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SOME GRÜSS TYPE INEQUALITIES IN INNER PRODUCT SPACES

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ABSTRACT. Some inequalities in inner product spaces $(H, \langle \cdot, \cdot \rangle)$ that provide upper bounds for the quantities

$$\left| \langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle \right| \text{ and } \left| \langle x, y \rangle - \frac{1}{2} \left[\langle x, f \rangle \langle f, y \rangle + \langle x, e \rangle \langle e, y \rangle \right] \right|,$$

where $e, f \in H$ with $\|e\| = \|f\| = 1$ and x, y are vectors in H satisfying some appropriate assumptions are given. Applications for discrete and integral inequalities are provided as well.

Key words and phrases: Schwarz's inequality, Triangle inequality, Buzano's inequality, Grüss type inequalities, Integral inequalities.

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1. INTRODUCTION

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\| \|y\| \geq |\langle x, y \rangle| \text{ for any } x, y \in H. \tag{1.1}$$

The equality case holds in (1.1) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that $x = \lambda y$.

In 1985 the author [4] (see also [19]) established the following refinement of (1.1):

$$\|x\| \|y\| \geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \geq |\langle x, y \rangle| \tag{1.2}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Using the triangle inequality for modulus we have

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \geq |\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle|$$

and by (1.2) we get

$$\begin{aligned} \|x\| \|y\| &\geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \\ &\geq 2 |\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle|, \end{aligned}$$

which implies the *Buzano inequality* [2]

$$\frac{1}{2} [\|x\| \|y\| + |\langle x, y \rangle|] \geq |\langle x, e \rangle \langle e, y \rangle| \tag{1.3}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

In [5], the author has proved the following Grüss' type inequality in real or complex inner product spaces.

Theorem 1.1. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and $e \in H$, $\|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0 \textit{ and } \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0 \tag{1.4}$$

hold, then we have the inequality

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \varphi| |\Gamma - \gamma|. \tag{1.5}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

For other Schwarz, Buzano and Grüss related inequalities in inner product spaces, see [1]-[3], [4]-[13], [17]-[20], [22]-[29], and the monographs [14], [15] and [16].

Motivated by the above results, we establish in this paper other upper bounds for the quantities

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \text{ and } \left| \langle x, y \rangle - \frac{1}{2} [\langle x, f \rangle \langle f, y \rangle + \langle x, e \rangle \langle e, y \rangle] \right|$$

provided $e, f \in H$ with $\|e\| = \|f\| = 1$ and x, y are vectors in H satisfying some appropriate assumptions.

Natural applications for discrete inequalities, power series and integral inequalities are also given.

2. MAIN RESULTS

The following results hold:

Theorem 2.1. *Let* $(H, \langle \cdot, \cdot \rangle)$ *be an inner product space over the real or complex numbers field* $\mathbb{K}$*. If* $x, y, e \in H$ *with* $\|e\| = 1$*, then*

$$\begin{aligned} (2.1) \qquad & |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \\ & \leq \min \left\{ \|x\| \left(\|y\|^2 - |\langle y, e\rangle|^2\right)^{1/2}, \left(\|x\|^2 - |\langle x, e\rangle|^2\right)^{1/2} \|y\| \right\} \\ & \leq \frac{1}{2} \left[\|x\| \left(\|y\|^2 - |\langle y, e\rangle|^2\right)^{1/2} + \left(\|x\|^2 - |\langle x, e\rangle|^2\right)^{1/2} \|y\| \right] \\ & \leq \frac{1}{2} \left(\|x\|^2 + \|y\|^2\right)^{1/2} \left(\|x\|^2 + \|y\|^2 - |\langle x, e\rangle|^2 - |\langle y, e\rangle|^2\right)^{1/2}. \end{aligned}$$

The inequalities are sharp.

Proof. Using Schwarz inequality we have

$$(2.2) \qquad \|x\| \, \|y - \langle y, e\rangle e\| \geq |\langle x, y - \langle y, e\rangle e\rangle| = |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|$$

and

$$(2.3) \qquad \|x - \langle x, e\rangle e\| \, \|y\| \geq |\langle x - \langle x, e\rangle e, y\rangle| = |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Since

$$\|x - \langle x, e\rangle e\|^2 = \|x\|^2 - |\langle x, e\rangle|^2, \ \ \|y - \langle y, e\rangle e\|^2 = \|y\|^2 - |\langle y, e\rangle|^2$$

then by (2.2) and (2.3) we get

$$\begin{aligned} (2.4) \qquad & \min \left\{ \|x\| \left(\|y\|^2 - |\langle y, e\rangle|^2\right)^{1/2}, \left(\|x\|^2 - |\langle x, e\rangle|^2\right)^{1/2} \|y\| \right\} \\ & \geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \end{aligned}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

This proves the first inequality in (2.1).

Using the elementary inequality

$$\frac{1}{2}(a + b) \geq \min\{a, b\}$$

that holds for any real numbers $a, b \in \mathbb{R}$, we have the second inequality in (2.1).

By the Cauchy-Bunyakovsky-Schwarz inequality

$$(2.5) \qquad ac + bd \leq \left(a^2 + b^2\right)^{1/2} \left(c^2 + d^2\right)^{1/2} \text{ for } a, b, c, d \geq 0$$

we have

$$\begin{aligned} (2.6) \qquad & \|x\| \left(\|y\|^2 - |\langle y, e\rangle|^2\right)^{1/2} + \left(\|x\|^2 - |\langle x, e\rangle|^2\right)^{1/2} \|y\| \\ & \leq \left(\|x\|^2 + \|y\|^2\right)^{1/2} \left(\|x\|^2 + \|y\|^2 - |\langle x, e\rangle|^2 - |\langle y, e\rangle|^2\right)^{1/2} \end{aligned}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

This proves the last part of (2.1).

