WEAK TYPE INEQUALITIES FOR SOME OPERATORS ON GENERALIZED MORREY SPACES OVER METRIC MEASURE SPACES

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ABSTRACT. We discuss weak type inequalities for maximal and fractional integral operators on generalized Morrey spaces over metric measure spaces. Here the measure satisfies the so called growth condition. By taking into account the maximal operator, we obtain a Hedberg type inequality, which leads us to the weak type inequality for the fractional integral operator on the same spaces.

1. INTRODUCTION

Let X be a metric measure space which is equipped with distance function d and Borel measure μ. Here the measure μ satisfies the *growth condition*:

$$\mu(B(a,r)) \le C\, r^n \tag{1.1}$$

for every ball $B(a,r)$ with center $a \in X$ and radius $r > 0$. Researchers often refer this type of measure as a growth measure. In the equation (1.1), n is less than or equal to the dimension of X, meanwhile C is a positive constant which does not depend on x and r. In the rest of this paper, C will be used to denote positive constants that may be different from one line to another.

With the growth measure, we define the *maximal operator*

$$Mf(x) := \sup_{r>0} \frac{1}{\mu(B(x,2r))} \int_{B(x,r)} |f(y)|\, d\mu(y) \quad (x \in \mathrm{supp}(\mu)) \tag{1.2}$$

(as in [8]) and the *fractional integral operator*

$$I_\alpha f(x) := \int_X \frac{f(y)}{d(x,y)^{n-\alpha}}\, d\mu(y) \tag{1.3}$$

(as in [3]). Note that $0 < \alpha < n$. In [3], it has been proved that the fractional integral operator satisfies the weak type $(1,q)$ on Lebesgue spaces over metric measure spaces. Meanwhile, in [10], it has been proved the weak type $(1,1)$ inequality for the maximal operator on Morrey spaces over metric measure spaces. Morrey spaces over Euclidean spaces were first introduced by C.B. Morrey in [6]. For certain condition, Morrey spaces reduce to Lebesgue spaces. In this paper, we will generalize the results of [3] and [10] on the generalized Morrey spaces over metric measure spaces.

2. WEAK TYPE INEQUALITIES FOR THE MAXIMAL AND FRACTIONAL INTEGRAL OPERATORS

Denote by $\mathcal{B}(\mu)$ the set of all balls with positive μ-measure. Let $\phi : (0,\infty) \to (0,\infty)$ be an almost decreasing function, that is, there is a positive constant C such that the inequality $\phi(s) \ge C\phi(t)$ holds whenever $s < t$. For $1 \le p < \infty$, we define the *generalized Morrey space over metric measure spaces* $\mathcal{M}^{p,\phi}(\mu) := \mathcal{M}^{p,\phi}(X,\mu)$ to be the set of all functions in $f \in L^p_{loc}(\mu)$ such that

$$\|f\|_{\mathcal{M}^{p,\phi}(\mu)} := \sup_{B \in \mathcal{B}(\mu)} \frac{1}{\phi(\mu(2B))} \left(\frac{1}{\mu(2B)} \int_B |f(y)|^p\, d\mu(y) \right)^{1/p} < \infty.$$

To find our results on generalized Morrey space over metric measure spaces, we provide the following lemma which is adopted from [2].

Lemma 2.1. *Let w be any positive weight on X. Then, for any function $f \in L^1_{loc}(\mu)$, we have the inequality*

$$\int_{\{x \in X : Mf(x) > \gamma\}} w(x)\, d\mu(x) \le \frac{C}{\gamma} \int_X |f(y)| Mw(y)\, d\mu(y). \tag{2.1}$$

Our first result is the following lemma. Analogous results on the Euclidean space $\mathbb{R}^d$ can be found in, for examples, [4, 7, 9].

