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COMMUTATORS OF HARDY TYPE OPERATORS

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ABSTRACT. The note deals with commutaors of the Hardy operator, Hardy type operators on Morrey spaces on R^+. We have proved that the commutators generated by Hardy operator and Hardy type operators with a BMO function b are bounded on the Morrey spaces.

Key words and phrases: Commutator; Hardy operator; Hardy type operator; Morrey space.

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1. INTRODUCTION

Let R^+ denote the set of positive real numbers. For $p \in (0,\infty)$ and $0 < \lambda < 1$, Morrey space on R^+, $L^{p,\lambda}(R^+)$ consists of all functions $f \in L^p_{loc}(R^+)$ with

$$\|f\|_{L^{p,\lambda}(R^+)} = \left(\sup_{I\subset R^+} \frac{1}{|I|^\lambda}\int_I |f(x)|^p dx\right)^{1/p} < \infty$$

where $I = (a,b] \subset R^+,\ \ 0 < a < b < +\infty$, is a bounded interval on R^+ and $|I|$ denotes the length of I.

For $f \in L^p_{loc}(R^+)$, the Hardy operator is defined as

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt, \qquad x > 0.$$

Also for $0 < \alpha < 1$, one Hardy type operator is defined by

$$H^\alpha f(x) = \frac{1}{x^{1-\alpha}}\int_0^x f(t)dt, \qquad x > 0,$$

and the other one is defined as

$$H_\alpha f(x) = \frac{1}{(1-\alpha)x}\int_{\alpha x}^x f(t)dt, \quad x > 0.$$

Hardy et al. in [4], proved that

$$\|Hf\|_{L^p(R^+)} \le \frac{p}{p-1}\|f\|_{L^p(R^+)}$$

for $1 < p < \infty$ and the constant $p/(p-1)$ is sharp.

Let $b(x)$ be a real measurable, locally integrable function on R^+. Then we define the commutators

$$[H,b] = \frac{1}{x}\int_0^x (b(x)-b(t))f(t)dt, \quad x > 0$$

and

$$[H^\alpha,b] = \frac{1}{x^{1-\alpha}}\int_0^x (b(x)-b(t))f(t)dt, \quad x > 0$$

In this paper, we have first shown that the Hardy operator is bounded on $L^{p,\lambda}(R^+)$ for $1 < p < \infty$ by simple calculations and with certain conditions the inverse is also true. Then we have proved that the Hardy type operator is bounded on $L^{p,\lambda}(R^+)$. Finally we have obtained that when b is in BMO, the space of bounded mean oscillation on R^+, both commutators $[H,b]$ and $[H^\alpha,b]$ are bounded on $L^{p,\lambda}(R^+)$.

Throughout this paper, C is used for a positive constant which is not depending on the main factors and C might be different at each occurance.

2. MAIN RESULTS AND THEIR PROOFS

For our convenience and reference, all main results and useful results are stated in this section and proofs are also given with some remarks. In order to simplify the proof of the following theorem, we will have a lemma.

Lemma 2.1. *For $0 < \lambda < 1$, if $t > 0$, then we have*

$$\|f(\cdot t)\|_{L^{p,\lambda}(R^+)} \le \frac{1}{t^{(1-\lambda)/p}}\|f(\cdot)\|_{L^{p,\lambda}(R^+)}.$$

Proof. By the definition of Morrey space and substitution, we have

$$\begin{aligned}\|f(\cdot t)\|_{L^{p,\lambda}(R^+)} &= \left(\sup_{I\subset R^+}\frac{1}{|I|^\lambda}\int_I |f(ts)|^p ds\right)^{1/p}\\ &= \left(\sup_{I\subset R^+}\frac{1}{|I|^\lambda}\int_{tI} |f(x)|^p \frac{dx}{t}\right)^{1/p}\\ &= \left(\frac{1}{t^{1-\lambda}}\sup_{tI\subset R^+}\frac{1}{|tI|^\lambda}\int_{tI} |f(x)|^p dx\right)^{1/p}\\ &\le \frac{1}{t^{(1-\lambda)/p}}\|f(\cdot)\|_{L^{p,\lambda}(R^+)}.\end{aligned}$$

∎

One of the main results is as follows.