Observe that if we take in (2.1) $y = x$, then we get from all inequalities that

$$\|x\|^2 - |\langle x, e\rangle|^2 \leq \left(\|x\|^2 - |\langle x, e\rangle|^2\right)^{1/2} \|x\|,$$

which is sharp since for $x \perp y$, $\langle x, e\rangle = 0$ it reduces to an equality. ∎

Remark 2.1. If we use the triangle inequality

$$|\langle x,e\rangle\langle e,y\rangle|-|\langle x,y\rangle|\le|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|$$

then we get from (2.1)

$$\begin{aligned}(2.7)\qquad |\langle x,e\rangle\langle e,y\rangle| &\le |\langle x,y\rangle| \\ &\quad+\min\left\{\|x\|\left(\|y\|^2-|\langle y,e\rangle|^2\right)^{1/2},\left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2}\|y\|\right\}\end{aligned}$$

for any $x,y,e\in H$ with $\|e\|=1$.

The following lemma holds, see [6]:

Lemma 2.2. *Let a,x,A be vectors in the inner product space $(H,\langle\cdot,\cdot\rangle)$ over $\mathbb{K}$ with $a\ne A$. Then*

$$\operatorname{Re}\langle A-x,x-a\rangle\ge 0 \tag{2.8}$$

if and only if

$$\left\|x-\frac{a+A}{2}\right\|\le\frac{1}{2}\|A-a\|. \tag{2.9}$$

Proof. Define

$$I_1:=\operatorname{Re}\langle A-x,x-a\rangle\ \text{ and }\ I_2:=\frac{1}{4}\|A-a\|^2-\left\|x-\frac{a+A}{2}\right\|^2.$$

A simple calculation shows that

$$I_1=I_2=\operatorname{Re}\left[\langle x,a\rangle+\langle A,x\rangle\right]-\operatorname{Re}\langle A,a\rangle-\|x\|^2$$

and thus, obviously, $I_1\ge 0$ iff $I_2\ge 0$ showing the required equivalence. ∎

The following corollary is obvious:

Corollary 2.3. *Let $x,e\in H$ with $\|e\|=1$ and $\delta,\Delta\in\mathbb{K}$ with $\delta\ne\Delta$. Then*

$$\operatorname{Re}\langle \Delta e-x,x-\delta e\rangle\ge 0 \tag{2.10}$$

iff

$$\left\|x-\frac{\delta+\Delta}{2}\cdot e\right\|\le\frac{1}{2}|\Delta-\delta|. \tag{2.11}$$

Remark 2.2. If $H=\mathbb{C}$, then $\operatorname{Re}\left[(A-x)(\bar{x}-\bar{a})\right]\ge 0$ if and only if $\left|x-\frac{a+A}{2}\right|\le\frac{1}{2}|A-a|$, where $a,x,A\in\mathbb{C}$. If $H=\mathbb{R}$, and $A>a$ then $a\le x\le A$ if and only if $\left|x-\frac{a+A}{2}\right|\le\frac{1}{2}(A-a)$.

The following lemma is of interest [6].

Lemma 2.4. *Let $x,e\in H$ with $\|e\|=1$. Then one has the following representation*

$$\|x\|^2-|\langle x,e\rangle|^2=\inf_{\lambda\in\mathbb{K}}\|x-\lambda e\|^2\ge 0. \tag{2.12}$$

Proof. Observe, for any $\lambda\in\mathbb{K}$, that

$$\begin{aligned}\langle x-\lambda e,x-\langle x,e\rangle e\rangle &=\|x\|^2-|\langle x,e\rangle|^2-\lambda\left[\langle e,x\rangle-\langle e,x\rangle\|e\|^2\right]\\ &=\|x\|^2-|\langle x,e\rangle|^2.\end{aligned}$$

Using Schwarz's inequality, we have

$$\begin{aligned}\left[\|x\|^2-|\langle x,e\rangle|^2\right]^2 &= |\langle x-\lambda e, x-\langle x,e\rangle e\rangle|^2 \le \|x-\lambda e\|^2 \|x-\langle x,e\rangle e\|^2 \\ &= \|x-\lambda e\|^2 \left[\|x\|^2-|\langle x,e\rangle|^2\right],\end{aligned}$$

giving the bound

$$\|x\|^2-|\langle x,e\rangle|^2 \le \|x-\lambda e\|^2, \qquad \lambda\in\mathbb{K}. \tag{2.13}$$

Taking the infimum in (2.13) over $\lambda\in\mathbb{K}$, we deduce

$$\|x\|^2-|\langle x,e\rangle|^2 \le \inf_{\lambda\in\mathbb{K}} \|x-\lambda e\|^2.$$

Since, for $\lambda_0=\langle x,e\rangle$, we get $\|x-\lambda_0 e\|^2=\|x\|^2-|\langle x,e\rangle|^2$, then the representation (2.12) is proved. ∎

The following result also holds:

