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ON A CLASS OF MEROMORPHIC FUNCTIONS OF JANOWSKI TYPE RELATED WITH A CONVOLUTION OPERATOR

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ABSTRACT. In this paper, we have introduced and studied new operator $Q^k_{\lambda,m,\gamma}$ by the Hadamard product (or convolution) of two linear operators D^k_λ and $I_{m,\gamma}$, then using this operator to study and investigate a new subclass of meromorphic functions of Janowski type,giving the coefficient bounds, a sufficient condition for a function to belong to the considered class and also a convolution property. The results presented provide generalizations of results given in earlier works.

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1. INTRODUCTION

Let $A = A(\Delta)$ denote the class of all analytic functions in the open unit disc $\Delta := \{z \in C : |z| < 1\}$.

Consider

$$\Omega = \{w \in A : w(0) = 0 \; and \; |w(z)| < 1\}(z \in \Delta), \tag{1.1}$$

the class of Schwarz functions.
For $0 \leq \alpha < 1$,let

$$P(\alpha) = \{p \in A : p(0) = 1 \; and \; Re(p(z)) > \alpha\}(z \in \Delta). \tag{1.2}$$

Note that $P = P(0)$ is the well-known Caratheodory class of functions. The classes of Schwarz and Caratheodory functions play an extremely important role in the theory of analytic functions and have been studied by many authors. It is easy to see that

$$p \in P(\alpha) \; if \; and \; only \; if \; \frac{p(z) - \alpha}{1 - \alpha} \in P(\alpha). \tag{1.3}$$

Any two analytic functions f and g, the function f is subordinate to the function g, are written $f \prec g$, provided there is an analytic function $w(z)$ defined on Δ with $w(0) = 0$ and $\mid w(z) \mid < 1$ such that $f(z) = g(w(z))$. In particular, if the function g is univalent in Δ, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Let $\sum$ denote the class of all meromorphic univalent functions having the form:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n. \tag{1.4}$$

which are analytic in the punctured unit disk $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \Delta - \{0\}$. Denote by$\sum^+$ the subclass of $\sum$ consisting of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \; (a_n \geq 0), z \in \Delta^*. \tag{1.5}$$

A function f is meromorphically multivalent starlike of order $\alpha(0 \leq \alpha < 1)$ see [1] if

$$-Re(\frac{zf'(z)}{f(z)}) > \alpha, z \in \Delta^*.$$

The class of all such functions is denoted $\sum^*(\alpha)$. If f is given by (1.4) and g is given by

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n,$$

we define the Hadamard product or convolution of f and g by

$$f(z) * g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n = g(z) * f(z) \; (z \in \Delta^*).$$

Now, for functions $f(z) \in \sum$ in the form (1.4) we define the linear operatorD_λ^k studied by A. S. Juma and Fateh S. Aziz [4] for $p = 1, D_\lambda^k : \sum \rightarrow \sum$ as follows

$$D_\lambda^0 f(z) = f(z)$$

and

$$D_0^k f(z) = f(z) \; (0 \leq \lambda < 1; k \in N_0 = N \cup 0; z \in \Delta^*)$$
$$D_\lambda^1 f(z) = D_\lambda = (1 + \lambda) f(z) + \lambda n f(z),$$

or

$$D_\lambda = \frac{1}{z} + \sum_{n=0}^{\infty}(1+\lambda+\lambda n)a_n z^n,$$

therefore,

$$D_\lambda^k f(z) = \frac{1}{z} + \sum_{n=0}^{\infty}(1+\lambda+\lambda n)^k a_n z^n, \tag{1.6}$$

$$D_\lambda^k f(z) = \frac{1}{z} + \sum_{n=0}^{\infty}\mu(\lambda,n)a_n z^n, \tag{1.7}$$

where

$$\mu(\lambda,n) = (1+\lambda+\lambda n)^k.$$

By simple calculations and using (1.7), we can easily verify that

$$\lambda z[D_\lambda^k f(z)]' = D_\lambda^{k+1} f(z) - (1+\lambda)D_\lambda^k f(z). \tag{1.8}$$

In [17] Yuan, et. all defined an operator $I_{m,\gamma} : \sum \rightarrow \sum$ as follows:

$$I_{m,\gamma} f(z) = f_{m,\gamma}(z) * f(z), \tag{1.9}$$

where

$$f_{m,\gamma}(z) * \frac{1}{z(1-z)^{m+1}} = \frac{1}{z(1-z)^{\gamma}}, (m > -1, z \in \Delta^*). \tag{1.10}$$

