the function f on compact subsets of $D_m \setminus \{\alpha_1, \dots, \alpha_m\}$ as $n \to \infty$. Moreover, for sufficiently large n the Padé approximant $[n/m]$ has exactly m poles and for each pole α_j of f of multiplicity l, $l \ge 1$, exactly l poles of $[n/m]$ converges to α_j as $n \to \infty$.

The general case was investigated by Gonchar [8] (for definition of σ-almost uniform convergence of rational functions see subsection 4.2).

Gonchar's theorem

Let $m \ge 1$. The sequence $\{[n/m]\}_{n=0}^{\infty}$ converges σ-almost uniformly inside D_m as $n \to \infty$.

Let n be a nonnegative integer. For an analytic function f given by a power series (1.1) we denote by $K_{n.m}$ the symmetric $m \times m$ matrix constructed by the coefficients of (1.1):

$$K_{n.m} = \begin{bmatrix} f_{n+m-1} & f_{n+m-2} & \cdots & f_n \\ f_{n+m-2} & f_{n+m-3} & \cdots & f_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ f_n & f_{n-1} & \cdots & f_{n-m+1} \end{bmatrix} \tag{1.3}$$

(we set $f_k = 0$ for $k < 0$). We remark that $K_{n,1} = f_n$ for all n. For any positive integer m let

$$l_m = \limsup_{n\to\infty} |\det(K_{n.m})|^{1/n},$$

and let $l_0 = 1$ for $m = 0$. The Hadamard's theorem [10], [5] states that the radii R_m can be expressed in terms of l_m.

Hadamard's theorem

We have

$$R_m = \frac{l_m}{l_{m+1}}, \quad m = 0, 1, \dots.$$

Here we assume that $R_m = \infty$, if $l_1, \ldots, l_m \neq 0$ and $l_{m+1} = 0$. It directly follows from Hadamard's theorem that for $m \geq 1$

$$\limsup_{n\to\infty} |\det(K_{n,m})|^{1/n} = \frac{1}{R_0 \ldots R_{m-1}}. \tag{1.4}$$

1.3. **Auxiliary results. The AAK theorem.** Let $G = \{z : |z| < 1\}$ be the open unit disk with the boundary Γ. We assume that Γ is positively oriented with respect to G. Denote by Γ_R the circle with center at the point $z = 0$ and radius $R > 0$. For any compact set K in the complex plane $\mathbf{C}$ and any continuous function φ on K denote by $||\varphi||_K$ the norm of φ in the uniform metric on K:

$$||\varphi||_K = \max_{z\in K} |\varphi(z)|.$$

Let $L_p(\Gamma)$, $1 \leq p < \infty$, be the Lebesgue space of functions φ measurable on Γ, with the norm

$$\|\varphi\|_p = \left(\frac{1}{2\pi}\int_\Gamma |\varphi(t)|^p\, ds\right)^{1/p}.$$

Denote by $\langle\varphi, \psi\rangle_2$ the inner product in the Hilbert space $L_2(\Gamma)$:

$$\langle\varphi, \psi\rangle_2 = \frac{1}{2\pi}\int_\Gamma (\varphi\overline{\psi})(t) ds, \quad \varphi, \psi \in L_2(\Gamma).$$

Let $L_\infty(\Gamma)$ be the space of essentially bounded on Γ functions φ, endowed with the norm

$$\|\varphi\|_\infty = \operatorname*{ess\,sup}_\Gamma |\varphi(t)| < \infty.$$

It easy to see that for any $p \in \mathbf{P}_m$,

$$||p||_2 \leq ||p||_\infty \leq (m+1)||p||_2. \tag{1.5}$$

Let $H_p(G), 1 \leq p \leq \infty$, be the Hardy space of analytic functions on G. Here and in what follows we consider $H_p(G)$ as a subspace of the space $L_p(\Gamma)$ (see [6], [12] for more details).

We represent $L_2(\Gamma)$ as the direct sum $L_2(\Gamma) = H_2(G) \oplus H_2^\perp(G)$, where $H_2^\perp(G)$ is the orthogonal complement of $H_2(G)$ in $L_2(\Gamma)$. We mention the following characteristic of the subspace $H_2^\perp(G)$:

Let $a \in L_2(\Gamma)$. Then $a \in H_2^\perp(G)$ if and only if there exists a function $b \in H_2(G)$ such that

$$a(t) = \frac{\overline{b(t)}}{t} = \overline{b(t)}\frac{ids}{dt} \quad \textit{a.e. on} \quad \Gamma.$$

Let a function g be continuous on Γ. The Hankel operator $A_g : H_2(G) \to H_2^\perp(G)$ with symbol g is the composition of the operator of multiplication by g and the orthogonal projection $\mathbf{P}_-$ from $L_2(\Gamma)$ onto $H_2^\perp(G)$:

$$A_g\varphi = \mathbf{P}_-(\varphi g), \quad \varphi \in H_2(G).$$

Note that A_g is a compact operator.

