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ON A PROBLEM ON PERIODIC FUNCTIONS

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ABSTRACT. Given a continuous periodic real function f with n translates $f_1, ..., f_n$, where $f_i(x) = f(x + a_i), i = 1, ..., n$. We solve a problem by Erdos and Chang and show that there are rational numbers r, s such that $f(r) \geq f_i(r), f(s) \leq f_i(s), i = 1, ..., n$. No restrictions on the constants or any further restriction on the function f are necessary as was imposed earlier.

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1. INTRODUCTION AND RESULTS

Let $[a, b]$ be a closed interval and let F be a family of continuous non negative functions such that for every positive integer n we have $\int_a^b f^n dx = \int_a^b g^n dx$ for all $f, g \in F$. Then we say that F is a family of *integro-equivalent functions (i.e.* for short) on $[a, b]$. If, furthermore, $f(a) = f(b), g(a) = g(b)$ for all $f, g \in F$ then we say that f and g are *closed integro equivalent functions (cie* for short).

Remark 1.1. It follows easily that if F is a family of ie or cie functions and if a, b are constants such that $af + b \geq 0$ for all $f \in F$ then the family $aF + b = \{af + b : f \in F\}$ is an ie or cie family. Also this property is kept to hold if we choose the constants a, b such that $af + b > 0$ for all $f \in F$ and if we take the family $\{1/(af + b) : f \in F\}$. We just use power series expansions. Similarly if we take any analytic function h such that $h(f)$ is analytic then the family $h(F) = \{h(f) : f \in F\}$ is an ie or cie family depending on F.

We give some examples.

Example 1.1.

(1) *If f is a p-periodic function and c is any real number then f and $f(x+c)$ are cie functions on any interval $[a, a + p]$.*

(2) *The functions $x, 1 - x$ are ie functions on $[0, 1]$.*

(3) *Consider the function $f(x) = \begin{cases} 2x, 0 \leq x \leq 1 \\ 3-x, 1 \leq x \leq 3 \end{cases}$ and the function $g(x) = \begin{cases} x, 0 \leq x \leq 2 \\ 6-2x, 2 \leq x \leq 3 \end{cases}$ .Then f and g are cie functions on $[0, 3]$ as it can be easily verified.*

(4) *Let $f(x) = x, g(x) = ax + b$ be two functions on $[0, 1]$. Let us find a, b such that $f(x)$and $g(x) = ax + b$ are ie functions on $[0, 1]$. Thus*

$$\int_0^1 x^n dx = \int_0^1 (ax + b)^n dx$$

for all positive integers n. Then we have

$$\frac{1}{n+1} = \frac{1}{a(n+1)}(a + b)^{n+1} - \frac{1}{a(n+1)} b^{n+1}.$$

It follows that $a = [(a + b)^{n+1} - b^{n+1})]$ for all n. Taking $n = 1$ and $n = 2$ we get $a = -1, b = 1$ and $g(x) = -x + 1$.

(5) *Let $f(x)$ be a continuous function defined on $[a, b]$ such that $f(a) = f(b)$. Let us take the function g with a graph being the reflection of the graph of $f(x)$ in the line $x = (a + b)/2$. Thus $g(x) = f(b + a - x), a \leq x \leq b$. Then f and g are cie functions which need not be translates of any periodic function.*

(6) *Let $f(x\ )$ be a 1-periodic real and continuous function. Let $a_i, i = 1, .., n$ be distinct constants in the interval $(0, 1)$ and let $f_i(x) = f(x+a_i), i = 1, ..., n$ be the corresponding translates of $f(x)$. Then the family $F = \{f_i : i = 1, ..., n\}$ is a cie family.*

Let $f(x\ )$ be a 1-periodic real and continuous function. We assume that $t(x)$ is not locally constant. Let $a_i, i = 1, .., n$ be distinct constants in the interval $(0, 1)$ and let $f_i(x) = f(x + a_i), i = 1, ..., n$ be the corresponding translates of $f(x)$. A problem first posed by P. Erdos and C. Chang, (see Hwang [1]) asks if there is a rational function r such that $f(r) \leq \min(f_i(r)), i = 1, ..., n$. Hwang in [1] proved the following partial answer.