From(1.9) and (1.10), we obtain

$$I_{m,\gamma} f(z) = \frac{1}{z} + \sum_{n=0}^{\infty}\frac{(\gamma)_{n+1}}{(m+1)_{n+1}}a_n z^n, \tag{1.11}$$

where $(a)_n$ is the Pochhammer symbol defined as:

$$(a)_0 = 1, (a)_n = a(a+1)(a+2).....(a+n-1), (n \in N).$$

For functions belonging to $\sum$ we define the linear operator $Q^k_{\lambda,m,\gamma}$ as the Hadamard product (or convolution) of the operators D^k_λ and $I_{m,\gamma}$ as follows:

$$Q^k_{\lambda,m,\gamma} f(z) = I_{m,\gamma} f(z) * D^k_\lambda f(z)$$

$$= \frac{1}{z} + \sum_{n=0}^{\infty}\frac{(\gamma)_{n+1}}{(m+1)_{n+1}}\mu(\lambda,n)a_n z^n,$$

or

$$Q^k_{\lambda,m,\gamma} f(z) = \frac{1}{z} + \sum_{n=0}^{\infty}\rho(\lambda,m,\gamma,n)a_n z^n, \tag{1.12}$$

where

$$\rho(\lambda,m,\gamma,n) = \frac{(\gamma)_{n+1}}{(m+1)_{n+1}}\mu(\lambda,n).$$

Now,f is meromorphic close-to-convex function if there exist a function $g \in \sum^*$ such that the following condition holds:

$$-Re(\frac{zf'(z)}{g(z)}) > 0, z \in \Delta^*.$$

Gao and Zhou [2] introduced and studied the class L_s consisting of all the functions h in A such that

$$Re(\frac{z^2h'(z)}{g(z)g(-z)}) < 0, z \in \Delta^*,$$

where g belongs to the class $N^*(1/2)$ of analytic starlike functions of order $\frac{1}{2}$ (i.e. the class of functions $g \in A$ which satisfy the inequality$Re(zg'(z)/g(z)) > 1/2, z \in \Delta$.) For several recently investigated classes of analytic functions related to the above mentioned class L_s, we refer to [3], [5], [12], [14] or [16]. Wang et al. [15] considered the class ML of meromorphic functions f from $\sum$ which satisfy the inequality

$$Re(\frac{f'(z)}{g(z)g(-z)}) > 0, z \in \Delta^*,$$

where $g \in \sum^*(1/2)$: More recently, Sim and Kwon [10] investigated the class $\sum(A, B)$ of functions $f \in \sum$ with the property

$$(\frac{f'(z)}{g(z)g(-z)}) \prec \frac{1 + Az}{1 + Bz},$$

with $(-1 \leq B < A \leq 1)$ and $g \in \sum^*(1/2)$, and in a similar manner, Soni and Kant [11] introduced and discussed $ML(s, A, B)$ consisting of functions $f \in \sum$ for which the following subordination holds:

$$(\frac{-f'(z)}{sg(z)g(-z)}) \prec \frac{1 + Az}{1 + Bz},$$

where $0 < |s| \leq 1$ and A, B and g are defined as above. Motivated by the aforementioned works, we introduce and investigate the following subclass of meromorphic functions:

Definition 1.1. For fixed parameters A, $B(-1 \leq B < A \leq 1)$and $0 \leq \alpha \leq 1$, we say that a function $f \in \sum$ of the form(1.4) is in the class $D^k_\lambda[A, B; \alpha]$, if there exists $g \in \sum^*(1/2)$, such that the following subordination is satisfied:

$$\frac{(1-2\alpha)(Q^k_{\lambda,m,\gamma}f(z))' - \alpha z(Q^k_{\lambda,m,\gamma}f(z))''}{g(z)g(-z)} \prec \frac{1 + Az}{1 + Bz}. \tag{1.13}$$

The class $D^k_\lambda[A, B; \alpha]$, provides a generalization of the classes studied by Wang, Sun and Xu [15] (the case $\alpha = 0; A = -1 and\ B = 1$) and Sim and Kwon [10] (the case $\alpha = 0$).

Recently Shi, Yi and Wang [13] obtained results on the class $D^k_\lambda[A, B; \alpha]$, when $\alpha \leq 0$.