Let $A : X \to Y$ be a compact linear operator, where X and Y are the Hilbert spaces. For any nonnegative integer n denote by $s_n(A)$ the *n-th singular number* of the operator A:

$$s_n(A) = \inf_K \|A - K\|,$$

where the infimum is taken over all linear operators $K : X \to Y$ of rank at most n, and $\|\cdot\|$ is the norm of the corresponding linear operator. We remark that the sequence $\{s_n(A)\}$, $n =$

$0, 1, 2, \ldots$, coincides with the sequence of eigenvalues (counting multiplicity) of the operator $(A^*A)^{1/2}$, where $A^* : Y \to X$ is the adjoint of A. The following formula is valid:

$$s_n(A) = \inf_{X_{-n}} \|A|_{X_{-n}}\|, \tag{1.6}$$

where the infimum is taken over all subspaces X_{-n} of codimension n of X (see [7], [11] for more details).

Let g be continuous on Γ. The Adamyan-Arov-Krein theorem [1], [2] establishes a connection between a singular numbers $s_n(A_g)$ of the Hankel operator A_g and the errors Δ_n of the meromorphic approximation of g in the space $L_\infty(\Gamma)$ by functions from the class $\mathbf{M}_n(G) = \mathbf{R}_{n,n} + H_\infty(G)$, where

$$\mathbf{R}_{n,m} = \{r = p/q,\ p \in \mathbf{P}_n(G), q \in \mathbf{P}_m,\ q \not\equiv 0\}.$$

Let

$$\Delta_n = \inf_{h \in \mathbf{M}_n(G)} \|f - h\|_\infty.$$

The AAK theorem states that for all $n = 0, 1, 2, \ldots,$

$$s_n(A_g) = \Delta_n.$$

We note that in [14] a generalization of the AAK theorem for multiply connected domains is proved.

2. The discrete Hankel operator

2.1. **Definition.** As above, we assume that f is analytic in $D_0 = \{z : |z| < R_0\},\ R_0 > 1$. Fix nonnegative integers n and m. For any polynomial $\alpha \in \mathbf{P}_m$ we represent the product αf in the open disk D_0 as a sum of the power series:

$$(\alpha f)(z) = \sum_{j=0}^{\infty} (\alpha f)_j z^j,$$

where $(\alpha f)_j$ is the j-th coefficient of the power series of the function αf. We can rewrite the last formula as

$$(\alpha f - \beta)(z) = \sum_{j=n+1}^{n+m+1} (\alpha f)_j z^j + z^{n+m+2} c(z), \tag{2.1}$$

where

$$\beta(z) = \sum_{j=0}^{n} (\alpha f)_j z^j. \tag{2.2}$$

and

$$c(z) = \sum_{j=0}^{\infty} (\alpha f)_{j+n+m+2} z^j. \tag{2.3}$$

Therefore, on the unit circle Γ we obtain that

$$\frac{(\alpha f - \beta)(t)}{t^{n+m+2}} = \frac{\overline{p(t)}}{t} + c(t), \tag{2.4}$$

where

$$p(t) = \sum_{j=0}^{m} \overline{(\alpha f)_{j+n+1}} t^{m-j}. \tag{2.5}$$

We now define an operator $D_{f,m,n}$. Let $H^{\perp}_{2,m}(G)$ be a $(m+1)$-dimensional subspace of $H^{\perp}_{2}(G)$ defined as follows:

$$H^{\perp}_{2,m}(G)=\{a\in L_2(\Gamma):a(t)=\frac{\overline{b(t)}}{t}=\overline{b(t)}\frac{ids}{dt}\quad\text{a.e. on}\quad\Gamma,\ \ b\in\mathbf{P}_m\}.$$

The operator $D_{f,m,n}:\mathbf{P}_m\to H^{\perp}_{2,m}(G)$ is the composition of the operator of multiplication by a function f/t^{n+m+2} and the orthogonal projection $\mathbf{P}_m$ from $L_2(\Gamma)$ onto $H^{\perp}_{2,m}(G)$:

$$D_{f,m,n}\alpha=\mathbf{P}_m\left(\frac{\alpha f}{t^{n+m+2}}\right),\quad \alpha\in\mathbf{P}_m.$$

We have the following formula

$$(D_{f,m,n}\alpha)(t)=\frac{(\alpha f-\beta)(t)}{t^{n+m+2}}-c(t),\quad t\in\Gamma, \tag{2.6}$$

where β and c are given by (2.2) and (2.3), respectively. We remark that the function c is analytic in D_0.

Let us consider an antilinear operator $B_{f,m,n}:\mathbf{P}_m\to\mathbf{P}_m$ that is defined by

$$B_{f,m,n}\alpha=p,\quad \alpha\in\mathbf{P}_m,$$

where the polynomial p is given by (2.5) (compare with [15], where the corresponding operator $B_g:H_2(G)\to H_2(G)$ is defined for the Hankel operator A_g). We can now write

$$D_{f,m,n}\alpha=\frac{\overline{B_{f,m,n}\alpha}}{t}=\overline{B_{f,m,n}\alpha}\,\frac{ids}{dt}\quad\text{on}\quad\Gamma. \tag{2.7}$$

By (2.4),

$$(\alpha f-\beta)(t)=\overline{(B_{f,m,n}\alpha)(t)}t^{n+m+1}+t^{n+m+2}c(t),\quad t\in\Gamma,$$

and

$$\frac{(\alpha f-\beta)(t)}{t^{n+m+2}}=\overline{(B_{f,m,n}\alpha)(t)}\,\frac{ids}{dt}+c(t),\ \ t\in\Gamma. \tag{2.8}$$