Corollary 2.5. *Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $e\in H$, $\|e\|=1$. If $\varphi,\gamma,\Phi,\Gamma$ are real or complex numbers and x,y are vectors in H such that the conditions*

$$\operatorname{Re}\langle \Phi e-x, x-\varphi e\rangle \ge 0,\ \ \operatorname{Re}\langle \Gamma e-y, y-\gamma e\rangle \ge 0 \tag{2.14}$$

hold, or, equivalently, the following assumptions

$$\left\|x-\frac{\varphi+\Phi}{2}e\right\| \le \frac{1}{2}|\Phi-\varphi|, \quad \left\|y-\frac{\gamma+\Gamma}{2}e\right\| \le \frac{1}{2}|\Gamma-\gamma| \tag{2.15}$$

are valid, then one has the inequality

$$\begin{aligned}|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle| &\le \frac{1}{2}\min\{\|x\|\,|\Gamma-\gamma|,|\Phi-\varphi|\,\|y\|\} \\ &\le \frac{1}{4}\left[\|x\|\,|\Gamma-\gamma|+|\Phi-\varphi|\,\|y\|\right] \\ &\le \frac{1}{4}\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(|\Phi-\varphi|^2+|\Gamma-\gamma|^2\right)^{1/2}.\end{aligned} \tag{2.16}$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible.

Proof. Using the inequality (2.1) and Lemma 2.4 we have

$$\begin{aligned}&|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle| \\ &\le \min\left\{\|x\|\left(\|y\|^2-|\langle y,e\rangle|^2\right)^{1/2}, \left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2}\|y\|\right\} \\ &= \min\left\{\|x\|\inf_{\eta\in\mathbb{K}}\|y-\eta e\|, \inf_{\lambda\in\mathbb{K}}\|x-\lambda e\|\,\|y\|\right\} \\ &\le \frac{1}{2}\min\{\|x\|\,|\Gamma-\gamma|,|\Phi-\varphi|\,\|y\|\},\end{aligned}$$

which proves the first inequality in (2.16).

The rest follows as in the proof of Theorem 2.1.

For the sharpness of the constants, we take $y=x$ in (2.16) to get

$$\|x\|^2-|\langle x,e\rangle|^2 \le \frac{1}{2}\|x\|\,|\Phi-\varphi|$$

provided

$$\left\|x-\frac{\varphi+\Phi}{2}e\right\| \le \frac{1}{2}|\Phi-\varphi|.$$

Moreover, if we take $\varphi = -\Phi$, then we have the inequality

$$\|x\|^2 - |\langle x, e\rangle|^2 \leq \|x\| \, |\Phi| \tag{2.17}$$

provided $\|x\| \leq |\Phi|$.

Let $x = \Phi m$ with $m \in H$, $\|m\| = 1$ and $m \perp e$. Then $\|x\| = |\Phi|$, $\|x\|^2 - |\langle x, e\rangle|^2 = |\Phi|$ and the equality case is realized in (2.17). ∎

The following result also holds:

Theorem 2.6. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. If $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$, then*

$$\begin{aligned} &\left| \langle x, y\rangle - \frac{1}{2} \left[\langle x, f\rangle \langle f, y\rangle + \langle x, e\rangle \langle e, y\rangle \right] \right| \\ &\leq \frac{1}{2} \left[\|x\| \left(\|y\|^2 - |\langle y, f\rangle|^2 \right)^{1/2} + \left(\|x\|^2 - |\langle x, e\rangle|^2 \right)^{1/2} \|y\| \right] \\ &\leq \frac{1}{2} \left(\|x\|^2 + \|y\|^2 \right)^{1/2} \left(\|x\|^2 + \|y\|^2 - |\langle x, e\rangle|^2 - |\langle y, f\rangle|^2 \right)^{1/2} \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} &|\langle x, f\rangle \langle f, y\rangle - \langle x, e\rangle \langle e, y\rangle| \\ &\leq \|x\| \left(\|y\|^2 - |\langle y, f\rangle|^2 \right)^{1/2} + \left(\|x\|^2 - |\langle x, e\rangle|^2 \right)^{1/2} \|y\| \\ &\leq \left(\|x\|^2 + \|y\|^2 \right)^{1/2} \left(\|x\|^2 + \|y\|^2 - |\langle x, e\rangle|^2 - |\langle y, f\rangle|^2 \right)^{1/2}. \end{aligned} \tag{2.19}$$

The inequalities (2.18) are sharp.

Proof. Using Schwarz inequality we have

$$\|x\| \, \|y - \langle y, f\rangle f\| \geq |\langle x, y - \langle y, f\rangle f\rangle| = |\langle x, y\rangle - \langle x, f\rangle \langle f, y\rangle| \tag{2.20}$$

and

$$\|x - \langle x, e\rangle e\| \, \|y\| \geq |\langle x - \langle x, e\rangle e, y\rangle| = |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \tag{2.21}$$

for any $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$.

If we add the inequalities (2.20) and (2.21) and use the triangle inequality, then we get

$$\begin{aligned} &\|x\| \, \|y - \langle y, f\rangle f\| + \|x - \langle x, e\rangle e\| \, \|y\| \\ &\geq |\langle x, y\rangle - \langle x, f\rangle \langle f, y\rangle| + |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \\ &\geq \begin{cases} |2 \langle x, y\rangle - \langle x, f\rangle \langle f, y\rangle - \langle x, e\rangle \langle e, y\rangle|, \\ \\ |\langle x, f\rangle \langle f, y\rangle - \langle x, e\rangle \langle e, y\rangle| \end{cases} \end{aligned} \tag{2.22}$$

for any $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$.