Remark 1.1. If $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, then $D^k_\lambda[A_1, B_1; \alpha] \subseteq D^k_\lambda[A_2, B_2; \alpha]$. To prove this, let $f \in D^k_\lambda[A_1, B_1; \alpha]$. Then

$$\frac{(1-2\alpha)(Q^k_{\lambda,m,\gamma}f(z))' - \alpha z(Q^k_{\lambda,m,\gamma}f(z))''}{g(z)g(-z)} \prec \frac{1 + A_1z}{1 + B_1z}.$$

But since $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, the following subordination is true:

$$\frac{1 + A_1z}{1 + B_1z} \prec \frac{1 + A_2z}{1 + B_2z} \tag{1.14}$$

Clearly that, when $-1 < B_2 \leq B_1$, the images of Δ under these two functions are two circles orthogonal on the real axis and also we have that

$$\min_{z\in\partial\Delta} Re\frac{1 + A_2z}{1 + B_2z} = \frac{1 - A_2}{1 - B_2} \leq \min_{z\in\partial\Delta} Re\frac{1 + A_1z}{1 + B_1z} = \frac{1 - A_1}{1 - B_1}$$

$$\leq \max_{z\in\partial\Delta} Re\frac{1+A_1z}{1+B_1z} = \frac{1-A_1}{1-B_1} \leq \max_{z\in\partial\Delta} Re\frac{1+A_2z}{1+B_2z} = \frac{1-A_2}{1-B_2},$$

which shows that the image of Δ under $(1+A_1z)/(1+B_1z)$ is included in the image of Δ under $(1+A_1z)/(1+B_1z)$, and so the subordination(1.14) holds. A similar argument shows the subordination is also true when $-1 = B_1 = B_2$ or $-1 = B_1 < B_2$. It therefore follows that $f \in D_\lambda^k[A_2, B_2; \alpha]$.

2. PRELIMINARY LEMMAS

The following lemmas are needed for proving our results.

Lemma 2.1. [15]
Let $g \in \sum^*(1/2)$. *Then* $-zg(z)g(-z) \in \sum^*$.

Lemma 2.2. [15]
Let $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \in \sum^*(1/2)$.*Then*

$$|B_{2n-1}| \leq \frac{1}{n}\ (n \in N = 1, 2, ...),$$

where

$$B_{2n-1} = 2b_{2n-1} + 2b_1b_{2n-3} - 2b_2b_{2n-4} + ... + (-1)^{n-1}b_{n-2}b_n + (-1)^n b_{n-1}^2. \tag{2.1}$$

Lemma 2.3. [7]

Let $G \in \sum^*$. *Then for* $|z| = r, 0 < r < 1$,*we have*

$$\frac{(1-r)^2}{r} \leq |G(z)| \leq \frac{(1+r)^2}{r}.$$

We need at this moment to give the definition of prestarlike functions. For $\beta < 1$, *the class* $\mathcal{H}(\beta)$ *of prestarlike functions of order* β *consists of all the normalized analytic functions* f *which satisfy the follows*

$$f(z) * \frac{z}{(1-z)^{2-2\beta}} \in N^*(\beta).$$

The class $\mathcal{H}(1)$ *is formed with analytic normalized functions f for which the inequality* $Re(f(z) = z) > 1/2$ *hold true*

Lemma 2.4. [9]
Let $\beta \leq 1, f \in \mathcal{H}(\beta)$ *and* $g \in N^*(\beta)$.*Then*

$$\frac{f * gF}{f * g}(\Delta) \subset \overline{co}(F(\Delta)),$$

where F *is an analytic function in* Δ *and* $\overline{co}(F(\Delta))$ *denotes the closed convex hull of* $F(\Delta)$.

Lemma 2.5. [8]
Let $h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$ *and* $k(z) = 1 + \sum_{n=1}^{\infty} k_n z^n$ *be two analytic functions in* Δ *If* k *is convex and* $h \prec k$, *then*

$$|h_n| \leq |k_1| (n \in N).$$

3. MAIN RESULTS

The following result gives a sufficient condition for a function to belong to the investigated class$D_\lambda^k[A,B;\alpha]$.

Theorem 3.1. *Let* $-1 \le B < A \le 1, 0 \le \alpha \le 1$ *and*

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n.$$

If $f \in \sum$ *given by (1.4) satisfies the condition*

$$\sum_{n=1}^{\infty}[(1+|B|)|1-\alpha-\alpha n||\rho(\lambda,m,\gamma,n)a_n|n + (1+|A|)|B_{2n-1}|] < A-B, \tag{3.1}$$

where the coefficients B_{2n-1} *are given by (2.1), then* $f \in D_\lambda^k[A,B;\alpha]$.