2.2. **Symmetric bilinear form.** Define the bilinear symmetric form $[u,v]$ for a pair of polynomials $u,v\in\mathbf{P}_m$:

$$[u,v]=\frac{1}{2\pi i}\int_{\Gamma}\frac{(uvf)(t)dt}{t^{n+m+2}}.$$

From (2.6) and (2.7) it may be concluded that

$$[u,v]=\langle v,B_{f,m,n}u\rangle_2=\langle u,B_{f,m,n}v\rangle_2\,, \tag{2.9}$$

and

$$[u,v]=\left\langle D_{f,m,n}u,\overline{v}\,\frac{ids}{dt}\right\rangle_2=\left\langle D_{f,m,n}v,\overline{u}\,\frac{ids}{dt}\right\rangle_2. \tag{2.10}$$

Lemma 2.1. *We have*

$$B^2_{f,m,n}=D^*_{f,m,n}D_{f,m,n}, \tag{2.11}$$

where $D^*_{f,m,n}:H^{\perp}_{2,m}(G)\to\mathbf{P}_m$ is the adjoint of $D_{f,m,n}$.

Proof. Applying (2.7), we can assert that for any $u,v\in\mathbf{P}_m$,

$$\langle D_{f,m,n}u,D_{f,m,n}v\rangle_2=\langle B_{f,m,n}v,B_{f,m,n}u\rangle_2$$

and (see (2.9)),

$$\left\langle u,D^*_{f,m,n}D_{f,m,n}v\right\rangle_2=\left\langle u,B^2_{f,m,n}v\right\rangle_2.$$

From this we get (2.11). ∎

Let $s_{k,m,n} = s_{k,m,n}(D_{f,m,n}),\ k = 0, \ldots, m,$

$$s_{0,m,n} \geq s_{1,m,n} \geq \cdots \geq s_{m,m,n},$$

be the singular numbers numbers of $D_{f,m,n}$, that are eigenvalues of $(D^*_{f,m,n} D_{f,m,n})^{1/2}$, and let $Q_{k,m,n},\ k = 0, \ldots, m,$ be associated orthonormal polynomials (compare with [15]):

$$B_{f,m,n} Q_{k,m,n} = s_{k,m,n} Q_{k,m,n} \tag{2.12}$$

and

$$D_{f,m,n} Q_{k,m,n} = s_{k,m,n} \overline{Q}_{k,m,n}\, \frac{ids}{dt}.$$

By (2.8) and (2.12),

$$\frac{(Q_{k,m,n} f - P_{k,m,n})(t)}{t^{n+m+2}} = s_{k,m,n} \overline{Q_{k,m,n}(t)} \frac{ids}{dt} + C_{k,m,n}(t), \quad t \in \Gamma, \tag{2.13}$$

where

$$P_{k,m,n}(t) = \sum_{j=0}^{n} (Q_{k,m,n} f)_j t^j$$

and

$$C_{k,m,n}(t) = \sum_{j=0}^{\infty} (Q_{k,m,n} f)_{j+m+n+2} t^j.$$

From (2.13), using the Cauchy integral theorem, we obtain that for any polynomial $\alpha \in \mathbf{P}_m$,

$$[\alpha, Q_{k,m,n}] = \frac{1}{2\pi i} \int_\Gamma \frac{(\alpha Q_{k,m,n} f)(t) dt}{t^{n+m+2}} = \frac{s_{k,m,n}}{2\pi} \int_\Gamma (\alpha \overline{Q}_{k,m,n})(t)\, ds. \tag{2.14}$$

We can rewrite (2.14) in the form

$$[\alpha, Q_{k,m,n}] = s_{k,m,n} \langle \alpha, Q_{k,m,n} \rangle_2, \quad \alpha \in \mathbf{P}_m. \tag{2.15}$$

So, the polynomials $Q_{k,m,n}$ are characterized by the double orthogonality conditions:

$$[Q_{i,m,n}, Q_{j,m,n}] = s_{i,m,n} \delta_{ij},\ \ \langle Q_{i,m,n}, Q_{j,m,n} \rangle_2 = \delta_{ij}, \tag{2.16}$$

where δ_{ij} is the Kroneker symbol.

2.3. **A symmetric matrix.** Set

$$Q_{k,m,n}(z) = \sum_{i=0}^{m} a_{i,k} z^i.$$

Letting $\alpha(t) = t^l, l = 0, \ldots, m$ in (2.14), and taking into account that

$$\frac{1}{2\pi} \int_\Gamma t^i \overline{t^j}\, ds = \delta_{ij}, \tag{2.17}$$

and

$$f_i = \frac{1}{2\pi i} \int_\Gamma \frac{f(t) dt}{t^{i+1}},$$

we get

$$\sum_{i=0}^{m} a_{i,k} f_{n+m+1-(i+l)} = s_{k,m,n} \overline{a_{l,k}}. \tag{2.18}$$