Theorem 1.1 (Hwang)**.** *Let $f(x)$ be a continuous function of period 1 and let $d_j, j = 1, ..., n$ be constants such that $d_j - d_1, j = 1, ..., n$ are rational (for example if $f(x)$ has a finite number* *of extremum points in the interval* $[0,1]$ *then this condition is satisfied). Then there are rational numbers* r,s *such that*

$$f(r)\leq\min(f_i(r)),f(s)\geq\max(f_i(s)),i=1,...,n.$$

We will show in this note that if $f_1,...,f_n$ are closely permutable continuous functions then there are rational numbers r,s such that

$$f(r)\leq\min(f_i(r)),f(s)\geq\max(f_i(s)),i=1,...,n.$$

No further restrictions are to be made on these functions. In particular if $f(x)$ is a continuous function of period 1 and if $a_j,j=1,...,n$ are arbitrary constants (with no restrictions) then there are rational numbers r,s such that

$$f(r)\leq\min(f_i(r)),f(r)\geq\max(f_i(s)),i=1,...,n.$$

Thus the answer to Erdus-Chang Problem is in the affirmative under the sole condition that $f(x)$ is continuous and periodic.

Let $f_i,i=1,...,m,$ be ie $\pi/2-$periodic functions. We assume, without loss of generality, that the f_i are not locally constant, and that the f_i can be made to lie between any two distinct constants by taking proper constants a,b and considering af_i+b instead of f_i. We assume all integrals are from 0 to $\pi/2$. Under these conditions we prove

Lemma 1.2. *For all positive integers* n *we have*

$$\int\sin^n f_1dx=\int\sin^n f_jdx=\int\cos^n f_1dx=\int\cos^n f_jdx,$$

for all $j=1,..,n.$

Proof. $\int\sin^n f_jdx=\int(f_j-f_j^3/3!+...)^ndx=\int\ (f_1-f_1^3/3!+...)^ndx=\int\sin^n f_1dx.$ Similarly $\int\cos^n f_jdx=\int\cos^n f_1dx$. Also we have, by periodicity,

$$\int\sin^n f_1dx=\int\cos^n(f_j-\pi/2)dx=\int\sin^n f_jdx.$$

This completes the proof. ∎

Remark 1.2. For the sake of completeness we prove the generalized Holder's inequality.

The reader may prefer to skip reading this remark. Let $g_1,...,g_n$ be nonnegative bounded and integrable functions defined on the interval $[a,b]$. Let $g=g_1...g_n$. Then

$$\int_a^b gdx\leq(\int_a^b g_1^ndx)^{1/n}....(\int_a^b g_n^ndx)^{1/n}$$

The proof is clear for the case of $n=2$; for it is just Schwartz' inequality. So we assume it is true for $n-1$ and consider the case n. Let $f=f_1...f_n$. We then have

$$\int_a^b fdx\leq\int_a^b(f_1^n)^{1/n}(f_2^{n/(n-1)}...f_n^{n/(n-1)})^{(n-1)/n}dx$$

$$\leq(\int_a^b f_1^ndx)^{1/n}(\int_a^b f_2^{n/(n-1)}...f_n^{n/(n-1)}dx)^{(n-1)/n}$$

$$\leq(\int_a^b f_1^ndx)^{1/n}[(\int_a^b f_2^{n.(n-1)/(n-1)}dx)^{1/(n-1)}...(\int_a^b f_n^{n.(n-1)/(n-1)}dx)^{1/(n-1)}]^{(n-1)/n}$$

$$\leq(\int_a^b f_1^ndx)^{1/n}...(\int_a^b f_n^ndx)^{1/n}.$$

Definition 1.1. Consider a monomial $\pm x_1 x_2 ... x_n$. If we substitute $\sin x$ or $\cos x$ for each variable $x_i, i = 1, ..., n$ in the monomial we call the resulting expression a *trigonometric* $n-$ *monomial*. A sum of m distinct trigonometric $n-$monomials is called a *trigonometric homogeneous* $n-$*polynomial of rank* m *(for short* $thn(m)$ polynomial).