Proof. To prove $f \in D_\lambda^k[A,B;\alpha]$, it suffices to show that

$$|\frac{(1-2\alpha)(Q_{\lambda,m,\gamma}^k f(z))' - \alpha z(Q_{\lambda,m,\gamma}^k f(z))''}{g(z)g(-z)} - 1| <$$

$$|B - A\frac{(1-2\alpha)(Q_{\lambda,m,\gamma}^k f(z))' - \alpha z(Q_{\lambda,m,\gamma}^k f(z))''}{g(z)g(-z)}|. \tag{3.2}$$

Let

$$G(z) = -zg(z)g(-z). \tag{3.3}$$

It is obvious that $G(-z) = -G(z)$ and by Lemma 2.1 we deduce that G is a meromorphic odd starlike function. Therefore, G has the form

$$G(z) = \frac{1}{z} + \sum_{n=1}^{\infty} B_{2n-1} z^{2n-1}, \tag{3.4}$$

where the coefficients B_{2n-1} are given by (2.1).

On taking

$$w = |(1-2\alpha)z(Q_{\lambda,m,\gamma}^k f(z))' - \alpha z^2(Q_{\lambda,m,\gamma}^k f(z))'' +$$
$$G(z)| - |AG(z) + B[(1-2\alpha)(Q_{\lambda,m,\gamma}^k f(z))' - \alpha z(Q_{\lambda,m,\gamma}^k f(z))'']|$$
$$= |\sum_{n=1}^{\infty}(1-\alpha-\alpha n)n\rho(\lambda,m,\gamma,n)a_n z^n + \sum_{n=1}^{\infty} B_{2n-1}z^{2n-1}|$$
$$-|(A-B)\frac{1}{z} + \sum_{n=0}^{\infty} AB_{2n-1}z^{2n-1} + \sum_{n=1}^{\infty} B(1-\alpha-\alpha n)n\rho(\lambda,m,\gamma,n)a_n z^n|,$$

from (3.1) we get the inequalities

$$w \le \sum_{n=1}^{\infty}|(1-\alpha-\alpha n)\rho(\lambda,m,\gamma,n)a_n|n|z|^n + \sum_{n=1}^{\infty}|B_{2n-1}||z|^{2n-1}$$
$$-[(A-B)\frac{1}{z} + \sum_{n=1}^{\infty}|AB_{2n-1}|z|^{2n-1} + \sum_{n=1}^{\infty}|B(1-\alpha-\alpha n)\rho(\lambda,m,\gamma,n)a_n|n|z|^n]$$
$$= -(A-B)\frac{1}{z}\sum_{n=1}^{\infty}(1+|B|)|(1-\alpha-\alpha n)\rho(\lambda,m,\gamma,n)a_n|n|z|^n + \sum_{n=1}^{\infty}(1+|A|)|B_{2n-1}||z|^{2n-1}$$

$$< -(A-B) + \sum_{n=1}^{\infty}(1+|B|)|(1-\alpha-\alpha n)\rho(\lambda,m,\gamma,n)a_n|n + \sum_{n=1}^{\infty}(1+|A|)|B_{2n-1}| \leq 0.$$

Thus, since $w < 0$, relation (3.2) holds true and we conclude that$f \in D_{\lambda}^{k}[A,B;\alpha]$. ∎

We next determine the coefficient estimates for functions in $D_{\lambda}^{k}[A,B;\alpha]$.

Theorem 3.2. *Let $-1 \leq B < A \leq 1, 0 \leq \alpha \leq 1$ and$f \in D_{\lambda}^{k}[A,B;\alpha]$ given by (1.4)*

$$|\rho(\lambda,m,\gamma,1)a_1| \leq 1$$

$$|\rho(\lambda,m,\gamma,2n)a_{2n}| \leq \frac{A-b}{2n|1-(2n+1)\alpha|}(1+\sum_{k=1}^{n-1}\frac{1}{k})\ (n \in N) \tag{3.5}$$

and

$$|\rho(\lambda,m,\gamma,2n+1)a_{2n+1}| \leq \frac{A-b}{(2n+1)|1-(2n+2)\alpha|}(1+\sum_{k=1}^{n}\frac{1}{k})\ (n \in N) \tag{3.6}$$