Since

$$\|x - \langle x, e\rangle e\| = \left(\|x\|^2 - |\langle x, e\rangle|^2 \right)^{1/2}, \ \ \|y - \langle y, f\rangle f\| = \left(\|y\|^2 - |\langle y, f\rangle|^2 \right)^{1/2}$$

then by (2.22) we have

$$\begin{aligned} &\|x\| \left(\|y\|^2 - |\langle y, f\rangle|^2 \right)^{1/2} + \left(\|x\|^2 - |\langle x, e\rangle|^2 \right)^{1/2} \|y\| \\ &\geq \begin{cases} |2 \langle x, y\rangle - \langle x, f\rangle \langle f, y\rangle - \langle x, e\rangle \langle e, y\rangle|, \\ \\ |\langle x, f\rangle \langle f, y\rangle - \langle x, e\rangle \langle e, y\rangle| \end{cases} \end{aligned} \tag{2.23}$$

for any $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$.

By employing the inequality (2.5) we get

$$\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(\|y\|^2-|\langle y,f\rangle|^2+\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2} \geq \|x\|\left(\|y\|^2-|\langle y,f\rangle|^2\right)^{1/2}+\left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2}\|y\| \tag{2.24}$$

for any $x,y,e,f\in H$ with $\|e\|=\|f\|=1.$

Making use of (2.23) and (2.24) we get the desired inequalities (2.18) and (2.19).

If we take $f=e$ in (2.18) then we get

$$\begin{aligned}&|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\\ &\leq\frac{1}{2}\left[\|x\|\left(\|y\|^2-|\langle y,e\rangle|^2\right)^{1/2}+\left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2}\|y\|\right]\\ &\leq\frac{1}{2}\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(\|x\|^2+\|y\|^2-|\langle x,e\rangle|^2-|\langle y,e\rangle|^2\right)^{1/2},\end{aligned}$$

which by Theorem 2.1 are sharp. ∎

Remark 2.3. If we use the triangle inequality

$$\begin{aligned}&\frac{1}{2}|\langle x,f\rangle\langle f,y\rangle+\langle x,e\rangle\langle e,y\rangle|-|\langle x,y\rangle|\\ &\leq\left|\langle x,y\rangle-\frac{1}{2}\left[\langle x,f\rangle\langle f,y\rangle+\langle x,e\rangle\langle e,y\rangle\right]\right|,\end{aligned}$$

then we get from (2.1)

$$\begin{aligned}&\frac{1}{2}|\langle x,f\rangle\langle f,y\rangle+\langle x,e\rangle\langle e,y\rangle|\\ &\leq|\langle x,y\rangle|+\frac{1}{2}\left[\|x\|\left(\|y\|^2-|\langle y,f\rangle|^2\right)^{1/2}+\left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2}\|y\|\right]\end{aligned} \tag{2.25}$$

for any $x,y,e,f\in H$ with $\|e\|=\|f\|=1.$

Corollary 2.7. *Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $e,f\in H,\|e\|=\|f\|=1.$ If $\varphi,\gamma,\Phi,\Gamma$ are real or complex numbers and x,y are vectors in H such that the conditions*

$$\operatorname{Re}\langle\Phi e-x,x-\varphi e\rangle\geq 0,\ \ \operatorname{Re}\langle\Gamma f-y,y-\gamma f\rangle\geq 0 \tag{2.26}$$

hold, or, equivalently, the following assumptions

$$\left\|x-\frac{\varphi+\Phi}{2}e\right\|\leq\frac{1}{2}|\Phi-\varphi|,\quad\left\|y-\frac{\gamma+\Gamma}{2}f\right\|\leq\frac{1}{2}|\Gamma-\gamma| \tag{2.27}$$

are valid, then one has the inequalities

$$\begin{aligned}&\left|\langle x,y\rangle-\frac{1}{2}\left[\langle x,f\rangle\langle f,y\rangle+\langle x,e\rangle\langle e,y\rangle\right]\right|\\ &\leq\frac{1}{4}\left[\|x\||\Gamma-\gamma|+|\Phi-\varphi|\|y\|\right]\\ &\leq\frac{1}{4}\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(|\Gamma-\gamma|^2+|\Phi-\varphi|^2\right)^{1/2}\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}&|\langle x,f\rangle\langle f,y\rangle-\langle x,e\rangle\langle e,y\rangle|\\ &\leq\frac{1}{2}\left[\|x\||\Gamma-\gamma|+|\Phi-\varphi|\|y\|\right]\\ &\leq\frac{1}{2}\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(|\Gamma-\gamma|^2+|\Phi-\varphi|^2\right)^{1/2}.\end{aligned} \tag{2.29}$$

The constant $\frac{1}{4}$ in the right hand side of (2.28) is sharp.

3. Applications for Sequences and Power Series

Consider the Hilbert space $\mathbb{C}^n$ endowed with the inner product $\langle\cdot,\cdot\rangle_{\mathbf{p}}:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}$ defined by

$$\langle\mathbf{x},\mathbf{y}\rangle_{\mathbf{p}}:=\sum_{j=1}^{n}p_jx_j\overline{y}_j,$$

where $\mathbf{p}=(p_1,...,p_n)$ is a probability distribution, i.e. $p_j\geq 0$, $j\in\{1,...,n\}$ with $\sum_{j=1}^{n}p_j=1$ and

$$\mathbf{x}=(x_1,...,x_n)\,,\ \mathbf{y}=(y_1,...,y_n)\in\mathbb{C}^n.$$

Assume that $\mathbf{e}=(e_1,...,e_n)\,,\mathbf{f}=(f_1,...,f_n)\in\mathbb{C}^n$ with

$$\sum_{j=1}^{n}p_j\left|e_j\right|^2=\sum_{j=1}^{n}p_j\left|f_j\right|^2=1. \tag{3.1}$$