Lemma 1.3. *Each of the expressions*

$$\sin(a_1 + a_2 + ... + a_n), \cos(a_1 + ... + a_n)$$

is a $thn(\ 2^{n-1})$.

Proof. From $\sin(a_1 + a_2) = \sin a_1 \cos a_2 + \cos a_1 \sin a_2$, and $\cos(a_1 + a_2) = \cos a_1 \cos a_2 - \sin a_1 \sin a_2$ and so the assertion is true for $n = 2$. Assume it is true for $n - 1$. Now $\sin(a_1 + a_2 + ... + a_n) = \sin(a_1 + (a_2 + ... + a_n)$ $= \sin a_1 \cos(a_2 + ... + a_n) + \cos a_1 \sin(a_2 + ... + a_n)$. Using induction we see that the latter sum is $thn(2^{n-1}) + thn(2^{n-1}) = thn(2^{n-1})$. This completes the proof.

Remark 1.3.

(1) Let f, f_1, f_2 be ie functions on $[0, \pi/2]$ and let $0 < f < \pi/6$. Then it is impossible to have $f < f_1 + f_2$. For a proof we argue as follows. From the hypothesis it follows that $0 < f, f_1, f_2 < \pi/6$. Assume that $f < f_1 + f_2$. Now $0 < f < f_1 + f_2 < \pi/3 < \pi/2$. Since $\sin x$ is increasing on the interval $[0, \pi/2]$ we have $\sin f < \sin(f_1 + f_2) = \sin f_1 \cos f_2 + \cos f_1 \sin f_2$. Taking integrals (as usual from 0 to $\pi/2$) and using Holder's inequality and Remark 1.1 we get

$$\begin{aligned}\int \sin f dx &\leq \int \sin f_1 \cos f_2 dx + \int \cos f_1 \sin f_2 dx \\ &\leq 2(\int \sin^2 f dx)^{1/2}(\int \sin^2 f dx)^{1/2} = 2\int \sin^2 f dx.\end{aligned}$$

It follows that $\int \sin f(2\sin f - 1)dx \geq 0$. But $2\sin f - 1 < 0$ since $0 < f < \pi/6$. This contradiction completes the proof.

(2) Let f, f_1, f_2, f_3 be ie functions on $[0, \pi/2]$and let $0 < f < \pi/8$. Then it is impossible to have $f \leq f_1 + f_2 + f_3$. For a proof we argue as follows. Assume that $f \leq f_1 + f_2 + f_3$. It follows that $0 < f, f_1, f_2, f_3 < \pi/8$. Assume that $0 < f \leq f_1 + f_2 + f_3 < 3\pi/8 < \pi/2$. Since $\sin x$ is increasing on the interval $[0, \pi/2]$, $\sin f \leq \sin(f_1 + f_2 + f_3)$ and so $\int \sin f dx < 4\int \sin^3 f dx$. Thus $\int \sin f(4\sin^2 f - 1)dx > 0$. But $4\sin^2 f - 1 < 0$ because $0 < f < \pi/6$. This contradiction completes the proof.

∎

Lemma 1.4. *Let* $f, f_1, f_2, ..., f_n$ *be ie functions on* $[0, \pi/2]$*and let* $0 < f < \pi/2n$. *Then it is impossible to have* $f \leq f_1 + f_2 + ... + f_n$.