Let

$$A_k = \begin{bmatrix} a_{0,k} \\ \vdots \\ a_{m,k} \end{bmatrix}.$$

By (1.3), $K_{n+1.m+1}$ is the following symmetric matrix constructed from the coefficients f_i:

$$K_{n+1.m+1} = \begin{bmatrix} f_{n+m+1} & f_{n+m} & \cdots & f_{n+1} \\ f_{n+m} & f_{n+m-1} & \cdots & f_n \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+1} & f_n & \cdots & f_{n-m+1} \end{bmatrix}.$$

Using (2.18) and (2.16), we can see that for all $k = 0, \ldots, m$, $s_{k,m,n}$ and A_k are singular values and singular vectors of the matrix $K_{n+1,m+1}$:

$$K_{n+1,m+1}A_k = s_{k,m,n}\overline{A_k},$$

and vectors $A_0, \ldots, A_m$, are orthonormal

$$\langle A_i, A_j \rangle = \delta_{ij},$$

where $< u, v >$ is the inner product in the space $\mathbf{C}^{m+1}$.

2.4. **An integral equation.** Let φ be any function in the Hardy space $H_1(G)$. We can represent φ as a sum of its Taylor series:

$$\varphi(z) = \sum_{i=0}^{\infty} \varphi_i z^i, \quad z \in G.$$

Let

$$S_m(\varphi)(z) = \sum_{i=0}^{m} \varphi_i z^i$$

be the m-th partial sum of the Taylor series of φ. By (2.15) and (2.17), for $\varphi \in H_1(G)$,

$$[S_m(\varphi), Q_{k,m,n}] = s_{k,m,n}\langle \varphi, Q_{k,m,n} \rangle_2. \tag{2.19}$$

Let $z \in G$ and

$$\varphi(t) = \frac{1}{1 - \overline{z}t}, \quad t \in G.$$

Then

$$S_m(\varphi)(t) = \frac{1 - (\overline{z}t)^{m+1}}{1 - \overline{z}t}, \quad t \in G.$$

By (2.19) and (2.17),

$$s_{k,m,n}\overline{Q_{k,m,n}(z)} = \frac{1}{2\pi i}\int_\Gamma \frac{(1 - (\overline{z}t)^{m+1})Q_{k,m,n}(t)f(t)}{(1 - \overline{z}t)t^{n+m+2}}\,dt, \quad z \in G.$$

2.5. **Integral formulas for singular numbers.** We now present some integral formulas for the singular numbers $s_{k,m,n}$ of our operator $D_{f,m,n}$ which follow from (1.6) and (2.10).

Lemma 2.2. *The following formulas hold for the singular numbers* $s_{k,m,n}$:

$$s_{0,m,n} = \sup_{u,v} \left|\frac{1}{2\pi i}\int_\Gamma \frac{(uvf)(t)dt}{t^{n+m+2}}\right| \tag{2.20}$$

and

$$s_{k,m,n} = \inf_{u_1,\ldots,u_k \in \mathbf{P}_m} \{\sup_{u,v} \left|\frac{1}{2\pi i}\int_\Gamma \frac{(uvf)(t)dt}{t^{n+m+2}}\right|\}, \quad k = 1, \ldots, m, \tag{2.21}$$

where the suprema in (2.20) *and* (2.21) *are taken over all polynomials* $u, v \in \mathbf{P}_m$, $\|u\|_2 = 1$, *and* $||v||_2 = 1$, *with the polynomials* u *in* (2.21) *satisfying the conditions* $< u, u_i >_2 = 0$, $i = 1, \ldots, k$.

3. AN ANALOGUE OF THE AAK THEOREM

3.1. **Estimates of singular numbers.** Section 3 is devoted to the study of an analogue of the AAK theorem for the operator $D_{f,m,n}$. As above, let f be analytic in the open disk $\{z : |z| < R_0\},\ R_0 > 1$, and let n and m be nonnegative integers. According to the formula (1.6), we can write

$$s_{m,m,n} = \inf_{\alpha\in\mathbf{P}_m,\alpha\not\equiv 0} \frac{\|D_{f,m,n}\alpha\|_2}{\|\alpha\|_2}. \tag{3.1}$$

Denote by $H_{2,n}(G)$ and $H_{\infty,n}(G)$ the subspaces of $H_2(G)$ and $H_\infty(G)$, consisting of analytic functions φ from $H_2(G)$ and $H_\infty(G)$, respectively, such that

$$\varphi(z) = \sum_{k=n}^{\infty} \varphi_k z^k$$

in some neighborhood of $z = 0$ (each function φ has a zero of order at least n at $z = 0$). According to the definition of the operator $D_{f,m,n}$ and formula (2.6), the equality

$$t^{n+m+2} D_{f,m,n}\alpha = (\alpha f - \beta - \omega)(t), \quad t \in \Gamma,$$

holds for any polynomial $\alpha \in \mathbf{P}_m$, where $\beta \in \mathbf{P}_m$ and $\omega \in H_{2,n+m+2}(G)$, are uniquely determined by the relation

$$\|D_{f,m,n}\alpha\|_2 = \|\alpha f - \beta - \omega\|_2 = \inf_{\beta'\in\mathbf{P}_n,\omega'\in H_{2,n+m+2}(G)} \|\alpha f - \beta' - \omega'\|_2. \tag{3.2}$$

Let l and k be nonnegative integers. Let us consider the following meromorphic approximation problem

$$\Delta_{l,k} = \Delta_{l,k}(f) = \inf \left\| f - \frac{p+\omega}{q} \right\|_\infty, \tag{3.3}$$

where the infimum is over the collection of $p \in \mathbf{P}_l,\ q \in \mathbf{P}_k,\ q \not\equiv 0,$ and $\omega \in H_{\infty,n+m+2}(G)$.