Then for any $\mathbf{x}=(x_1,...,x_n)\,,\mathbf{y}=(y_1,...,y_n)\in\mathbb{C}^n$ we have from (2.1) the inequality

$$\begin{aligned}
&\left|\sum_{j=1}^{n}p_jx_j\overline{y}_j-\sum_{j=1}^{n}p_jx_j\overline{e}_j\sum_{j=1}^{n}p_je_j\overline{y}_j\right| \\
&\leq\min\left\{\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2\right)^{1/2}\left(\sum_{j=1}^{n}p_j\left|y_j\right|^2-\left|\sum_{j=1}^{n}p_jy_j\overline{e}_j\right|^2\right)^{1/2},\right. \\
&\left.\left(\sum_{j=1}^{n}p_j\left|y_j\right|^2\right)^{1/2}\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2-\left|\sum_{j=1}^{n}p_jx_j\overline{e}_j\right|^2\right)^{1/2}\right\} \\
&\leq\frac{1}{2}\left[\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2\right)^{1/2}\left(\sum_{j=1}^{n}p_j\left|y_j\right|^2-\left|\sum_{j=1}^{n}p_jy_j\overline{e}_j\right|^2\right)^{1/2}\right. \\
&\left.+\left(\sum_{j=1}^{n}p_j\left|y_j\right|^2\right)^{1/2}\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2-\left|\sum_{j=1}^{n}p_jx_j\overline{e}_j\right|^2\right)^{1/2}\right] \\
&\leq\frac{1}{2}\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2+\sum_{j=1}^{n}p_j\left|y_j\right|^2\right)^{1/2} \\
&\times\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2+\sum_{j=1}^{n}p_j\left|y_j\right|^2-\left|\sum_{j=1}^{n}p_jx_j\overline{e}_j\right|^2-\left|\sum_{j=1}^{n}p_jy_j\overline{e}_j\right|^2\right)^{1/2}
\end{aligned} \tag{3.2}$$

while from (2.18) we get

$$\begin{aligned}
(3.3)\quad & \left|\sum_{j=1}^{n}p_jx_j\overline{y}_j-\frac{1}{2}\left[\sum_{j=1}^{n}p_jx_j\overline{e}_j\sum_{j=1}^{n}p_je_j\overline{y}_j+\sum_{j=1}^{n}p_jx_j\overline{f}_j\sum_{j=1}^{n}p_jf_j\overline{y}_j\right]\right| \\
&\le \frac{1}{2}\left[\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2\right)^{1/2}\left(\sum_{j=1}^{n}p_j\left|y_j\right|^2-\left|\sum_{j=1}^{n}p_jy_j\overline{f}_j\right|^2\right)^{1/2}\right. \\
&\left. +\left(\sum_{j=1}^{n}p_j\left|y_j\right|^2\right)^{1/2}\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2-\left|\sum_{j=1}^{n}p_jx_j\overline{e}_j\right|^2\right)^{1/2}\right] \\
&\le \frac{1}{2}\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2+\sum_{j=1}^{n}p_j\left|y_j\right|^2\right)^{1/2} \\
&\times\left(\sum_{j=1}^{n}p_j\left|x_j\right|^2+\sum_{j=1}^{n}p_j\left|y_j\right|^2-\left|\sum_{j=1}^{n}p_jx_j\overline{e}_j\right|^2-\left|\sum_{j=1}^{n}p_jy_j\overline{f}_j\right|^2\right)^{1/2}.
\end{aligned}$$

If we denote by $\mathcal{C}(0,1)$ the unit circle of radius 1 in $\mathbb{C}$, namely $\mathcal{C}(0,1)=\{z\in\mathbb{C}|\ |z|=1\}$, then for $\mathbf{e}=(e_1,...,e_n)$, $\mathbf{f}=(f_1,...,f_n)\in\mathbb{C}^n$ with $e_j,f_j\in\mathcal{C}(0,1)$ for any $j\in\{1,...,n\}$ we have that the condition (3.1) holds true and therefore the inequalities (3.2) and (3.3) are valid.

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$.

The most important power series with nonnegative coefficients that can be used to illustrate the above results are:

$$\begin{aligned}
(3.4)\quad & \exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n,\ z\in\mathbb{C},\ \frac{1}{1-z}=\sum_{n=0}^{\infty}z^n,\ z\in D(0,1), \\
& \ln\frac{1}{1-z}=\sum_{n=1}^{\infty}\frac{1}{n}z^n,\ z\in D(0,1),\ \cosh z=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n},\ z\in\mathbb{C}, \\
& \sinh z=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1},\ z\in\mathbb{C}.
\end{aligned}$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\begin{aligned}
(3.5)\quad & \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1), \\
& \sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!}z^{2n+1},\ z\in D(0,1),
\end{aligned}$$

$$\tanh^{-1}(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \; z \in D(0,1),$$

$$ {}_2F_1(\alpha, \beta, \gamma, z) := \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha, \beta, \gamma > 0$$
$$z \in D(0,1),$$

where Γ is *Gamma function*.