Lemma 2.2. *For $t > 0$, let $g : t \to t^n$ and $\Phi(t) = \phi(g(t))$. If Φ satisfies $\int_r^\infty \frac{\Phi(t)}{t} dt \le C^*\Phi(r)$ for $r > 0$ and $C^* > 0$, then for any function f on $\mathcal{M}^{1,\phi}(\mu)$ and any ball $B(a,r)$ on X we have*

$$\int_X |f(x)| M\chi_{B(a,r)}(x)\, d\mu(x) \le Cr^n \phi(r^n) \|f\|_{\mathcal{M}^{1,\phi}(\mu)}. \tag{2.2}$$

Proof. Let $\chi_{B(a,r)}$ denote the characteristic function of $B(a,r)$. Then, for any $f \in \mathcal{M}^{1,\phi}(\mu)$, we have

$$
\int_X |f(x)| M\chi_{B(a,r)}(x) \, d\mu(x)
$$

$$
\leq \int_{B(a,2r)} |f(x)| M\chi_{B(a,r)}(x) \, d\mu(x) + \sum_{k=1}^{\infty} \int_{B(a,2^{k+1}r)\setminus B(a,2^k r)} |f(x)| M\chi_{B(a,r)}(x) \, d\mu(x)
$$

$$
\leq C \int_{B(a,2r)} |f(x)| + \sum_{k=1}^{\infty} \int_{B(a,2^{k+1}r)\setminus B(a,2^k r)} 2^{-kn}|f(x)| \, d\mu(x)
$$

$$
\leq C \left[\mu(B(a,4r))\phi(\mu(B(a,4r)))\|f\|_{\mathcal{M}^{1,\phi}(\mu)} \right]
$$

$$
+ C \left[\sum_{k=1}^{\infty} 2^{-kn}\mu(B(a,2^{k+2}r))\phi(\mu(B(a,2^{k+2}r)))\|f\|_{\mathcal{M}^{1,\phi}(\mu)} \right]
$$

$$
= C \sum_{k=0}^{\infty} 2^{-kn}\mu(B(a,2^{k+2}r))\phi(\mu(B(a,2^{k+2}r)))\|f\|_{\mathcal{M}^{1,\phi}(\mu)}.
$$

By the growth condition, we find that

$$
\int_X |f(x)| M\chi_{B(a,r)}(x) \, d\mu(x) \leq C \sum_{k=0}^{\infty} 2^{-kn}(2^{k+2}r)^n \phi(C_1(2^{k+2}r)^n)\|f\|_{\mathcal{M}^{1,\phi}(\mu)}
$$

$$
= Cr^n \|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{k=0}^{\infty} \phi(C_1(2^{k+2}r)^n)
$$

$$
= Cr^n \|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{k=0}^{\infty} \Phi(2\sqrt[n]{C_1}(2^{k+1}r)).
$$

As ϕ is almost decreasing, Φ is also almost decreasing. Hence,

$$
\int_X |f(x)| M\chi_{B(a,r)}(x) \, d\mu(x) \leq Cr^n \|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{k=0}^{\infty} \Phi(\sqrt[n]{C_1}(2^{k+1}r))
$$

$$
= Cr^n \|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{k=0}^{\infty} \Phi(C_2(2^{k+1}r)).
$$

Now, notice that for $2^k r \leq t \leq 2^{k+1}r$ we have

$$
\Phi(C_2(2^{k+1}r)) \leq C_3\Phi(C_2(t)
$$

and

$$
\Phi(C_2 t) = \Phi(\sqrt[n]{C_1}(t) = \phi(C_1{}^n t^n) \leq C_4\phi(t^n) = C_4\Phi(t).
$$

