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C^*-VALUED METRIC PROJECTION AND MOORE-PENROSE INVERSE ON HILBERT C^*-MODULES

M. ESHAGHI GORDJI, H. FATHI AND S.A.R. HOSSEINIOUN

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DEPARTMENT OF MATHEMATICS, SEMNAN UNIVERSITY, P.O. BOX 35195-363, SEMNAN, IRAN; CENTER OF EXCELLENCE IN NONLINEAR ANALYSIS AND APPLICATIONS (CENAA), SEMNAN UNIVERSITY, IRAN.
Madjid.Eshaghi@gmail.com

DEPARTMENT OF MATHEMATICS, SHAHID BEHESHTI UNIVERSITY, TEHRAN, IRAN.
Hedayat.fathi@yahoo.com

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF ARKANSAS, FAYETTEVILLE, ARKANSAS 72701, USA.
shossein@uark.net

ABSTRACT. Let t be a regular operator between Hilbert C^*-modules and $t^\dagger$ be its Moore-Penrose inverse. We give some characterizations for $t^\dagger$ based on C^*-valued metric projection. Moore-Penrose inverse of bounded operators and elements of a C^*-algebra is studied as a special case.

Key words and phrases: Hilbert C^*-module, C^*-valued metric projection, Moore-Penrose inverse.

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1. INTRODUCTION

A Hilbert C^*-module is a generalization of a Hilbert space for which the inner product takes its values in a C^*-algebra instead of the complex numbers. Exactly a pre-Hilbert C^*-module over a (not necessarily unital) C*-algebra A is a right A-module E equipped with an A-valued mapping $\langle .,. \rangle : E \times E \longrightarrow A$ with following properties:

(i) $\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle \qquad (x, y, z \in E, \lambda \in \mathbb{C})$,
(ii) $\langle x, ya\rangle = \langle x, y\rangle a \qquad (x, y \in E, a \in A)$,
(iii) $\langle y, x\rangle = \langle x, y\rangle^* \qquad (x, y \in E)$,
(iv) $\langle x, x\rangle \geq 0,\ (x \in E)$ and if $\langle x, x\rangle = 0$ then $x = 0$.

The map $\langle .,. \rangle$ is called the A-valued inner product on E. A pre-Hilbert A-module $(E\ ,\ \langle .,.\rangle)$ is called Hilbert A-module if it is complete with respect to the norm $\| \ . \ \| = \| \ \langle .,. \rangle \ \|_A^{\frac{1}{2}}$. We always suppose that the linear structures of A and E are compatible.

Although Hilbert C^*-modules behave like Hilbert spaces in some ways, some fundamental and familiar Hilbert space properties do not hold. For example, given a closed submodule V of a Hilbert C^*-module E, define $V^\perp = \{x \in E : \langle x, y\rangle = 0, \forall y \in V\}$. Then $V^\perp$ is a closed submodule, but usually $E \neq V \oplus V^\perp$.

If I is a closed right ideal of C^*-algebra A, then I is a Hilbert A-module if we define

$$\langle a, b\rangle = a^*b \qquad (a, b \in I).$$

In particular any C^*-algebra is a Hilbert module over itself. On the other hand any Hilbert module over the complex field $\mathbb{C}$ is a Hilbert space. Thus Hilbert C^*-modules generalize both C^*-algebras and Hilbert spaces.

As a convention, throughout the present paper we assume that A is an arbitrary C^*-algebra and E, F are Hilbert A-modules. Since we deal with bounded and unbounded operators at the same time, we denote bounded operators by capital letters and unbounded operators by lower case letters. We use the denotations $Dom(.), Ker(.)$ and $Ran(.)$ for domain, kernel and range of operators, respectively.