Proposition 3.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $0 < p < R$, $u, v \in \mathcal{C}(0,1)$ and $x, y \in \mathbb{C}$ with $p|x|^2, p|y|^2 < R$ then we have the inequalities*

$$\begin{aligned}
(3.6) \qquad & \left| \frac{f(px\overline{y})}{f(p)} - \frac{f(px\overline{u})}{f(p)} \frac{f(pu\overline{y})}{f(p)} \right| \\
&\leq \min \left\{ \left(\frac{f(p|x|^2)}{f(p)} \right)^{1/2} \left(\frac{f(p|y|^2)}{f(p)} - \left| \frac{f(py\overline{u})}{f(p)} \right|^2 \right)^{1/2}, \right. \\
&\left. \left(\frac{f(p|y|^2)}{f(p)} \right)^{1/2} \left(\frac{f(p|x|^2)}{f(p)} - \left| \frac{f(px\overline{u})}{f(p)} \right|^2 \right)^{1/2} \right\} \\
&\leq \frac{1}{2} \left[\left(\frac{f(p|x|^2)}{f(p)} \right)^{1/2} \left(\frac{f(p|y|^2)}{f(p)} - \left| \frac{f(py\overline{u})}{f(p)} \right|^2 \right)^{1/2} \right. \\
&\left. + \left(\frac{f(p|y|^2)}{f(p)} \right)^{1/2} \left(\frac{f(p|x|^2)}{f(p)} - \left| \frac{f(px\overline{u})}{f(p)} \right|^2 \right)^{1/2} \right] \\
&\leq \frac{1}{2} \left(\frac{f(p|x|^2)}{f(p)} + \frac{f(p|y|^2)}{f(p)} \right)^{1/2} \\
&\times \left(\frac{f(p|x|^2)}{f(p)} + \frac{f(p|y|^2)}{f(p)} - \left| \frac{f(px\overline{u})}{f(p)} \right|^2 - \left| \frac{f(py\overline{u})}{f(p)} \right|^2 \right)^{1/2}
\end{aligned}$$

and

$$(3.7)\qquad \left|\frac{f(px\overline{y})}{f(p)}-\frac{1}{2}\left[\frac{f(px\overline{u})}{f(p)}\frac{f(pu\overline{y})}{f(p)}+\frac{f(px\overline{v})}{f(p)}\frac{f(pv\overline{y})}{f(p)}\right]\right|$$
$$\leq \frac{1}{2}\left[\left(\frac{f\left(p|x|^2\right)}{f(p)}\right)^{1/2}\left(\frac{f\left(p|y|^2\right)}{f(p)}-\left|\frac{f(py\overline{v})}{f(p)}\right|^2\right)^{1/2}\right.$$
$$\left.+\left(\frac{f\left(p|y|^2\right)}{f(p)}\right)^{1/2}\left(\frac{f\left(p|x|^2\right)}{f(p)}-\left|\frac{f(px\overline{u})}{f(p)}\right|^2\right)^{1/2}\right]$$
$$\leq \frac{1}{2}\left(\frac{f\left(p|x|^2\right)}{f(p)}+\frac{f\left(p|y|^2\right)}{f(p)}\right)^{1/2}$$
$$\times\left(\frac{f\left(p|x|^2\right)}{f(p)}+\frac{f\left(p|y|^2\right)}{f(p)}-\left|\frac{f(px\overline{u})}{f(p)}\right|^2-\left|\frac{f(py\overline{v})}{f(p)}\right|^2\right)^{1/2}.$$

Proof. If $u,v\in\mathcal{C}(0,1)$ then for any $n\geq 0$ we have $u^n,v^n\in\mathcal{C}(0,1)$. Observe that for any $m\geq 1$ we have that

$$\frac{\sum_{n=0}^m a_np^n|u^n|^2}{\sum_{n=0}^m a_np^n}=\frac{\sum_{n=0}^m a_np^n|v^n|^2}{\sum_{n=0}^m a_np^n}=\frac{\sum_{n=0}^m a_np^n}{\sum_{n=0}^m a_np^n}=1.$$

Using the inequality (3.2) we have

$$(3.8)\qquad \left|\frac{\sum_{n=0}^m a_np^n(x\overline{y})^n}{\sum_{n=0}^m a_np^n}-\frac{\sum_{n=0}^m a_np^n(x\overline{u})^n}{\sum_{n=0}^m a_np^n}\frac{\sum_{n=0}^m a_np^n(u\overline{y})^n}{\sum_{n=0}^m a_np^n}\right|$$
$$\leq\min\left\{\left(\frac{\sum_{n=0}^m a_np^n|x|^{2n}}{\sum_{n=0}^m a_np^n}\right)^{1/2}\left(\frac{\sum_{n=0}^m a_np^n|y|^{2n}}{\sum_{n=0}^m a_np^n}-\left|\frac{\sum_{n=0}^m a_np^n(y\overline{u})^n}{\sum_{n=0}^m a_np^n}\right|^2\right)^{1/2},\right.$$
$$\left.\left(\frac{\sum_{n=0}^m a_np^n|y|^{2n}}{\sum_{n=0}^m a_np^n}\right)^{1/2}\left(\frac{\sum_{n=0}^m a_np^n|x|^{2n}}{\sum_{n=0}^m a_np^n}-\left|\frac{\sum_{n=0}^m a_np^n(x\overline{u})^n}{\sum_{n=0}^m a_np^n}\right|^2\right)^{1/2}\right\}$$
$$\leq\frac{1}{2}\left[\left(\frac{\sum_{n=0}^m a_np^n|x|^{2n}}{\sum_{n=0}^m a_np^n}\right)^{1/2}\left(\frac{\sum_{n=0}^m a_np^n|y|^{2n}}{\sum_{n=0}^m a_np^n}-\left|\frac{\sum_{n=0}^m a_np^n(y\overline{u})^n}{\sum_{n=0}^m a_np^n}\right|^2\right)^{1/2}\right.$$
$$\left.+\left(\frac{\sum_{n=0}^m a_np^n|y|^{2n}}{\sum_{n=0}^m a_np^n}\right)^{1/2}\left(\frac{\sum_{n=0}^m a_np^n|x|^{2n}}{\sum_{n=0}^m a_np^n}-\left|\frac{\sum_{n=0}^m a_np^n(x\overline{u})^n}{\sum_{n=0}^m a_np^n}\right|^2\right)^{1/2}\right]$$
$$\leq\frac{1}{2}\left(\frac{\sum_{n=0}^m a_np^n|x|^{2n}}{\sum_{n=0}^m a_np^n}+\frac{\sum_{n=0}^m a_np^n|y|^{2n}}{\sum_{n=0}^m a_np^n}\right)^{1/2}$$
$$\times\left(\frac{\sum_{n=0}^m a_np^n|x|^{2n}}{\sum_{n=0}^m a_np^n}+\frac{\sum_{n=0}^m a_np^n|y|^{2n}}{\sum_{n=0}^m a_np^n}-\left|\frac{\sum_{n=0}^m a_np^n(x\overline{u})^n}{\sum_{n=0}^m a_np^n}\right|^2-\left|\frac{\sum_{n=0}^m a_np^n(y\overline{u})^n}{\sum_{n=0}^m a_np^n}\right|^2\right)^{1/2}$$