As a result, we obtain

$$
\begin{aligned}
\int_X |f(x)| M\chi_{B(a,r)}(x)\, d\mu(x)
&\le C r^n \|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{k=0}^{\infty} \int_{2^k r}^{2^{k+1} r} \frac{\Phi(C_2 t)}{t}\, dt \\
&\le C r^n \|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{k=0}^{\infty} \int_{2^k r}^{2^{k+1} r} \frac{\Phi(t)}{t}\, dt \\
&\le C r^n \|f\|_{\mathcal{M}^{1,\phi}(\mu)} \int_{r}^{\infty} \frac{\Phi(t)}{t}\, dt \\
&\le C r^n \Phi(r) \|f\|_{\mathcal{M}^{1,\phi}(\mu)} \\
&\le C r^n \phi(r^n) \|f\|_{\mathcal{M}^{1,\phi}(\mu)}.
\end{aligned}
$$

So, we have done. ∎

The above lemma enables us to prove the following weak type (1,1) for the maximal operator.

Theorem 2.3. *Let ϕ satisfies the condition given in Lemma 2.2. For any $\gamma > 0$ and any ball $B(a,r)$, we have*

$$
(2.3) \qquad \mu\{x \in B(a,r) : Mf(x) > \gamma\} \le \frac{C}{\gamma} r^n \phi(r^n) \|f)\|_{\mathcal{M}^{1,\phi}(\mu)}.
$$

Proof. By applying Lemma 2.1 and 2.2, we obtain

$$
\begin{aligned}
\mu\left(\{x \in B(a,r) : Mf(x) > \gamma\}\right)
&= \int_{\{x \in X : Mf(x) > \gamma\}} \chi_{B(a,r)}\, d\mu(x) \\
&\le \frac{C}{\gamma} \int_X |f(x)| M\chi_{B(a,r)}(x)\, d\mu(x) \\
&\le \frac{C}{\gamma} r^n \phi(r^n) \|f\|_{\mathcal{M}^{1,\phi}(\mu)},
\end{aligned}
$$

as desired. ∎

Now, we will present the weak type $(1, q)$ inequality for the fractional integral operator.

Theorem 2.4. *Let $0 < \alpha < n$, $0 \le \lambda \le n - \alpha$ for some λ, and $\frac{1}{q} = 1 - \frac{\alpha}{n-\lambda}$. Also, for $t > 0$, let $g : t \to t^n$ and $\Phi(t) = \phi(g(t))$. If Φ satisfies the condition $\int_R^{\infty} t^{\alpha-1} \Phi(t)\, dt \le C R^{\lambda+\alpha-n}$ for any positive R, then the inequality*

$$
(2.4) \qquad \mu\left(\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\}\right) \le C r^n \phi(r^n) \left(\frac{\|f\|_{\mathcal{M}^{1,\phi}(\mu)}}{\gamma}\right)^q
$$

holds for any f in $\|f\|_{\mathcal{M}^{1,\phi}(\mu)}$, any ball $B(a,r)$ in X, and any positive γ.

Proof. For every $x \in B(a,r)$, we have

$$
\begin{aligned}
|I_\alpha f(x)| &\le \int_{B(x,R)} \frac{|f(y)|}{d(x,y)^{n-\alpha}}\, d\mu(y) + \int_{X \setminus B(x,R)} \frac{|f(y)|}{d(x,y)^{n-\alpha}}\, d\mu(y) \\
&= A_1 + A_2.
\end{aligned}
$$

The estimate for A_2 is as follows.

$$A_2 \le \int_{d(x,y) \ge R} \frac{|f(y)|}{d(x,y)^{n-\alpha}} \, d\mu(y)$$

$$\le \sum_{k=0}^{\infty} \int_{B(x,2^{k+1}R) \setminus B(x,2^k R)} \frac{|f(y)|}{d(x,y)^{n-\alpha}} \, d\mu(y)$$

$$\le \sum_{k=0}^{\infty} \frac{1}{(2^k R)^{n-\alpha}} \int_{B(x,2^{k+1}R)} |f(y)| \, d\mu(y)$$

$$= \sum_{k=0}^{\infty} \frac{2^{2n}(2^k R)^\alpha}{(2^{k+2}R)^n} \int_{B(x,2^{k+1}R)} |f(y)| \, d\mu(y).$$