We denote the set of all bounded A-linear maps $T : E \to F$ for which there is a map $T^* : F \to E$ such that the equality $\langle Tx, y\rangle = \langle x, T^*y\rangle$ holds for any $x \in E, y \in F$ by $B(E, F)$. The operator T^* is called the adjoint operator of T. It follows that T (and also T^*) is a bounded, linear, A-module map in the sense of $T(xa) = (Tx)a$ for any $x \in E, a \in A$. We abbreviate $B(E, E)$, to $B(E)$.

The concept of Moore-Penrose inverse, first defined by E.H. Moore in the framework of finite matrix, in decade $1910 - 1920$ [10] (see also [11]) ;called by him the "general reciprocal". Then this concept more or less was forgotten and then was rediscovered by R. Penrose in an algebraic form in 1955 [13]. The equivalence of the two definitions was pointed out by Rado [14]. Definition of Moore-Penrose inverse for linear operators between Hilbert spaces and Hilbert C^*-modules is similar:

Definition 1.1. Let $T \in B(E, F)$. The Moore-Penrose inverse $T^\dagger$ of T (if exists) is an element S of $B(F, E)$ which satisfies:

$$TST = T, STS = S, (TS)^* = TS, (ST)^* = ST.$$

Xu and Sheng ([17] Theorem 2.2) have shown that bounded adjointable A-linear operator T between two Hilbert A-modules admits a bounded Moore-Penrose inverse if and only if the operator T has closed range. In which case, $T^\dagger$ exists uniquely with $T^\dagger TT^* = T^*$ and $T^\dagger = 0$ on $Rant(T)^\perp$.

Also we have an explicit representation of Moore-Penrose inverse in $B(E,F)$. If $T \in B(E,F)$ has Moore-Penrose inverse $T^\dagger \in B(F,E)$ then we have

$$T^\dagger = \lim_{\alpha\to 0^+}(\alpha 1 + T^*T)^{-1}T^* = \lim_{\alpha\to 0^+} T^*(\alpha 1 + TT^*)^{-1}.$$

If we define $\mid x \mid = \langle x,x\rangle^{\frac{1}{2}}$, then $\mid . \mid$ is called A-valued "norm". This is not actually a norm, since in general the triangle inequality $\mid x+y \mid \leq \mid x \mid + \mid y \mid$ is not satisfied (a simple example presented in [5]).

Kolarec[7] proved that if $x,y \in E$ and $|x|,|y| \in Z(A)$-the center of A- then

$$|x+y| \leq |x| + |y|.$$

Also Jiang [6] showed that A-valued triangle inequality is satisfied if and only if $\overline{\langle E,E\rangle}$ is commutative, where

$$\langle E,E\rangle = span\{\langle x,y\rangle : x,y \in E\}.$$

In this paper using A-valued norm, we define A-valued metric projection, after some useful results we characterize Moore-penrose inverse of a densely defined operator by A-valued metric projection. Note that in our approach we do not use the triangle inequality.

2. PRELIMINARIES

In this section we recall some definitions and basic facts about regular operators on Hilbert A-modules. More details and properties can be found in ([9] chapters 9 and 10) .

We will use the notation $t : Dom(t) \subseteq E \to F$ to indicate that t is an A-linear operator whose $Dom(t)$ is a dense submodule of E and whose range is in F. Given $t,s : Dom(t), Dom(s) \subseteq E \to F$, we write $s \subseteq t$ if $Dom(s) \subseteq Dom(t)$ and $s(x) = t(x)$ for all $x \in Dom(s)$. A densely defined operator $t : Dom(t) \subseteq E \to F$ is called closed if its graph $G(t) = \{(x,t(x)) : x \in Dom(t)\}$ is a closed submodule of the Hilbert A-module $E \oplus F$. If t is closable, the operator $s : Dom(s) \subseteq E \to F$ with the property $G(s) = \overline{G(t)}$ is called the closure of t denoted by $s = \bar{t}$. The operator t is the smallest closed operator that contains t.