Now we can state an analogue of the AAK theorem for the operator $D_{f,m,n}$. Theorem 3.1 says that the m-th singular number of the operator $D_{f,m,n,}$ can be characterized as an error of the best meromorphic approximation of f in the space $L_\infty(\Gamma)$ by functions $(p+\omega)/q, p \in \mathbf{P}_n,\ q \in \mathbf{P}_m,\ q \not\equiv 0,$ and $\omega \in H_{\infty,n+m+2}(G)$.

Theorem 3.1. *We have*

$$s_{m,m,n}(D_{f,m,n}) = \Delta_{n,m}. \tag{3.4}$$

Proof. With the help of (3.2) we can deduce that for any $\alpha \in \mathbf{P}_m, \alpha \not\equiv 0, \beta' \in \mathbf{P}_n,\ \omega' \in H_{\infty,n+m+2}(G)$,

$$\|D_{f,m,n}\alpha\|_2 \le \|\alpha\|_2 \cdot \left\| f - \frac{\beta'+\omega'}{\alpha} \right\|_\infty.$$

So, by (3.1) and (3.3),

$$s_{m,m,n} \le \Delta_{n,m}. \tag{3.5}$$

Let us now turn to the formula (2.13) for $k = m$

$$\frac{(Q_{m,m,n}f - P_{m,m,n})(t)}{t^{n+m+2}} = s_{m,m,n}\overline{Q_{m,m,n}(t)}\frac{ids}{dt} + C_{m,m,n}(t), \quad t \in \Gamma, \tag{3.6}$$

where $C_{m,m,n}$ is analytic in $\{z : |z| < R_0\}, R_0 > 1$. It follows easily from (3.6) that the function

$$(P_{m,m,n} + t^{n+m+2}C_{m,m,n})/Q_{m,m,n}$$

is analytic on Γ. Moreover,

$$\left| f(t) - \frac{P_{m,m,n}(t) + t^{n+m+2} C_{m,m,n}(t)}{Q_{m,m,n}} \right| = s_{m,m,n}, \ t \in \Gamma. \tag{3.7}$$

We can conclude from this that the equality is attained in (3.5). This completes the proof. ∎

Before stating and proving the next result of the paper, we note that by (2.21) the following formula for the m-th singular number $s_{m.m.n}, m \geq 1$,holds:

$$s_{m,m,n} = \inf_{u \in \mathbf{P}_m, ||u||_2 = 1} \ \sup_{v \in \mathbf{P}_m, ||v||_2 = 1} \left| \frac{1}{2\pi i} \int_\Gamma \frac{(uvf)(t)dt}{t^{n+m+2}} \right|. \tag{3.8}$$

Now we can proceed to estimates of the singular number $s_{k,m,n}$ in terms of the errors of meromorphic approximation for the general case when $k = 0, \ldots, m$. Let us mention that for $k = m$ in formula (3.9) given below we obtain equality (3.4).

Theorem 3.2. *For $0 \leq k \leq m$ we have*

$$\Delta_{n+m-k,k} \leq s_{k,m,n} \leq \Delta_{n-m+k,k}. \tag{3.9}$$

Proof. We first prove the inequalities

$$s_{k,m,n} \leq \Delta_{n-m+k,k}, \quad k = 0, \ldots, m, \tag{3.10}$$

confining ourselves to the case $m \geq 1$ and $k \geq 1$; the case $k = 0$ can be treated analogously. Fix any $q \in \mathbf{P}_k, q \not\equiv 0, p \in \mathbf{P}_{n-m+k}$ and $\omega \in H_{\infty, n+m+2}(G)$. Using formula (2.21) we get

$$s_{k,m,n} \leq \sup_{u,v} \left| \frac{1}{2\pi i} \int_\Gamma \frac{(uqvf)(t)dt}{t^{n+m+2}} \right|, \tag{3.11}$$

where the supremum in (3.11) is taken over all polynomials $u \in \mathbf{P}_{m-k}, v \in \mathbf{P}_m$, $||uq||_2 = 1$ and $||v||_2 = 1$. Since

$$\frac{1}{2\pi i} \int_\Gamma \frac{(uqvf)(t)dt}{t^{n+m+2}} = \frac{1}{2\pi i} \int_\Gamma \frac{(uqv)(t)(f - (p+\omega)/q)(t)}{t^{n+m+2}} \, dt,$$

with help of (3.11), we obtain that

$$s_{k,m,n} \leq \left\| f - \frac{p + \omega}{q} \right\|_\infty.$$

Then from the last relation and (3.3) we get the required inequality (3.10).