Since μ satisfies the growth condition and ϕ is almost decreasing, we have

$$A_2 = \sum_{k=0}^{\infty} \frac{2^{2n}(2^k R)^\alpha}{\mu(B(x,2^{k+2}R))} \int_{B(x,2^{k+1}R)} |f(y)| \, d\mu(y)$$

$$\le C\|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{k=0}^{\infty} (2^k R)^\alpha \phi(C_1 (2^{k+2}R)^n)$$

$$\le C\|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{k=0}^{\infty} (2^k R)^\alpha \Phi(2\sqrt[n]{C_1}(2^{k+1}R))$$

$$\le C\|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{k=0}^{\infty} (2^k R)^\alpha \Phi(C_2 (2^{k+1}R)).$$

Now, for $2^k R \le t \le 2^{k+1}R$ with $k = 1, 2, ...$, we have

$$\Phi(C_2 2^{k+1}R) \le C_3 \Phi(C_2 t) \le C_4 \Phi(C_2 2^k R)$$

and

$$(2^k R)^\alpha \le t^\alpha \le (2^{k+1}R)^\alpha.$$

Furthermore,

$$\Phi(C_2 t) = \Phi(\sqrt[n]{C_1} t) = \phi(C_1{}^n t^n) \le C_4 \phi(t^n) = C_4 \Phi(t),$$

which gives us

$$(2^k R)^\alpha \Phi(C_2(2^{k+1}R)) \le C \int_{2^k R}^{2^{k+1}R} t^{\alpha-1} \Phi(C_2 t) \, dt \le C \int_{2^k R}^{2^{k+1}R} t^{\alpha-1} \Phi(t) \, dt.$$

As a result, we obtain

$$A_2 \le \int_{d(x,y) \ge R} \frac{|f(y)|}{d(x,y)^{n-\alpha}} \, d\mu(y)$$

$$\le C\|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{k=0}^{\infty} \int_{2^k R}^{2^{k+1}R} t^{\alpha-1} \Phi(t) \, dt$$

$$\le C\|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{k=0}^{\infty} \int_{R}^{\infty} t^{\alpha-1} \Phi(t) \, dt$$

$$\le C R^{\lambda+\alpha-n} \|f\|_{\mathcal{M}^{1,\phi}(\mu)}.$$

Let $CR^{\lambda+\alpha-n}\|f\|_{\mathcal{M}^{1,\phi}(\mu)} := \frac{\gamma}{2}$. Since $A_2 \leq \frac{\gamma}{2}$, we have $\left\{x \in B(a,r) : A_2 > \frac{\gamma}{2}\right\} = \emptyset$. This means that $\mu\left(\left\{x \in B(a,r) : A_2 > \frac{\gamma}{2}\right\}\right) = 0$. Finally, by using the fact that

$$\left\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\right\} \subset \left\{x \in B(a,r) : A_1 > \frac{\gamma}{2}\right\} \cup \left\{x \in B(a,r) : A_2 > \frac{\gamma}{2}\right\},$$

and Lemma 2.2, we can obtain the weak estimate $(1,q)$ for I_α:

$$\begin{aligned}
\mu(\{x \in B(a,r) &: |I_\alpha f(x)| > \gamma\}) \\
&\leq \mu\left(\left\{x \in B(a,r) : A_1 > \frac{\gamma}{2}\right\}\right) \\
&= \mu\left(\left\{x \in B(a,r) : \int_{B(x,R)} \frac{|f(y)|}{d(x,y)^{n-\alpha}}\, d\mu(y) > \frac{\gamma}{2}\right\}\right) \\
&\leq \frac{2}{\gamma} \int_{B(a,r)} \int_{B(x,R)} \frac{|f(y)|}{d(x,y)^{n-\alpha}}\, d\mu(y)\, d\mu(x) \\
&\leq \frac{2}{\gamma} \int_X \int_{B(x,R)} \frac{|f(y)|}{d(x,y)^{n-\alpha}} \chi_{B(a,r)}(x)\, d\mu(y)\, d\mu(x) \\
&= \frac{2}{\gamma} \int_X |f(y)| \left(\int_{B(y,R)} \frac{\chi_{B(a,r)}(x)}{d(x,y)^{n-\alpha}}\, d\mu(x)\right)\, d\mu(y) \\
&= \frac{2}{\gamma} \int_X |f(y)| \left(\sum_{-\infty}^{-1} \int_{B(y,2^{k+1}R)\setminus B(y,2^k R)} \frac{\chi_{B(a,r)}(x)}{d(x,y)^{n-\alpha}}\, d\mu(x)\right)\, d\mu(y) \\
&\leq \frac{2}{\gamma} \int_X |f(y)| \left(\sum_{-\infty}^{-1} \frac{1}{(2^k R)^{n-\alpha}} \int_{B(y,2^{k+1}R)} \chi_{B(a,r)}(x)\, d\mu(x)\right)\, d\mu(y) \\
&\leq \frac{C}{\gamma} \int_X |f(y)| \left(\sum_{-\infty}^{-1} \frac{(2^k R)^\alpha}{\mu(B(y,2^{k+2}R))} \int_{B(y,2^{k+1}R)} \chi_{B(a,r)}(x); \mu(x)\right)\, d\mu(y) \\
&\leq \frac{C}{\gamma} \int_X |f(y)| R^\alpha M\chi_{B(a,r)}(y)\, d\mu(y) \\
&\leq \frac{C}{\gamma} R^\alpha r^n \phi(r^n)\|f\|_{\mathcal{M}^{1,\phi}(\mu)}.
\end{aligned}$$

Notice that

$$\frac{R^\alpha}{\gamma}\|f\|_{\mathcal{M}^{1,\phi}(\mu)} = CR^{n-\lambda} = \left(\frac{\|f\|_{\mathcal{M}^{1,\phi}(\mu)}}{\gamma}\right)^q.$$

This equation give us

$$\begin{aligned}
\mu\left(\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\}\right) &\leq Cr^n \phi(r^n)\left(\frac{R^\alpha\|f\|_{\mathcal{M}^{1,\phi}(\mu)}}{\gamma}\right) \\
&= Cr^n \phi(r^n)\left(\frac{\|f\|_{\mathcal{M}^{1,\phi}(\mu)}}{\gamma}\right)^q,
\end{aligned}$$

which is the desired inequality. ∎

3. HEDBERG TYPE INEQUALITY

Hedberg type inequality estimates the fractional integral operator by taking into account the maximal operator. The original work on the Hedberg inequality can be found in [5]. Here we present one on the generalized Morrey spaces.

Theorem 3.1. *For $t > 0$, let $g : t \to t^n$ and $\Phi(t) = \phi(g(t))$. If $0 < \alpha < n$, $0 \le \lambda \le n - \alpha$ for some λ, and $\int_R^\infty t^{\alpha-1}\Phi(t)\, dt \le CR^{\lambda+\alpha-n}$ for any positive R, then we have*

(3.1)
$$|I_\alpha f(x)| \le C(Mf(x))^{1-\alpha/(n-\lambda)}\|f\|_{\mathcal{M}^{1,\phi}(\mu)^{\alpha/(n-\lambda)}}.$$

Proof. Let $x \in X$ and $f \in \mathcal{M}^{1,\phi}(\mu)$. Then, for any positive R, we can write

$$|I_\alpha f(x)| \le \int_{B(x,R)} \frac{|f(y)|}{d(x,y)^{n-\alpha}}\, d\mu(y) + \int_{X \setminus B(x,R)} \frac{|f(y)|}{d(x,y)^{n-\alpha}}\, d\mu(y)$$
$$= A_1 + A_2.$$