A densely defined operator $t : Dom(t) \subseteq E \to F$ is called adjointable if it possesses a densely defined map $t^* : Dom(t^*) \subseteq F \to E$ with the domain
$Dom(t^*) = \{y \in F :$ there exists $z \in E$ such that $\langle t(x),y\rangle_F = \langle x,z\rangle_E$ for any $x \in Dom(t)\}$. which satisfies the property $\langle t(x),y\rangle_F = \langle x,t^*(y)\rangle_E$, for any $x \in Dom(t), y \in Dom(t^*)$. This property implies that t^* is a closed A-linear map.

The composition of two densely defined operators t,s is the unbounded operator ts with $Dom(ts) = \{x \in Dom(s) : s(x) \in Dom(t)\}$ given by $(ts)(x) = t(s(x))$ for all $x \in Dom(ts)$. The operator ts is not necessarily densely defined. Suppose two densely defined operators t,s are adjointable, then $s^*t^* \subseteq (ts)^*$. If T is a bounded adjointable operator, then $s^*T^* = (Ts)^*$.

A densely defined closed A-linear map $t : Dom(t) \subseteq E \to F$ is called regular if it is adjointable and the operator $1+t^*t$ has a dense range. We denote the set of all regular operators from E to F by $R(E,F)$. A criterion of regularity via the graph of densely defined operators has been given in [3]. In fact a densely defined operator t with a densely defined adjoint operator is regular if and only if its graph is orthogonally complemented in $E \oplus F$. If t is regular then t^* is regular and $t = t^{**}$, moreover t^*t is regular and selfadjoint (cf. [9], Corollaries 9.4, 9.6 and Proposition 9.9).

A regular operator t has closed range if and only if its adjoint operator t^* has closed range, and then for $|t| := |(t^*t)|^{\frac{1}{2}}$ the orthogonal sum decompositions $E = Ker(t) \oplus \overline{Ran(t^*)} = Ker(|t|) \oplus \overline{Ran(|t|)}, F = Ker(t^*) \oplus \overline{Ran(t)} = Ker(|t^*|) \oplus \overline{Ran(|t^*|)}$ exist, (cf. Proposition 1.2 of [3] and Result 7.19 of [8]).

The concept of Moore-Penrose inverses of unbounded regular operators, is defined in [4]

Definition 2.1. Let $t \in R(E,F)$ be a regular operator between two Hilbert A-modules E, F over some fixed C^*-algebra A. A regular operator $s \in R(F,E)$ is called the Moore-Penrose inverse of t if

$$tst = t, \quad sts = s, \quad (ts)^* = \overline{ts}, \quad (st)^* = \overline{st}.$$

If a regular operator t has a generalized inverse s, then the above definition implies that $Ran(t) \subseteq Dom(s)$ and $Ran(s) \subseteq Dom(t)$. A fundamental existence theorem for Moore-Penrose inverse is given in [4] as follows:

Theorem 2.2. *If $t \in R(E,F)$, then the following conditions are equivalent:*

(i) *t and t^* have unique Moore-Penrose inverses which are adjoint to each other, $t^\dagger$ and $t^{\dagger *}$.*
(ii) *$E = Ker(t) \oplus \overline{Ran(t^*)}$ and $F = Ker(t^*) \oplus \overline{Ran(t)}$.*

In this situation, $\overline{t^* t^{\dagger *}}$ and $\overline{tt^\dagger}$ are the projections onto $\overline{Ran(t^*)} = \overline{Ran(t^*t)}$ and $\overline{Ran(t)}$, respectively.

Groetsch's representation for the Moore-Penrose inverse of unbounded regular operator t between Hilbert C^*-modules is given in [15]:

Theorem 2.3. *Suppose $t \in R(E,F)$ is a regular operator and t and t^* possess the Moore-Penrose inverses $t^\dagger$ and $t^{\dagger *}$. Then*

(i) $t^\dagger = \lim_{w \to 0^+} t^*(w1 + tt^*)^{-1} = \lim_{w \to 0^+} \overline{(w1 + t^*t)^{-1} t^*}$ *on* $Dom(t^\dagger)$.
(ii) $t^{\dagger *} = \lim_{w \to 0^+} t(w1 + t^*t)^{-1} = \lim_{w \to 0^+} \overline{(w1 + tt^*)^{-1} t}$ *on* $Dom(t^{\dagger *})$.