We now proceed to the proof of the inequalities

$$\Delta_{n+m-k,k} \leq s_{k,m,n}, \ k = 0, \ldots, m.$$

As above, the arguments will be based on the formulas (2.20) and (2.21). We confine ourselves to the case when $k \geq 1$. Since for any $u_1, \ldots, u_k \in \mathbf{P}_m$ there exists $u \in \mathbf{P}_k, u \not\equiv 0$, such that $< u, u_i >_2 = 0$, $i = 1, \ldots, k$, we get, by (2.21),

$$\inf_{u \in \mathbf{P}_k, ||u||_2 = 1} \ \sup_{v \in \mathbf{P}_k, ||v||_2 = 1} \left| \frac{1}{2\pi i} \int_\Gamma \frac{(uvf)(t)dt}{t^{n+m+2}} \right| \leq s_{k,m,n}. \tag{3.12}$$

By formula (3.8), the expression on the left-hand side of (3.12) is k-th singular number of the operator $D_{f,k,n+m-k}$. Applying now Theorem 3.1 we get the desired inequality

$$\Delta_{n+m-k,k} \leq s_{k,m,n},$$

which completes the proof of the theorem. ∎

4. ASYMPTOTICS OF THE SINGULAR NUMBERS AND CONVERGENCE OF RATIONAL APPROXIMANTS

4.1. **Asymptotics of singular numbers.** In this subsection we fix nonnegative integers m and $k,\ 0 \leq k \leq m$, and investigate asymptotics of the singular numbers $s_{k,m,n}$ as $n \to \infty$.

Now we formulate one of our main results.

Theorem 4.1. *For any* $k,\ \ 0 \leq k \leq m$, *we have*

$$\limsup_{n\to\infty} s_{k,m,n}^{1/n} = \frac{1}{R_k}. \tag{4.1}$$

Proof. We first prove that for any $k,\ \ 0 \leq k \leq m$,

$$\limsup_{n\to\infty} s_{k,m,n}^{1/n} \leq \frac{1}{R_k}. \tag{4.2}$$

Without loss of generality we can assume that $1 \leq k \leq m$. We choose and fix $1 < R < R_k$ close enough to R_k such that all poles of f in D_k are inside of the circle $\Gamma_R = \{z : |z| = R\}$. Let $u \in \mathbf{P}_m, ||u||_2 = 1$, be any polynomial with zeros at poles of f in D_k (counting multiplicities). Let us estimate the following integral

$$I = \frac{1}{2\pi i}\int_\Gamma \frac{(uvf)(t)dt}{t^{n+m+2}},$$

where $v \in \mathbf{P}_m$ and $||v||_2 = 1$. Since the function uf is analytic in the open disk D_k, by the Cauchy integral theorem,

$$I = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{(uvf)(t)dt}{t^{n+m+2}}.$$

Therefore, we get

$$|I| \leq \frac{||u||_{\Gamma_R}||v||_{\Gamma_R}||f||_{\Gamma_R}}{R^{n+m+1}}. \tag{4.3}$$

By the Bernstein-Walsh lemma for estimating the growth of polynomials (see, for example, [19]), we obtain that

$$||v||_{\Gamma_R} \leq ||v||_\infty R^m \tag{4.4}$$

and

$$||u||_{\Gamma_R} \leq ||u||_\infty R^m. \tag{4.5}$$

Taking into account now (4.4) and (4.5), we can conclude from (4.3) that

$$|I| \leq \frac{||u||_\infty||v||_\infty||f||_{\Gamma_R}}{R^{n-m+1}},$$

and, by (1.5),

$$|I| \leq (m+1)^2\frac{||u||_2||v||_2||f||_{\Gamma_R}}{R^{n-m+1}} = (m+1)^2\frac{||f||_{\Gamma_R}}{R^{n-m+1}}.$$

It remains to observe that the last inequality and formula for singular numbers (2.21) imply that

$$\limsup_{n\to\infty} s_{k,m,n}^{1/n} \leq \frac{1}{R}.$$

Passing to the limit as $R \to R_k$ on the right-hand side of the last inequality, we obtain (4.2).

Since $s_{i,m,n},\ i = 0, \dots, m$ are singular numbers of matrix $K_{n+1,m+1}$, we can write the following formula for the product of the singular numbers:

$$\det(K_{n+1,m+1}) = \prod_{i=0}^{m} s_{i,m,n}. \tag{4.6}$$

Using (4.6) and (1.4), we get

$$\limsup_{n\to\infty} \prod_{i=0}^{m} s_{i,m,n}^{1/n} = \frac{1}{R_0 \dots R_m}.$$

From this, by (4.2), we obtain immediately (4.1). Thus, Theorem 4.1 is proved.
∎

4.2. **Convergence of rational approximants.** In subsections 4.2–4.3 we investigate the convergence of the rational approximants $P_{k,m,n}/Q_{k,m,n}$ to f with k and m fixed and as $n \to \infty$. First we define σ convergence of a sequence of rational functions introduced by Gonchar (see [8] and [9]).

A sequence of rational functions $\{r_n\}$ *converges σ-almost uniformly* inside an open set U as $n \to \infty$ to a function $g : U \to \overline{\mathbf{C}}$, if for any compact set $K \subset U$ and for any $\varepsilon > 0$ there exists an open set $U_\varepsilon,\ \sigma(U_\varepsilon) < \varepsilon$, such that the sequence $\{r_n\}$ converges uniformly to g on $K \setminus U_\varepsilon$ as $n \to \infty$. By definition,

$$\sigma(U_\varepsilon) = \inf \sum_i d_i,$$

where the infimum is taken over all covering $\{W_i\}$ of U_ε by open disks $\{W_i\}$ with the diameter d_i. Let us mention some consequences (see [8] and [9] for more details).