By applying the growth condition of μ, we find an estimate for A_1, that is

$$A_1 = \int_{d(x,y)<R} \frac{|f(y)|}{d(x,y)^{n-\alpha}}\, d\mu(y)$$
$$= \sum_{k=-\infty}^{-1} \int_{2^kR<d(x,y)\le 2^{k+1}R} \frac{|f(y)|}{d(x,y)^{n-\alpha}}\, d\mu(y)$$
$$\le \sum_{k=-\infty}^{-1} \frac{1}{(2^kR)^{n-\alpha}} \int_{B(x,2^{k+1}R)} |f(y)|\, d\mu(y)$$
$$\le C \sum_{k=-\infty}^{-1} \frac{R^\alpha 2^{k\alpha}}{\mu(B(x,2^{k+2}R))} \int_{B(x,2^{k+1}R)} |f(y)|\, d\mu(y)$$
$$\le CR^\alpha Mf(x).$$

Meanwhile, we estimate A_2 as follows:

$$A_2 = \int_{d(x,y)<R} \frac{|f(y)|}{d(x,y)^{n-\alpha}}\, d\mu(y)$$
$$= \sum_{k=0}^{\infty} \int_{2^kR\le d(x,y)<2^{k+1}R} \frac{|f(y)|}{d(x,y)^{n-\alpha}}\, d\mu(y)$$
$$\le \sum_{k=0}^{\infty} \frac{1}{(2^kR)^{n-\alpha}} \int_{B(x,2^{k+1}R)} |f(y)|\, d\mu(y)$$
$$= \sum_{k=0}^{\infty} \frac{(2^kR)^\alpha 2^{2n}}{(2^{k+2}R)^{n-\alpha}} \int_{B(x,2^{k+1}R)} |f(y)|\, d\mu(y).$$

Again, since μ is the growth measure, we obtain

$$A_2 \leq C \sum_{k=0}^{\infty} \frac{(2^k R)^\alpha}{\mu(B(x, 2^{k+2}R))} \int_{B(x,2^{k+1}R)} |f(y)| \, d\mu(y)$$

$$\leq C \sum_{k=0}^{\infty} (2^k R)^\alpha \phi(\mu(B(x, 2^{k+2}R))) \int_{B(x,2^{k+1}R)} |f(y)| \, d\mu(y)$$

$$\leq C\|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{k=0}^{\infty} (2^k R)^\alpha \phi(C_1(2^{k+2}R)^n)$$

$$= C\|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{k=0}^{\infty} (2^k R)^\alpha \Phi(2 \sqrt[n]{C_1}(2^{k+1}R))$$

$$\leq C\|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{k=0}^{\infty} 2^k R^\alpha \Phi(C_2(2^{k+1}R))$$

$$\leq C\|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{k=0}^{\infty} \int_{2^k R}^{2^{k+1}R} t^{\alpha-1}\Phi(t) \, dt$$

$$\leq C\|f\|_{\mathcal{M}^{1,\phi}(\mu)} \int_{R}^{\infty} t^{\alpha-1}\Phi(t) \, dt$$

$$\leq C R^{\lambda+\alpha-n}\|f\|_{\mathcal{M}^{1,\phi}(\mu)}.$$

Now, by letting $R = \left(\frac{M_f(x)}{\|f\|_{\mathcal{M}^{1,\phi}(\mu)}}\right)^{\frac{1}{\lambda-n}}$, we find that

$$|I_\alpha f(x)| \leq C R^\alpha \left(Mf(x) + R^{\lambda-n}\|f\|_{\mathcal{M}^{1,\phi}(\mu)}\right)$$

$$= C(Mf(x))^{1-\alpha/(n-\lambda)}\|f\|_{\mathcal{M}^{1,\phi}(\mu)}^{\alpha/(n-\lambda)}.$$

This completes the proof. ∎

By having the Hedberg type inequality in Theorem 3.1 and the weak type $(1,1)$ inequality for maximal operator in Theorem 2.3, we can have an alternative proof for the weak type $(1, q)$ for the fractional integral operator in Theorem 2.4. The alternative proof goes as follows.