The Gram operator of t is defined to be the operator t^*t and studied in [12]:

Theorem 2.4. *Suppose $t \in R(E,F)$, has closed range and $Ran(t) \subseteq Dom(t^*)$. Then*

(i) $t^\dagger = (t^*t)^\dagger t^* = t^*(tt^*)^\dagger$,
(ii) $(t^*t)^\dagger = t^\dagger t^{\dagger *}$.

Note, that bounded A-linear operators may admit generalized inverses in the set of regular operators even if they do not admit any bounded generalized inverse operator. For examples, consider contractive operators on Hilbert spaces with dense, but non-closed range.

3. MAIN PART

In [2] using A-valued norm we defined A-valued metric projection of closed convex subset $K \subset E$ as $P_K^A : E \longrightarrow K$,

$$P_K^A(x) = \{y_0 \in K : |\, x - y_0 \,|^2 = \inf_{y \in K} |\, x - y \,|^2\}.$$

We proved that $P_K^A(x)$ has at most one element. Also we presented a general characterization of A-valued metric projection as a C^*-valued variational inequality.

Proposition 3.1. *Let $(E, \langle ., . \rangle)$ be a Hilbert C^*-module and $K \subseteq E$ is a closed, convex and $x \in E$. The following statements are equivalent:*

(i) $p \in K$ *and* $Re\, \langle x - p \,,\, p - y \rangle \geq 0 \qquad (y \in K)$,
(ii) $p = P_K^A(x)$.

Moreover if K is closed submodule then $p = P_K^A(x)$ if and only if $x - p \in K^\perp$.

Above proposition shows that for a submodule V of E, $P_V^A(x)$ exists for every $x \in E$ if and only if V is complemented submodule. In fact behavior of C^*-valued metric projection operator on a closed complemented submodule is similar to metric projection operator on a closed subspace of a Hilbert space. We collect some basic facts about C^*-valued metric projections onto closed complemented submodules:

Corollary 3.2. *Let V be a closed complemented submodule of E and let $x \in E$. Then the following hold:*

(i) $P_V^A(x)$ *exists and is characterized by* $P_V^A(x) \in V$ *and* $x - P_V^A \in V^\perp$ *.*
(ii) $|P_V^A(x)|^2 = \langle P_V^A(x), x\rangle$.
(iii) $P_{V^\perp}^A = Id - P_V^A$.
(iv) P_V^A *is selfadjoint and thus is in* $B(E)$*,* $\|P_V^A\| = 1$ *if* $V \neq 0$*, and* $\|P_V^A\| = 0$ *if* $V = 0$.
(v) $|x|^2 = |P_V^A(x)|^2 + |P_{V^\perp}^A(x)|^2$.

In this paper we describe Moore-Penrose inverse of a regular operator by C^*-valued metric projection.

Lemma 3.3. *Let K be a nonempty closed convex subset of E, and let x and y be in E. Then* $P_{y+K}^A x = y + P_K^A(x-y)$.

Proof.

$$\begin{aligned} Re\langle x - y - P_K^A(x-y), y + P_K^A(x-y) - (y+z)\rangle &= Re\langle x - y - P_K^A(x-y), P_K^A(x-y) - z\rangle \\ &\geq 0. \end{aligned}$$

and Proposition 3.1 completes the proof. ∎

The next lemma is essential for our main theorem.