Let a sequence of rational functions $\{r_n\}$ converges σ-almost uniformly inside a domain U as $n \to \infty$ to a function $g : U \to \overline{\mathbf{C}}$. Then

1. if each rational function r_n has at most $m, m \geq 0,$ poles in U then the function g is meromorphic in U and has at most m poles in U. Moreover, if g has a pole of order $l, l \geq 1$ at the point $a \in U$ then at least l poles of r_n tend to a as $n \to \infty$;

2. if each function r_n has at most m poles in U and the function g has exactly m poles in U, then the poles of r_n in U tend to poles of g (each pole of g attracts as many poles of r_n as an order of a pole of g). Moreover, the sequence $\{r_n\}$ convergences uniformly on compact subsets of a domain U' as $n \to \infty$, where the domain U' obtained by deleting from U poles of g.

Fix nonnegative integers k and $m,\ 0 \leq k \leq m$. For any $\varepsilon > 0$ we set

$$U_\varepsilon = \cup_{n=0}^{\infty} U_{n,\varepsilon},$$

where $U_{n,\varepsilon}$, $n = 1, 2, \dots,$ is $\varepsilon/6mn^2$-neighborhood of the set of zeros $\alpha_{j,k,n}$ of $Q_{k,m,n}$ and $U_{0,\varepsilon}$ is $\varepsilon/6m$-neighborhood of the set of poles α_j of f in D_m. It is easy to see that $\sigma(U_\varepsilon) < \varepsilon$.

We represent $Q_{k,m,n}$ in the form

$$Q_{k,m,n}(z) = c_{k,m,n} \prod_{|\alpha_{j,k,n}|\leq 1} (z - \alpha_{j,k,n}) \prod_{|\alpha_{j,k,n}|>1} (\frac{z}{\alpha_{j,k,n}} - 1).$$

Since $||Q_{k,m,n}||_2 = 1$, by (1.5), we get

$$|c_{k,m,n}| > \frac{1}{C_1},$$

and, then, for any compact set K in the complex plane $\mathbf{C}$,

$$\min_{K \setminus U_\varepsilon} |Q_{k,m,n}| > C_2 n^{-2m}, \tag{4.7}$$

where C_1 and C_2 are positive constants, independent of n. Moreover, it follows from (1.5) and the normalization $||Q_{k,m,n}||_2 = 1$ that

$$||Q_{k,m,n}||_\infty \le m + 1.$$

Using now the Bernstein-Walsh lemma for estimating the growth of polynomials, we obtain that for any compact set K in the complex plane $\mathbf{C}$,

$$||Q_{k,m,n}||_K \le C, \tag{4.8}$$

where C is a positive constant, independent of n.

4.3. **Convergence theorem.** In this section we state and prove one of the main results of the paper related to convergence of the rational approximants.

Theorem 4.2. *Fix a positive integer m and an integer k, $0 \le k \le m$. The sequence $P_{k,m,n}/Q_{k,m,n}$ converges σ-almost uniformly inside D_k as $n \to \infty$.*

Let us mention the result that is an analogue of the Montessus de Ballore's theorem.

Corollary 4.3. *Let a function f have exactly m poles $\alpha_1, \ldots, \alpha_m$ in the disk D_m, $m \ge 1$. Then the sequence $P_{m,m,n}/Q_{m,m,n}$ converges uniformly to the function f on compact subsets of $D_m \setminus \{\alpha_1, \ldots, \alpha_m\}$ as $n \to \infty$. Moreover, for sufficiently large n the polynomial $Q_{m,m,n}$ has exactly degree m and for each pole α_j of f of multiplicity l, $l \ge 1$, exactly l zeros of $Q_{m,m,n}$ converge to α_j as $n \to \infty$.*

Proof. Let $q(z) = \prod_j (z - \alpha_j)^{\mu_j}$, where the product is taken of the poles α_j of f in D_{m+1} (μ_j is the order of a pole α_j). Note that a function $F = qf$ is analytic in D_{m+1} and $q \in \mathbf{P}_{m+1}$. We select R, $1 < R < R_{m+1}$, close to R_{m+1} such that the open disk $G_R = \{z : |z| < R\}$ contains all poles of f in D_{m+1}. We assume that the boundary Γ_R of G_R is positively oriented with respect to G_R. Using the Cauchy integral formula and (2.1) for $\alpha = Q_{k,m,n}$ and $\beta = P_{k,m,n}$, we get

$$\frac{(Q_{k,m,n}F - qP_{k,m,n} - qT_{k,m,n})(z)}{z^{n+m+2}} = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{(Q_{k,m,n}F - qT_{k,m,n})(t)dt}{t^{n+m+2}(t-z)}, \quad z \in G_R,$$

where

$$T_{k,m,n}(z) = \sum_{j=n+1}^{n+m+1} (Q_{k,m,n}f)_j z^j.$$

Consequently, for $z \in G_R$,

$$(f - \frac{P_{k,m,n} + T_{k,m,n}}{Q_{k,m,n}})(z) = \frac{z^{n+m+2}}{2\pi i (qQ_{k,m,n})(z)} \int_{\Gamma_R} \frac{(Q_{k,m,n}F - qT_{k,m,n})(t)dt}{t^{n+m+2}(t-z)}. \tag{4.9}$$