Theorem 2.2. *The Hardy operator H is bounded on $L^{p,\lambda}(R^+)$, that is,*

$$\|Hf\|_{L^{p,\lambda}(R^+)} \le C_{p,\lambda}\|f\|_{L^{p,\lambda}(R^+)}$$

where $C_{p,\lambda} = \frac{p}{p+\lambda-1}$.

Also the reverse Hardy inequality

$$\|Hf\|_{L^{p,\lambda}(R^+)} \ge \|f\|_{L^{p,\lambda}(R^+)}$$

holds for any positive, nonincreasing function $f \in L^{p,\lambda}(R^+)$.

Proof. For $x \in R^+$, we rewrite the Hardy operator

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt = \int_0^1 f(xs)ds.$$

For any $0 < \lambda < 1$ and $I \subset R^+$, by Lemma 2.1 and Minkowski's inequality for integral, we have

$$\begin{aligned}\frac{1}{|I|^\lambda}\int_I |(Hf)(x)|^p dx &= \frac{1}{|I|^\lambda}\int_I \left|\int_0^1 f(xt)dt\right|^p dx\\ &\le \frac{1}{|I|^\lambda}\left(\int_0^1\left[\int_I |f(xt)|^p dx\right]^{1/p} dt\right)^p\\ &\le \left(\int_0^1\left[\frac{1}{|I|^\lambda}\int_I |f(xt)|^p dx\right]^{1/p} dt\right)^p\\ &\le \left(\int_0^1 \frac{1}{t^{(1-\lambda)/p}}\|f\|_{L^{p,\lambda}(R^+)} dt\right)^p\\ &= (\|f\|_{L^{p,\lambda}(R^+)})^p\left(\int_0^1\frac{1}{t^{(1-\lambda)/p}}dt\right)^p\\ &= \left(\frac{p}{p+\lambda-1}\right)^p(\|f\|_{L^{p,\lambda}(R^+)})^p.\end{aligned}$$

Therefore

$$\|Hf\|_{L^{p,\lambda}(R^+)} \le C_{p,\lambda}\|f\|_{L^{p,\lambda}(R^+)},$$

where $C_{p,\lambda} = \frac{p}{p+\lambda-1}$.

Conversely, assume that $f \in L^{p,\lambda}(R^+)$ is positive and nonincreasing. Then we know that

$$(Hf)(x) = \frac{1}{x}\int_0^x f(t)dt \geq f(x).$$

The desired result follows immediately by the definition of $L^{p,\lambda}(R^+)$. ∎

The commutator $[T, b]$, T is Calderón-Zygmund operator and b is a BMO function, have been studied by many authors on different spaces, see [2], etc. In this paper we consider the commutator generated by Hardy operator H with a BMO function b, the commutator generated by Hardy type operator H^α with a BMO function b, and the commutator generated by the other Hardy type operator H_α with a BMO function b on Morrey spaces, $L^{p,\lambda}(R^+)$. Now we proceed to our next result regarding to the commutator of Hardy operator H and a BMO function b, $[H, b]$.

Theorem 2.3. *Let $p \in (1, \infty)$ and $b \in BMO$. Then $[H, b]$ is bounded on $L^{p,\lambda}(R^+)$, that is,*

$$\|[H, b]f\|_{L^{p,\lambda}(R^+)} \leq C\|b\|_{BMO}\|f\|_{L^{p,\lambda}(R^+)}$$

where the constant C depends on p and λ.

To prove Theorem 2.3, we will establish Lemma 2.4.

We define

$$H_r f(x) = \left(\frac{1}{x}\int_0^x |f(t)|^r dt\right)^{1/r} = (H(f^r)(x))^{1/r}$$

Lemma 2.4. *Let $1 < r < p < \infty$. Then H_r is bounded on $L^{p,\lambda}(R^+)$, i.e.*

$$\|H_r f\|_{L^{p,\lambda}(R^+)} \leq C\|f\|_{L^{p,\lambda}(R^+)}.$$

Proof. For any finite interval $I \subset R^+$, we consider

$$\begin{aligned}
\left(\frac{1}{|I|^\lambda}\int_I |H_r f(x)|^p dx\right)^{1/p} &= \left(\frac{1}{|I|^\lambda}\int_I \left|(H|f(x)|^r)^{p/r}\right| dx\right)^{1/p} \\
&\leq \|H(f^r)\|^{1/r}_{L^{\frac{p}{r},\lambda}(R^+)} \\
&\leq C\|f^r\|^{1/r}_{L^{\frac{p}{r},\lambda}(R^+)} \\
&= C\|f\|_{L^{p,\lambda}(R^+)}.
\end{aligned}$$