Lemma 3.4. *Let $t \in R(E,F)$, $y \in Dom(t^*)$ and $Ran(t) \subseteq Dom(t^*)$ then for every $x \in Dom(t^*t)$,*

$$t(x) = P_{\overline{Ran(t)}}^A(y) \Leftrightarrow t^*t(x) = t^*(y).$$

Proof.

$$\begin{aligned} t(x) = P_{\overline{Ran(t)}}^A(y) &\Leftrightarrow \langle r, t(x) - y\rangle = 0 \ (\forall r \in \overline{Ran(t)}) \\ &\Leftrightarrow \langle t(z), tx - y\rangle = 0 \ (\forall z \in Dom(t)) \\ &\Leftrightarrow \langle z, t^*(t(x) - y)\rangle = 0 \ (\forall z \in \overline{Dom(t)} = E) \\ &\Leftrightarrow t^*t(x) = t^*(y). \end{aligned}$$

∎

The next theorem, which is the main theorem of this paper, shows that if $t^\dagger \in R(E,F)$ is the Moore-Penrose inverse of $t \in R(E,F)$ and t^* has Moore-Penrose inverse, which is adjoint of $t^\dagger$ then for every $y \in F$, $t^\dagger(y)$ can be characterized by $P_{G_y}^A(0)$ for a special affine subspace G_y of E (cf. Theorem 2.2).

Theorem 3.5. *Suppose that $t \in R(E,F)$, $Ran(t) \subseteq Dom(t^*)$ and $Ran(t)$ and $Ran(t^*)$ are both orthogonally complemented in F and E respectively. Fix $y \in Dom(t^*)$. Set $G_y = \{x \in Dom(t^*t) : t^*t(x) = t^*(y)\}$. Then $t^\dagger(y) = P_{G_y}^A(0)$.*

Proof. Note that $P_{G_y}^A(0)$ is nonempty (in fact singelton) for every $y \in Dom(t^*)$. In fact the Lemma 3.4 shows that G_y is an affine subspace, but unlike Hilbert spaces in Hilbert C^*-modules, C^*-valued metric projection on affine subspaces may be empty. But for every $z \in G_y$ we have

$$\begin{aligned} G_y &= \{x \in Dom(t^*t) : t^*t(x) = t^*t(z)\} \\ &= \{x \in Dom(t^*t) : x - z \in Ker(t^*t)\} = z + Ker(t^*t) = z + Ker(t). \end{aligned}$$

So

$$G_y = z + Ker(t).$$

On the other hand by Lemma 3.3 we have

$$P^A_{G_y}(0) = P^A_{z+Ker(t)}(0) = z + P^A_{Ker(t)}(-z).$$

Now since $Ker(t)$ is complemented submodule so $P^A_{Ker(t)}$ is a single valued map. We show that $s(y) = P^A_{G_y}(0)$ satisfies the conditions of definition 2.1. First we prove some auxiliary relations:

(i) $G_y \cap Ker(t)^{\perp} = \{s(y)\}$:
since $G_y = z + Ker(t)(\forall z \in G_y)$, using Proposition 3.1 we have

$$\begin{aligned} z = s(y) &\Leftrightarrow z = P^A_{G_y}(0) \\ &\Leftrightarrow Re\langle x - z, z\rangle \geq 0 \ (\forall x \in G_y) \\ &\Leftrightarrow Re\langle x - z, z\rangle \geq 0 \ (\forall x \in z + Ker(t)) \\ &\Leftrightarrow z \in Ker(t)^{\perp}. \end{aligned}$$

(ii) $P^A_{\overline{Ran(t)}} = \overline{ts}$:
Fix $y \in Dom(s)$. Then $s(y) \in G_y = \{x \in Dom(t^*t) : t(x) = P^A_{\overline{Ran(t)}}(y)\}$. Which implies that $ts(y) = P^A_{\overline{Ran(t)}}(y)$ and continuity implies that $\overline{ts} = P^A_{\overline{Ran(t)}}$.