Proof. Assume that $f \leq f_1 + f_2 + ... + f_n$. Then $\int \sin f dx < 2^{n-1} \int \sin^n f dx$. Thus $\int \sin f(2^{n-1}\sin^{n-1} f - 1)dx > 0$. It follows that $(2^{n-1}\sin^{n-1} f - 1) < 0$ in the interval $[0, \pi/(2n)]$. This is a contradiction. The proof is complete. ∎

Proposition 1.5. *Let* $\{f, f_1, f_2, ..., f_n\}$ *be a set of ie non constant , non negative and continuous cie functions on* $[a, b]$. *Then there is* y *such that* $f(y) > \max f_i(y)$ *and there is* z *such that* $f(z) < \min f_i(z)$.

Proof. There is a uniform bound M for the elements of the cie family $F = \{f, f_1, f_2, ..., f_n\}$ and there is a change of variable and there are constants a, b that makes Lemma 1.4 applicable. Then there is y such that $f(y) > f_1 + f_2 + ... + f_n$.. Thus there is y such that $f(y) > \max f_i(y)$. Using the Remark 1.1 and taking reciprocals the second inequality follows: There is z such that $f(z) < \min f_i(z)$. ∎

Proposition 1.6. *Let f be a non locally constant real-valued continuous periodic function of period $\pi/2$. Let $a_1, ..., a_n \in (0,1)$ be distinct real numbers. Let $f_i(x) = f(x+a_i), i = 1, ..., n$ be n translates of f. Then there is y such that $f(y) > \max f_i(y)$ and there is z such that $f(z) < \min f_i(z)$.*

Proof. There are $a > 0, b$ such that $0 < F = af + b < \pi/(2n)$. Let $F_i(x) = F(x+a_i), i = 1, ..., n$ be the n translates of F. Then by Lemma 1.4 it is impossible to have $F \leq F_1 + F_2 + ... + F_n$. Thus there is $y \in (0, \pi/2)$ such that $F(y) > F_1(y) + F_2(y) + ... + F_n(y)$. It is clear then that $F(y) > F_i(y), f(y) > f_i(y), i = 1, ..., n$. The first part of the proposition follows. We choose real numbers $c, d, h, c > 0$ such that $G = 0 < \frac{d}{F+c} + h < \pi/(2n)$ and form the translates $G_i(x) = G(x+a_i), i = 1, ..., n$. Then by Lemma 1.4 it is impossible to have $G \leq G_1 + G_2 + ... + G_n$. Thus there is z such that $G(z) > G_1(z) + G_2(z) + ... + G_n(z)$. It is clear then that $G(z) > G_i(z), f(z) < f_i(z), i = 1, ..., n$. The second part of the proposition follows. This completes the proof. ∎

Theorem 1.7. *Let $f(x)$ be a continuous function of period 1 and let $d_j \in (0,1), j = 1, ..., n$.Then there are rational numbers r, s such that $f(r) \leq \min(f_i(r)), f(s) \geq \max(f_i(s)), i = 1, ..., n$.*

Proof. Let $g(x) = f(x\pi/2)$. Then g is $\pi/2$-periodic. Then there is z such that $g(z) < g_1(z), g_2(z)$. Also there is a rational number r such that if $r/(\pi/2) = s$ then $g(s) < g_1(s), g_2(s)$. Or, $f(r) = f(\frac{r}{\pi/2} . \frac{\pi}{2}) < f_1(\frac{r}{\pi/2} . \frac{\pi}{2}) = f_1(r), f_2(\frac{r}{\pi/2} . \frac{\pi}{2}) = f_2(r)$. Similarly there is a rational number t such that $f(t) > f_1(t), f_2(t)$. This can be generalized to n translates of f. This completes the proof. ∎

Problem 1. *Let $f_i, i = 1, ..., m$ be real-valued, continuous permutable functions defined on a cube l in R^n. Is there a rational point $p = (r_1, ..., r_m) \in I$ such that $f_1(p) \geq f_i(p), i = 1, ..., m$?*

REFERENCES

[1] [1] J. S. HWANG, A Problem on Continuous and Periodic Functions, *Pacific Journal of Mathematics*, **Vol. 117**, No. 1 (1985).