Note that we used Theorem 2.2 and the fact that if $f \in L^{p,\lambda}(R^+)$, then $f^r \in L^{p/r,\lambda}(R^+)$ in the middle of our proof. This completes the proof of Lemma 2.4.

∎

Proof of Theorem 2.3. For $1 < r < p < \infty$, by Hölder inequality we get

$$\begin{aligned}
|[H, b]f(x)| &= \left|\frac{1}{x}\int_0^x (b(x) - b(t))f(t)dt\right| \\
&\leq \left(\frac{1}{x}\int_0^x |b(x) - b(t)|^{r'} dt\right)^{1/r'} \left(\frac{1}{x}\int_0^x |f(t)|^r dt\right)^{1/r} \\
&\leq \|b\|_{BMO} H_r f(x).
\end{aligned}$$

where $\frac{1}{r} + \frac{1}{r'} = 1$.

From

$$\|[H, b]f\|_{L^{p,\lambda}(R^+)} \leq |b\|_{BMO}\|H_r f(x)\|_{L^{p,\lambda}(R^+)}$$

we are done with the proof of Theorem 2.3.

∎

Next we work with Hardy type operator, H^α and the corresponding commutator $[H^\alpha, b]$.

Theorem 2.5. *Let $0 < \alpha < 1$, $0 < \mu < 1$, $0 < \lambda < 1$, $1 < r < p < \infty$, and $\frac{1-\mu}{r} - \alpha = \frac{1-\lambda}{p}$. Then*

$$\|H^\alpha f\|_{L^{p,\lambda}(R^+)} \leq C\|f\|_{L^{r,\mu}(R^+)}.$$

Here C is a constant and is not depending on f.

Proof. By definition of H^α, we have

$$\begin{aligned}
|H^\alpha f(x)| &\leq \frac{1}{x^{1-\alpha}} \int_0^x |f(t)|dt \\
&\leq \frac{1}{x^{1-\alpha}} \left(\int_0^x dt\right)^{1/r'} \left(\int_0^x |f(t)|^r dt\right)^{1/r} \qquad \left(\frac{1}{r'} + \frac{1}{r} = 1\right) \\
&= \frac{1}{x^{1-\alpha-\frac{\mu}{r}-\frac{1}{r'}}} \left(\frac{1}{x^\mu}\int_0^x |f(t)|^r dt\right)^{1/r} \\
&\leq \frac{1}{x^{\frac{1-\mu}{r}-\alpha}} \|f\|_{L^{r,\mu}(R^+)}.
\end{aligned}$$

Therefore we have for any $I \subset R^+$

$$\begin{aligned}
\left[\frac{1}{|I|^\lambda}\int_I |H^\alpha f(x)|^p dx\right]^{1/p} &\leq \left[\frac{1}{|I|^\lambda}\int_I \frac{1}{x^{(\frac{1-\mu}{r}-\alpha)p}} \|f\|^p_{L^{r,\mu}(R^+)} dx\right]^{1/p} \\
&= \|f\|_{L^{r,\mu}(R^+)} \left[\frac{1}{|I|^\lambda}\int_I \frac{1}{x^{(\frac{1-\mu}{r}-\alpha)p}} dx\right]^{1/p} \\
&= \left[\frac{1}{|I|^\lambda}\int_I \frac{1}{x^{1-\lambda}} dx\right]^{1/p} \|f\|_{L^{r,\mu}(R^+)} \\
&\leq C\|f\|_{L^{r,\mu}(R^+)}
\end{aligned}$$

By the definition of $L^{p,\lambda}(R^+)$, the result follows.
∎

Theorem 2.6. *Let $0 < \alpha < 1$, $0 < \mu < 1$, $0 < \lambda < 1$, $1 < r < p < \infty$, $\frac{1-\mu}{r} - \alpha = \frac{1-\lambda}{p}$, and $b \in BMO$. Then*

$$\|[H^\alpha, b]f\|_{L^{p,\lambda}(R^+)} \leq C\|b\|_{BMO}\|f\|_{L^{r,\mu}(R^+)},$$

where C is a constant and is independent of f.