(iii) $P^A_{\overline{Ran(t^*)}} = \overline{st}$:
By i) and ii) we have

$$tst \subseteq P^A_{\overline{Ran(t)}}t = t = t(P^A_{Ker(t)} + P^A_{\overline{Ran(t^*)}}) = tP^A_{\overline{Ran(t^*)}}.$$

Hence $Ran(st - P^A_{\overline{Ran(t^*)}}) \subseteq Ker(t)$. On the other hand $Ran(s) \subseteq Ker(t)^{\perp} = \overline{Ran(t^*)}$, so

$$Ran(st - P^A_{\overline{Ran(t^*)}}) \subseteq \overline{Ran(t^*)} = Ker(t)^{\perp}.$$

altogether,

$$Ran(st - P^A_{\overline{Ran(t^*)}}) \subseteq Ker(t) \cap Ker(t)^{\perp} = \{0\}.$$

Which implies that $\overline{st} = P^A_{\overline{Ran(t^*)}}$.

(iv) $Ker(t^*) = Ker(s)$:
If $y \in Ker(t^*)$ then since $s(y) \in G_y$ so $s(y) \in Dom(t^*t) \subseteq Dom(t)$. Also $ts(y) = P^A_{\overline{Ran(t)}}(y) = 0$, which means $s(y) \in Ker(t)$. On the other hand by (i) $s(y) \in Ker(t)^{\perp}$. So $s(y) = 0$
If $y \in Ker(s)$ then $ts(y) = 0$, So $y \in Ker(ts) \subseteq Ker(P^A_{\overline{Ran(t)}}) = Ker(t^*)$.

Using above results,

$$(ts)^* = \overline{(ts)^*} = (\overline{ts})^* = (P^A_{\overline{Ran(t)}})^* = \overline{ts}.$$

similarly $(st)^* = \overline{st}$. Also by (ii) the operator ts acts on $Ran(t)$ as the identity operator so $tst = t$ and similarly $sts = s$ by (iii). It remains to prove that $s \in R(F, E)$. Consider the isometry $U \in B(E \oplus F, F \oplus E)$ by $U(x, y) = (y, x)$, then by Proposition 9.3 of [9] we have $F \oplus E = U(G(t)) \oplus G(-t^*)$ and so

$$\begin{aligned} F \oplus E = \{(t(x), x) : x \in Dom(t) \cap Ker(t)^{\perp}\} \oplus \{(0, x) : x \in Ker(t)\} \\ \oplus \{(y, -t^*(y)) : y \in Dom(t^*) \cap \overline{Ran(t)}\} \oplus \{(y, 0) : y \in Ker(t^*)\} \end{aligned}$$

For any $x \in Dom(t) \cap Ker(t)^{\perp}$, let $y = t(x)$ then by (i) we have $x = s(y)$. So

$$\{(t(x), x) : x \in Dom(t) \cap Ker(t)^{\perp}\} = \{(y, s(y)) : y \in \overline{Ran(t)}\}.$$

Using (iv) we have

$$\{(y,0) : y \in Ker(t^*)\} = \{(y, s(y)) : y \in Ker(t^*)\}.$$

By the same argument on $\tilde{s} : Dom(t) \subseteq E \to F,\ \tilde{s} = P^A_{C_x}(0)$ where $C_x = \{x \in Dom(tt^*) : tt^*(x) = t(y)\}$ we have

$$\{(y, -t^*(y)) : y \in Dom(t^*) \cap \overline{Ran(t)}\} \oplus \{(0,x) : x \in Ker(t)\}$$
$$= \{(-\tilde{s}(y), y) : y \in Dom(\tilde{s})\}$$

So $F \oplus E = G(s) \oplus VG(\tilde{s})$, where $V \in B(E \oplus F, F \oplus E)$ is an isometry defined by $V(x,y) = (y, -x)$. Now Corollary 3.2 of [9] implies that $s \in R(F,E)$. In particular $s^* = \tilde{s}$. ∎

Since $B(E,F)$ is a subset of $R(E,F)$, we have the following corollary.