We represent the right-hand side as a difference of two integrals:

$$I_{1,n}(z) = \frac{z^{n+m+2}}{2\pi i (qQ_{k,m,n})(z)} \int_{\Gamma_R} \frac{(Q_{k,m,n}F)(t)dt}{t^{n+m+2}(t-z)}, \quad z \in G_R,$$

and

$$I_{2,n}(z) = \frac{z^{n+m+2}}{2\pi i (qQ_{k,m,n})(z)} \int_{\Gamma_R} \frac{(qT_{k,m,n})(t)dt}{t^{n+m+2}(t-z)}, \quad z \in G_R.$$

Let K be any compact set in $D_k \cap G_R$. Set $r = ||z||_K$. We choose an arbitrary small positive ε. Denote by U_ε the corresponding open set. Let us estimate the first integral $I_{1,n}$. By (4.7) and (4.8),

$$||I_{1,n}||_{K\setminus U_\varepsilon} \le C\frac{r^n}{R^n}\frac{||Q_{k,m,n}||_{\Gamma_R}}{\min_{K\setminus U_\varepsilon}|Q_{k,m,n}|} \le C_1\frac{r^n}{R^n}n^{2m}$$

(in what follows $C, C_1, \ldots,$ will denote positive quantities not dependent on n). Consequently, letting $n \to \infty$ and then $R \to R_{m+1}$, we get

$$\limsup_{n\to\infty} ||I_{1,n}||^{1/n}_{K\setminus U_\varepsilon} \le \frac{r}{R_{m+1}} < 1. \tag{4.10}$$

To get an estimate of the second integral $I_{2,n}$, we first remark that on the unit circle Γ,

$$T_{k,m,n}(t) = s_{k,m,n}\overline{Q_{k,m,n}(t)}t^{n+m+1}.$$

So,

$$|T_{k,m,n}(t)| = s_{k,m,n}|Q_{k,m,n}(t)|, \quad t \in \Gamma,$$

and

$$||T_{k,m,n}||_2 = s_{k,m,n}||Q_{k,m,n}||_2 = s_{k,m,n}.$$

Since $T_{k,m,n}/z^{n+1}$ is a polynomial of degree at most m, using the Bernstein-Walsh lemma for estimating the growth of polynomials and inequalities (1.5), we get

$$||\frac{T_{k,m,n}}{z^{n+1}}||_{\Gamma_R} \le ||\frac{T_{k,m,n}}{z^{n+1}}||_\infty R^m \le (m+1)||T_{k,m,n}||_2 R^m = (m+1)s_{k,m,n}R^m. \tag{4.11}$$

Therefore,

$$||I_{2,n}||_{K\setminus U_\varepsilon} \le C_2\frac{r^n||T_{k,m,n}/t^{n+1}||_{\Gamma_R}}{\min_{K\setminus U_\varepsilon}|Q_{k,m,n}|} \le C_3 r^n s_{k,m,n}n^{2m}.$$

Hence, letting $n \to \infty$, by (4.1), we obtain that

$$\limsup_{n\to\infty} ||I_{2,n}||^{1/n}_{K\setminus U_\varepsilon} \le \frac{r}{R_k} < 1. \tag{4.12}$$

Let us estimate $T_{k,m,n}/Q_{k,m,n}$ on $K \setminus U_\varepsilon$. We have

$$||\frac{T_{k,m,n}}{Q_{k,m,n}}||_{K\setminus U_\varepsilon} \le \frac{r^{n+1}||T_{k,m,n}/z^{n+1}||_K}{\min_{K\setminus U_\varepsilon}|Q_{k,m,n}|} \le C_4 r^n n^{2m}||T_{k,m,n}/z^{n+1}||_K. \tag{4.13}$$

By the maximum principle of analytic functions,

$$||T_{k,m,n}/z^{n+1}||_K \le ||T_{k,m,n}/z^{n+1}||_{\Gamma_R}.$$

Using now (4.11) and (4.13), we get

$$||\frac{T_{k,m,n}}{Q_{k,m,n}}||_{K\setminus U_\varepsilon} \le C_5 r^n n^{2m} s_{k,m,n}.$$

Consequently, letting $n \to \infty$, we obtain (see also (4.1)) that

$$\limsup_{n\to\infty} ||\frac{T_{k,m,n}}{Q_{k,m,n}}||^{1/n}_{K\setminus U_\varepsilon} \le \frac{r}{R_k} < 1. \tag{4.14}$$

Now we can estimate $||f - P_{k,m,n}/Q_{k,m,n}||_{K\setminus U_\varepsilon}$. Using formula (4.9) and estimates (4.10), (4.12), and (4.14), we get

$$\limsup_{n\to\infty} ||f - P_{k,m,n}/Q_{k,m,n}||^{1/n}_{K\setminus U_\varepsilon} \le \frac{r}{R_k} < 1.$$

Thus, since ε was arbitrary, we can conclude that

$$P_{k,m,n}/Q_{k,m,n} \to f$$

σ-almost uniformly inside D_k as $n \to \infty$.

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