Proof. Since $f \in D_{\lambda}^{k}[A,B;\alpha]$, we know that

$$-\frac{(1-2\alpha)z(Q_{\lambda,m,\gamma}^{k}f(z))' - \alpha z^2(Q_{\lambda,m,\gamma}^{k}f(z))''}{G(z)} \prec \frac{1+Az}{1+Bz},$$

where $G(z) = -zg(z)g(-z)$ is given by (3.4). Setting

$$\phi(z) = -\frac{(1-2\alpha)z(Q_{\lambda,m,\gamma}^{k}f(z))' - \alpha z^2(Q_{\lambda,m,\gamma}^{k}f(z))''}{G(z)}, \tag{3.7}$$

it follows that $\phi(z) = 1 + d_1z + d_2z^2 + ...$ and $\phi(z) \prec \frac{1+Az}{1+Bz}$. Also, by Lemma 2.5, we remark that

$$|d_n| \leq A - B(n \in N). \tag{3.8}$$

Moreover, equation (3.7) gives

$$(1 + d_1z + d_2z^2 + ...)(\frac{1}{z} + B_1z + B_3z^3 + ...)$$

$$= \frac{1}{z} - (1-2\alpha)\rho(\lambda,m,\gamma,1)a_1z - 2(1-3\alpha)\rho(\lambda,m,\gamma,2)a_2z^2 - ... - 2n[1-(2n+1)\alpha]$$
$$\rho(\lambda,m,\gamma,2n)a_{2n}z^{2n} - (2n+1)[1-(2n+2)\alpha]\rho(\lambda,m,\gamma,2n+1)a_{2n+1}z^{2n+1} - ...,$$

from which we obtain $d_1 = 0$,

$$-2n[1-(2n+1)\alpha]\rho(\lambda,m,\gamma,2n)a_{2n} = d_3B_{2n-3} + ... + d_{2n-1}B_1 + d_{2n+1}(n \in N)$$

and

$$-(2n+1)[1-(2n+2)\alpha]\rho(\lambda,m,\gamma,2n+1)a_{2n+1} = d_2B_{2n-1} + d_4B_{2n-3} + ... + d_{2n}B_1 + d_{2n+2}(n \in N).$$

Moreover, by Lemma 2.2 and from (3.8), it results that

$$2n|1-(2n+1)\alpha||\rho(\lambda,m,\gamma,2n)a_{2n}| \leq (A-B)(\frac{1}{1-n} + ... + \frac{1}{2} + 1 + 1) \tag{3.9}$$

and

$$(2n+1)|1-(2n+2)\alpha||\rho(\lambda,m,\gamma,2n+1)a_{2n+1}| \leq (A-B)(\frac{1}{n} + \frac{1}{1-n} + ... + \frac{1}{2} + 1 + 1) \tag{3.10}$$

The conclusion in (3.5) and (3.6) follows now from (3.9) and (3.10), whereas the estimation $|\rho(\lambda,m,\gamma,1)a_1| \leq 1$ is true for any meromorphic function f univalent inΔ^*. ∎

Theorem 3.3. *If* $-1 \le B < A \le 1, 0 \le \alpha \le 1$ *and* $f \in D^k_\lambda[A, B; \alpha]$ *given by (1.4), then for* $|z| = r, 0 < r < 1$, *the following inequalities hold*

$$\frac{(1-r)^2}{r^2}\frac{1-Ar}{1-Br} \le |(1-2\alpha)(Q^k_{\lambda,m,\gamma}f(z))' - \alpha z(Q^k_{\lambda,m,\gamma}f(z))'| \le \frac{(1-r)^2}{r^2}\frac{1+Ar}{1+Br}. \tag{3.11}$$

Proof. Suppose $f \in D^k_\lambda[A, B; \alpha]$ and let

$$\phi(z) = -\frac{(1-2\alpha)z(Q^k_{\lambda,m,\gamma}f(z))' - \alpha z^2(Q^k_{\lambda,m,\gamma}f(z))''}{G(z)},$$

where G given in (3.4) is a meromorphically starlike function. Since

$$\phi(z) \prec \frac{1+Az}{1+Bz},$$

by the subordination principle we obtain for $|z| = r, 0 < r < 1$, that

$$\frac{1-Ar}{1-Br} \le |\phi(z)| \le \frac{1+Ar}{1+Br}.$$

Making use of Lemma 2.3, we readily obtain the desired inequalities, as asserted in (3.11) ∎

We provide next a convolution property of functions from the class $D^k_\lambda[A, B; \alpha]$ considered.