Since all the series whose partial sums are involved in inequality (3.8) are convergent, then by letting $m\to\infty$ in (3.8) we get the desired result (3.6).

The proof of the inequality (3.7) can be proved in the same way by utilizing (3.3) and we omit the details. ∎

Remark 3.1. The inequality (3.6) can provide some particular inequalities of interest. For instance, if we take $f(z)=\exp(z)$, $z\in\mathbb{C}$, then we get

$$\begin{aligned}
&\left|\exp\left[p\left(x\overline{y}-1\right)\right]-\exp\left[p\left(x\overline{u}+u\overline{y}-2\right)\right]\right| \\
&\leq \min\left\{\exp\left[\frac{1}{2}p\left(|x|^2-1\right)\right]\left(\exp\left[p\left(|y|^2-1\right)\right]-\left|\exp\left[p\left(y\overline{u}-1\right)\right]\right|^2\right)^{1/2},\right. \\
&\left.\exp\left[\frac{1}{2}p\left(|y|^2-1\right)\right]\left(\exp\left[p\left(|x|^2-1\right)\right]-\left|\exp\left[p\left(x\overline{u}-1\right)\right]\right|^2\right)^{1/2}\right\} \\
&\leq \frac{1}{2}\left[\exp\left[\frac{1}{2}p\left(|x|^2-1\right)\right]\left(\exp\left[p\left(|y|^2-1\right)\right]-\left|\exp\left[p\left(y\overline{u}-1\right)\right]\right|^2\right)^{1/2}\right. \\
&\left.+\exp\left[\frac{1}{2}p\left(|y|^2-1\right)\right]\left(\exp\left[p\left(|x|^2-1\right)\right]-\left|\exp\left[p\left(x\overline{u}-1\right)\right]\right|^2\right)^{1/2}\right] \\
&\leq \frac{1}{2}\left(\exp\left[\frac{1}{2}p\left(|x|^2-1\right)\right]+\exp\left[\frac{1}{2}p\left(|y|^2-1\right)\right]\right)^{1/2} \\
&\times\left(\exp\left[\frac{1}{2}p\left(|x|^2-1\right)\right]+\exp\left[\frac{1}{2}p\left(|y|^2-1\right)\right]\right. \\
&\left.-\left|\exp\left[p\left(x\overline{u}-1\right)\right]\right|^2-\left|\exp\left[p\left(y\overline{u}-1\right)\right]\right|^2\right)^{1/2}
\end{aligned}\tag{3.9}$$

for any $p>0, u\in\mathcal{C}(0,1)$ and $x,y\in\mathbb{C}$.

If we take $u=v=1$, then from (3.9) we get

$$\begin{aligned}
&\left|\exp\left[p\left(x\overline{y}-1\right)\right]-\exp\left[p\left(x+\overline{y}-2\right)\right]\right| \\
&\leq \min\left\{\exp\left[\frac{1}{2}p\left(|x|^2-1\right)\right]\left(\exp\left[p\left(|y|^2-1\right)\right]-\left|\exp\left[p\left(y-1\right)\right]\right|^2\right)^{1/2},\right. \\
&\left.\exp\left[\frac{1}{2}p\left(|y|^2-1\right)\right]\left(\exp\left[p\left(|x|^2-1\right)\right]-\left|\exp\left[p\left(x-1\right)\right]\right|^2\right)^{1/2}\right\} \\
&\leq \frac{1}{2}\left[\exp\left[\frac{1}{2}p\left(|x|^2-1\right)\right]\left(\exp\left[p\left(|y|^2-1\right)\right]-\left|\exp\left[p\left(y-1\right)\right]\right|^2\right)^{1/2}\right. \\
&\left.+\exp\left[\frac{1}{2}p\left(|y|^2-1\right)\right]\left(\exp\left[p\left(|x|^2-1\right)\right]-\left|\exp\left[p\left(x-1\right)\right]\right|^2\right)^{1/2}\right] \\
&\leq \frac{1}{2}\left(\exp\left[\frac{1}{2}p\left(|x|^2-1\right)\right]+\exp\left[\frac{1}{2}p\left(|y|^2-1\right)\right]\right)^{1/2} \\
&\times\left(\exp\left[\frac{1}{2}p\left(|x|^2-1\right)\right]+\exp\left[\frac{1}{2}p\left(|y|^2-1\right)\right]\right. \\
&\left.-\left|\exp\left[p\left(x-1\right)\right]\right|^2-\left|\exp\left[p\left(y-1\right)\right]\right|^2\right)^{1/2}
\end{aligned}\tag{3.10}$$

for any $p>0$ and $x,y\in\mathbb{C}$.