Corollary 3.6. *Suppose $T \in B(E,F)$ has closed range. For any $y \in F$ set $G_y = \{x \in E : T^*T(x) = T^*(y)\}$. Then $T^\dagger(y) = P^A_{G_y}(0)$.*

There is a large literature dealing with the Moore-Penrose inverse of an element of a C^*-algebra (for example see [1] and references in it). Here we have the following version of above theorem for C^*-algebras:

Corollary 3.7. *Let A be unital C^*-algebra and a be an element of A which aA is closed. Then $a^\dagger$ is a unique element of A which satisfies:*

$$|a^\dagger|^2 = \inf_{b \in G} |b|^2,$$

where $G = \{b \in A : a^*ab = a^*\}$.

Proof. Let E be A as a Hilbert A-module. We have $B(E) \simeq A$ given by $T \leftrightarrow T(1)$; cf.[9, Theorem 6]. Let $T(x) = ax$. Since aA is closed $Ran(T)$ is closed. Now Corollary 3.6 completes the proof.

∎

Next theorem is some characterizations of Moore- Penrose inverse.

Theorem 3.8. *Let $t \in R(E,F)$, $Ran(t) \subseteq Dom(t^*)$ and $Ran(t)$ and $Ran(t^*)$ are orthogonally complemented in F and E respectively. Let $\tilde{t} \in R(F,E)$, $Ran(t) \subseteq Dom(\tilde{t})$ and $Ran(\tilde{t}) \subseteq Dom(t)$. Then the following conditions are equivalent:*

(i) $\tilde{t} = t^\dagger$,

(ii) $Dom(\tilde{t}) = Dom(t^*)$, $\overline{t\tilde{t}} = P^A_{\overline{Ran(t)}}$ *and* $\overline{\tilde{t}t} = P^A_{\overline{Ran(t^*)}}$.

(iii) $Dom(\tilde{t}) = Dom(t^*)$, $\tilde{t}t|_{\overline{Ran(t^*)}} = Id$, *and* $\tilde{t}|_{Ran(t)^\perp} = 0$.

Proof. $(i) \Rightarrow (ii)$ is a consequence of Theorem 3.5 (note that by (iv) in proof of Theorem 3.5 $Ran(\tilde{t}) = Ran(t^*)$) .
$(ii) \Rightarrow (iii)$ is straightforward.
$(iii) \Rightarrow (i)$ Take $y \in Dom(\tilde{t})$. Set $y_1 = P^A_{\overline{Ran(t)}}(y)$ and $y_2 = P^A_{\overline{Ran(t)}^\perp}(y)$. Then there exists $x_1 \in Ker(t)^\perp$ such that $y_1 = t(x_1)$. Hence

$$\tilde{t}(y) = \tilde{t}(y_1) + \tilde{t}(y_2) = \tilde{t}(y_1) = \tilde{t}t(x_1) = x_1. \tag{3.-10}$$

So $\tilde{t}(y) \in Ker(t)^\perp$. Also it follows that $t\tilde{t}(y) = t(x_1) = y_1 = P^A_{\overline{Ran(t)}}(y)$ and Lemma 3.4 yields $t^*t\tilde{t}(y) = t^*(y)$. Now (i) in proof of the Theorem 3.5 implies that $\tilde{t}(y) = t^\dagger(y)$. ∎

Corollary 3.9. *Let $T \in B(E,F)$ and $\tilde{T} \in B(F,E)$ both have closed range.Then the following are equivalent:*

(i) $\tilde{T} = T^\dagger$,
(ii) $T\tilde{T} = P^A_{Ran(T)}$ *and* $\tilde{T}T = P^A_{Ran(T^*)}$.
(iii) $\tilde{T}T|_{Ran(T^*)} = Id$, *and* $\tilde{T}|_{Ran(T)^\perp} = 0$.