Theorem 3.4. *Let* $-1 \le B < A \le 1, 0 \le \alpha \le 1, \beta \le 1$ *and* $f \in D^k_\lambda[A, B; \alpha]$ *such that the corresponding function* $g \in \sum^*(1/2)$ *satisfies the condition*

$$-Re(\frac{zg'(z)}{g(z)}) < \frac{3}{2} - \frac{1}{2}\beta \ (z \in \Delta). \tag{3.12}$$

If $\psi \in \sum$ *with* $z^2\psi(z) \in \mathcal{H}(\beta)$, *then* $\psi * f \in D^k_\lambda[A, B; \alpha]$.

Proof. Let $f \in D^k_\lambda[A, B; \alpha]$ and let G and ϕ be given by

$$G(z) = -zg(z)g(-z)\ , \phi(z) = -\frac{(1-2\alpha)z(Q^k_{\lambda,m,\gamma}f(z))' - \alpha z^2(Q^k_{\lambda,m,\gamma}f(z))''}{G(z)}.$$

From the proof of Theorem 3.1, G is an odd meromorphic starlike function. Further, since

$$\frac{zG'(z)}{G(z)} = 1 + \frac{zg'(z)}{g(z)} - \frac{zg'(-z)}{g(-z)},$$

inequality (3.12) implies

$$-Re(\frac{zg'(z)}{g(z)} < 2 - \beta \ (z \in \Delta),$$

then we get $z^2G(z) \in N^*(\beta)$. Define $\nu(z) = (\psi * G)(z)$ and $J(z) = \sqrt(z\nu(z))/z$. Clearly that ν is also an odd and starlike meromorphic function. In order to prove this, let $F(z) = -zG'(z)/G(z)$. Since

$$-\frac{z(\psi * G)'(z)}{(\psi * G)(z)} = \frac{(\psi * -zG')(z)}{(\psi * G)(z)} = \frac{\psi(z) * G(z)F(z)}{(\psi * G)(z)} = \frac{z^2\psi(z) * z^2G(z)F(z)}{z^2\psi(z) * z^2G(z)},$$

with $z^2G(z) \in N^*(\beta)$ and $z^2\psi(z) \in \mathcal{H}(\beta)$, we deduce, by Lemma 2.4, that

$$-\frac{z(\psi * G)'(z)}{(\psi * G)(z)} \in \overline{co}(F(\Delta))$$

Because G is starlike meromorphic, we have $Re(F(z)) > 0$ and so the above relation yields $\psi * G$ is indeed also starlike. As a consequence, $J \in \sum^{*}(1/2)$and $\nu(z) = (-zJ'(z))/J(-z)$ Moreover, we have

$$-\frac{(1-2\alpha)[\psi * Q^{k}_{\lambda,m,\gamma}f]'(z) - \alpha z[\psi * Q^{k}_{\lambda,m,\gamma}f(z)]''}{J(z)J(-z)},$$

$$= \frac{\psi * [(1-2\alpha)z(Q^{k}_{\lambda,m,\gamma}f(z))' - \alpha z^2(Q^{k}_{\lambda,m,\gamma}f)''](z)}{\nu(z)} = \frac{\psi(z) * \phi(z)G(z)}{(\psi * G)(z)}$$

$$= \frac{z^2\psi(z) * \phi(z)z^2G(z)}{z^2\psi(z) * z^2G(z)}.$$

From Lemma 2.4 once again we deduce that

$$\frac{z^2\psi(z) * \phi(z)z^2G(z)}{z^2\psi(z) * z^2G(z)} \in \overline{co}(\phi(\Delta)) \tag{3.13}$$

Since $f \in D^{k}_{\lambda}[A, B; \alpha]$, we have

$$\phi(z) \prec \frac{1 + Az}{1 + Bz}. \tag{3.14}$$

The function $(1 + Az)/(1 + Bz)$ is convex and therefore equations (3.13) and(3.14) yield

$$\frac{z^2\psi(z) * \phi(z)z^2G(z)}{z^2\psi(z) * z^2G(z)} \prec \frac{1 + Az}{1 + Bz},$$

which is equivalent to

$$\frac{(1-2\alpha)[\psi * Q^{k}_{\lambda,m,\gamma}f]'(z) - \alpha z[\psi * Q^{k}_{\lambda,m,\gamma}f]''(z)}{J(z)J(-z)} \prec \frac{1 + Az}{1 + Bz}.$$

Thus $\psi * (Q^{k}_{\lambda,m,\gamma}f) \in D^{k}_{\lambda}[A, B; \alpha]$and so the proof is complete. ∎

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