Moreover, if we take in (3.10) $x=\overline{y}=z\in\mathbb{C}$, then we get

$$\begin{aligned}
&\left|\exp\left[p\left(z^2-1\right)\right]-\exp\left[2p\left(z-1\right)\right]\right| \\
&\leq \exp\left[\frac{1}{2}p\left(|z|^2-1\right)\right]\left(\exp\left[p\left(|z|^2-1\right)\right]-\left|\exp\left[p\left(z-1\right)\right]\right|^2\right)^{1/2}
\end{aligned}\tag{3.11}$$

for any $p>0$ and $z\in\mathbb{C}$.

4. APPLICATIONS FOR INTEGRALS

Consider $L^2[a,b]$ the Hilbert space of all complex valued functions f with $\int_a^b |f(t)|^2 dt < \infty$. The inner product is given by

$$\langle f,g\rangle_2 := \int_a^b f(t)\overline{g(t)}dt.$$

Assume that $h,k \in L^2[a,b]$ with

$$\int_a^b |h(t)|^2 dt = \int_a^b |k(t)|^2 dt = 1. \tag{4.1}$$

For instance, if $h(t) = \frac{1}{\sqrt{b-a}}\rho(t)$, $k(t) = \frac{1}{\sqrt{b-a}}\varphi(t)$ with $\rho(t), \varphi(t) \in \mathcal{C}(0,1)$ for almost any $t \in [a,b]$, then $h,k \in L^2[a,b]$ and the condition (4.1) is satisfied.

Proposition 4.1. *Assume that $h,k \in L^2[a,b]$ with the property (4.1). Then for any $f,g \in L^2[a,b]$ we have the inequality*

$$\begin{aligned}
&\Bigg|\int_a^b f(t)\overline{g(t)}dt \\
&-\frac{1}{2}\left[\int_a^b f(t)\overline{h(t)}dt\int_a^b h(t)\overline{g(t)}dt + \int_a^b f(t)\overline{k(t)}dt\int_a^b k(t)\overline{g(t)}dt\right]\Bigg| \\
&\le \frac{1}{2}\left(\int_a^b \left(|f(t)|^2 + |g(t)|^2\right)dt\right)^{1/2} \\
&\times\left(\int_a^b \left(|f(t)|^2 + |g(t)|^2\right)dt - \left|\int_a^b f(t)\overline{h(t)}dt\right|^2 - \left|\int_a^b g(t)\overline{k(t)}dt\right|^2\right)^{1/2}.
\end{aligned}\tag{4.2}$$

The proof follows by Theorem 2.6 for the inner product $\langle\cdot,\cdot\rangle_2$.

Remark 4.1. If $\rho(t), \varphi(t) \in \mathcal{C}(0,1)$ for almost any $t \in [a,b]$, then we have the following inequalities for integral means

$$\begin{aligned}
&\Bigg|\frac{1}{b-a}\int_a^b f(t)\overline{g(t)}dt \\
&-\frac{1}{2(b-a)^2}\left[\int_a^b f(t)\overline{\rho(t)}dt\int_a^b \rho(t)\overline{g(t)}dt\right. \\
&\left.+\int_a^b f(t)\overline{\varphi(t)}dt\int_a^b \varphi(t)\overline{g(t)}dt\right]\Bigg| \\
&\le \frac{1}{2}\left(\frac{1}{b-a}\int_a^b \left(|f(t)|^2 + |g(t)|^2\right)dt\right)^{1/2} \\
&\times\left[\frac{1}{b-a}\int_a^b \left(|f(t)|^2 + |g(t)|^2\right)dt\right. \\
&\left.-\left|\frac{1}{b-a}\int_a^b f(t)\overline{\rho(t)}dt\right|^2 - \left|\frac{1}{b-a}\int_a^b g(t)\overline{\varphi(t)}dt\right|^2\right]^{1/2}
\end{aligned}\tag{4.3}$$

for any $f,g \in L^2[a,b]$.

If we take $\rho(t)=1, \varphi(t)=sgn\left(t-\frac{a+b}{2}\right), t\in[a,b]$, then $\rho(t), \varphi(t)\in\mathcal{C}(0,1)$ for almost any $t\in[a,b]$ and then we get from (4.3)

$$\begin{aligned}(4.4)\quad & \left|\frac{1}{b-a}\int_a^b f(t)\overline{g(t)}dt\right.\\ & -\frac{1}{2(b-a)^2}\left[\int_a^b f(t)\,dt\int_a^b\overline{g(t)}dt\right.\\ & \left.\left.+\int_a^b f(t)\,sgn\left(t-\frac{a+b}{2}\right)dt\int_a^b sgn\left(t-\frac{a+b}{2}\right)\overline{g(t)}dt\right]\right|\\ & \leq\frac{1}{2}\left(\frac{1}{b-a}\int_a^b\left(|f(t)|^2+|g(t)|^2\right)dt\right)^{1/2}\\ & \times\left[\frac{1}{b-a}\int_a^b\left(|f(t)|^2+|g(t)|^2\right)dt\right.\\ & \left.-\left|\frac{1}{b-a}\int_a^b f(t)\,dt\right|^2-\left|\frac{1}{b-a}\int_a^b g(t)\,sgn\left(t-\frac{a+b}{2}\right)dt\right|^2\right]^{1/2}\end{aligned}$$

for any $f,g\in L^2[a,b]$.

On making use of Corollaries 2.5 and 2.7 one can state similar discrete and integral inequalities. However the details are not presented here.

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