Theorem 3.10. *Suppose $T \in B(E,F)$ has closed range. Then $T^\dagger = (T|_{Ker(T)^\perp})^{-1} P^A_{Ran(T)}$.*

Proof. We first show that the expression on the right side is well-defined, i.e., the restriction of T to $Ker(T)^\perp$ has a bounded inverse on $Ran(T)$. Let $T_0 := T|_{Ker(T)^\perp}$. Then T_0 is a bounded linear operator from the Hilbert A-module $Ker(T)^\perp$ to the Hilbert A-module $Ran(T)$. Moreover, T_0 is bijective. To see that T_0 is injective, suppose $T_0(x_1) = T_0(x_2)$ for some $x_i \in Ker(T)^\perp$. Then $x_1 - x_2 \in Ker(T)^\perp$ and $T(x_1 - x_2) = T_0(x_1 - x_2) = 0$, so $x_l - x_2 \in Ker(T)^\perp \cap Ker(T) = \{0\}$, or $x_1 = x_2$. To see that T_0 is surjective, let $y \in Ran(T)$. Then there exists $x \in E$ such that $y = T(x)$. So

$$y = T(P^A_{Ker(T)}(x) + P^A_{Ker(T)^\perp}(x)) = TP^A_{Ker(T)^\perp}(x) = T_0 P^A_{Ker(T)^\perp}(x) \in Ran(T_0).$$

Thus T_0 is surjective. By the bounded inverse theorem, we see that T_0 has a bounded inverse. Also $T \in B(E,F)$ implies that

$$T_0^{-1} \in B(Ran(T), Ker(T^\perp)).$$

Let $S := T_0^{-1} P^A_{Ran(T)}$. We will show that $S = T^\dagger$. To this end, first note that $S \in B(F, Ker(T)^\perp) \subseteq B(F,E)$. Fix any $y \in F$. Since the range of S is contained in $Ker(T)^\perp$, we have $S(y) \in Ker(T)^\perp$. Also, $TS(y) = T_0 S(y) = T_0 T_0^{-1} P^A_{Ran(T)}(y) = P^A_{Ran(T)}(y)$. Now (i) in the proof of Theorem 3.5 implies that $S(y) = T^\dagger(y)$. Since $y \in F$ was arbitrary, $S = T^\dagger$. ∎

We close this paper with a corollary about weighted Moore-Penrose inverse of an operator. For primary definition and results about weighted Moore-Penrose inverse of an operator on a Hilbert C^*-module see [16].

An element M of $B(F)$ is said to be positive definite, if M is positive and invertible in $B(F)$. For any positive definite $M \in B(F)$ we use the notation F_M to denote the Hilbert A-module with the inner-product given by

$$\langle x, y \rangle_M = \langle x, My \rangle, \quad x, y \in E,$$

and call F_M the weighted space (with respect to M).

For any positive definite element N of $B(E)$, $T \in B(E,F)$, if we regard T as an element of $B(E_N, H_M)$, then $T^\sharp = N^{-1}T^*M$. where $T^\sharp \in B(F_M, E_N)$ is the adjoint operator of $T \in B(E_N, F_M)$.

Definition 3.11. Let $T \in B(E,F)$ be arbitrary, and let $M \in B(F)$ and $N \in B(E)$ be two positive definite operators. The weighted Moore-Penrose inverse $T^\dagger_{MN}$ (if it exists) is the element $X \in B(F,E)$, which satisiňĄes

$$TXT = T, \; XTX = X, \; (MTX)^* = MTX, \; (NXT)^* = NXT.$$

If $M = I_F$ and $N = I_E$, then $T^\dagger_{MN} = T^\dagger$.

We know by (Theorem 1.3 [16]) that $T^\dagger_{MN}$ exists if and only if A has closed range.

The following corollary is an immediate consequence of Theorem 3.5. In fact if we put

$$G_y = \{x \in F : T^\sharp T(x) = T^\sharp(Y)\} + \{x \in F : N^{-1}T^*MT(x) = N^{-1}T^*M(y)\}.$$

Then we ahve

$$G_y\{x \in F : T^*MT(x) = T^*M(y)\}.$$

Corollary 3.12. *For any $T \in B(E,F)$ we have $T^\dagger_{MN} = P^A_{G_y}(0)$ which $G_y = \{x \in F : T^*MT(x) = T^*M(y)\}$.*

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