Proof. (of Theorem 2.4) It follows from Theorem 3.1 that if $|I_\alpha f(x)| > \gamma$, then we have

$$Mf(x) > \left(\frac{\gamma}{C\|f\|_{\mathcal{M}^{1,\phi}(\mu)}^{\alpha/(n-\lambda)}}\right)^q.$$

Now, by applying Theorem 2.3, we obtain

$$\mu(\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\})$$

$$\leq \mu\left(\left\{x \in B(a,r) : Mf(x) > \left(\frac{\gamma}{C\|f\|_{\mathcal{M}^{1,\phi}(\mu)}^{\alpha/(n-\lambda)}}\right)^q\right\}\right)$$

$$\leq C\left(\frac{\|f\|_{\mathcal{M}^{1,\phi}(\mu)}^{\alpha/(n-\lambda)}}{\gamma}\right)^q r^n \phi(r^n)\|f\|_{\mathcal{M}^{1,\phi}(\mu)}$$

$$= C r^n \phi(r^n) \left(\frac{\|f\|_{\mathcal{M}^{1,\phi}(\mu)}}{\gamma}\right)^q,$$

for any ball $B(a, r)$ in X. ∎

4. CONCLUDING REMARKS

We have proved weak type inequalities for the fractional integral operators on generalized Morrey spaces over metric measure spaces in two different ways, with and without the Hedberg type inequality. For the strong type inequalities, the use of Hedberg type inequality is common (see, for instance, [1]), although it is not always applicable (see, for instance, [11]). Note that the maximal operator plays an important role, especially when we choose to use the Herberg type inequality.

REFERENCES

[1] ERIDANI, H. GUNAWAN, and E. NAKAI, On generalized fractional integral operators, *Sci. Math. Jpn.,* **60** (2004), pp. 539–550.

[2] C. FEFFERMAN and E. M. STEIN, Some maximal inequalities, *Amer. J. Math*, **93** (1971), pp. 107–115.

[3] J. GARCÍA-CUERVA and A.E. GATTO, Boundedness properties of fractional integral operators associated to non-doubling measures, *Studia Math.* **162**, (2004), pp. 245–261.

[4] J. GARCÍA-CUERVA and J.M. MARTELL, Two weight norm inequalities for maximal operators and fractional integrals on non-homogeneous spaces, *Indiana Univ. Math. J.*, **50** (2001), pp. 1241–1280.

[5] L.I. HEDBERG, On certain convolution inequalities, *Proc. Amer. Math. Soc.*, **36** (1972), pp. 505–510.

[6] C.B. MORREY, Functions of several variables and absolute continuity, *Duke Math. J.,* **6** (1940), pp. 187–215.

[7] E. NAKAI, Hardy-Littlewood maximal operator, singular integral operators, and the Riesz potentials on generalized Morrey spaces, *Math. Nachr.*, **166** (1994), pp. 95–103.

[8] Y. SAWANO, Sharp estimates of the modified Hardy-Littlewood maximal operator on the nonhomogeneous space via covering lemmas, *Hokkaido Math. J.*, **34** (2005), pp. 435–458.

[9] I. SIHWANINGRUM, S. MARYANI, and H. GUNAWAN, Weak type inequalities for fractional integral operators on generalized non-homogeneous Morrey spaces, *Anal. Theory Appl.*, **28** (2012), No. 1, pp. 65–72.

[10] I. SIHWANINGRUM and Y. SAWANO, Weak and strong type estimates for fractional integral operator on Morrey spaces over metric measure spaces, *Eurasian Math. J.*, **4** (2013), pp. 76–81.

[11] I. SIHWANINGRUM, H.P. SURYAWAN, and H. GUNAWAN, Fractional integral operators and Olsen inequalities on non-homogeneous spaces, *Aust. J. Math. Anal. Appl.*, **7** (2010), No. 1, Article 14, 6 pp.