Proof. For the commutator $[H^\alpha, b]$, we have

$$\begin{aligned}
[H^\alpha, b] &= \frac{1}{x^{1-\alpha}} \int_0^x (b(x) - b(t))f(t)dt, \quad x > 0 \\
&\leq \frac{1}{x^{1-\alpha}} \left(\int_0^x |b(x) - b(t)|^{r'} dt\right)^{1/r'} \left(\int_0^x |f(t)|^r dt\right)^{1/r} \qquad \left(\frac{1}{r'} + \frac{1}{r} = 1\right) \\
&\leq \frac{1}{x^{\frac{1-\mu}{r}-\alpha}} \|b\|_{BMO}\|f\|_{L^{r,\mu}(R^+)}.
\end{aligned}$$

In the same manner as the proof of the previous Theorem, we have for any $I \subset R^+$

$$\begin{aligned}\left[\frac{1}{|I|^\lambda}\int_I |[H^\alpha, b]f(x)|^p dx\right]^{1/p} &\le \|b\|_{BMO}\|f\|_{L^{r,\mu}(R^+)}\left[\frac{1}{|I|^\lambda}\int_I \frac{1}{x^{(\frac{1-\mu}{r}-\alpha)p}}dx\right]^{1/p}\\ &= \left[\frac{1}{|I|^\lambda}\int_I \frac{1}{x^{1-\lambda}}dx\right]^{1/p}\|b\|_{BMO}\|f\|_{L^{r,\mu}(R^+)}\\ &\le C\|b\|_{BMO}\|f\|_{L^{r,\mu}(R^+)}\end{aligned}$$

∎

Note that when both μ and λ approach 0, both $L^{p,\mu}(R^+)$ and $L^{p,\lambda}(R^+)$ are $L^p(R^+)$ space. The results of Theorems 2.5 and 2.6 on $L^{p,\mu}(R^+)$ and $L^{p,\lambda}(R^+)$ agree to the those on $L^p(R^+)$.

For the other Hardy type operator H_α, we have

$$H_\alpha f(x) = \frac{1}{(1-\alpha)x}\int_{\alpha x}^{x} f(t)dt = \frac{1}{1-\alpha}\int_\alpha^1 f(xs)ds$$

and the following theorem.

Theorem 2.7. *For $0 < \alpha < 1$ and $0 < \lambda < 1$. H_α is bounded on $L^{p,\lambda}(R^+)$, i.e. for any $f \in L^{p,\lambda}(R^+)$*

$$\|H_\alpha f\|_{L^{p,\lambda}(R^+)} \le C\|f\|_{L^{p,\lambda}(R^+)}.$$

Proof. The proof here is similar to that of Theorem 2.2. For any interval $I \subset R^+$, we have from Minkowski's integral inequality

$$\begin{aligned}\left[\frac{1}{|I|^\lambda}\int_I |H_\alpha f(x)|^P dx\right]^{1/p} &\le \frac{1}{(1-\alpha)|I|^{\lambda/p}}\left[\int_I\left(\int_\alpha^1 |f(xs)|ds\right)^p dx\right]^{1/p}\\ &\le \frac{1}{(1-\alpha)|I|^{\lambda/p}}\int_I\left(\int_\alpha^1 |f(xs)|^p dx\right)^{1/p} ds\\ &= \frac{1}{(1-\alpha)}\int_\alpha^1 \frac{1}{s^{\frac{1-\lambda}{p}}}\left(\frac{1}{|sI|^\lambda}\int_{sI}|f(x)|^p dx\right)^{1/p} ds\\ &\le C\|f\|_{L^{p,\lambda}(R^+)}.\end{aligned}$$

This ends the proof.

∎

For the corresponding commutator, $[H_\alpha, b]$, we have the theorem below.

Theorem 2.8. *Let $1 < p < \infty$. Then $[H_\alpha, b]$ is bounded for any $b(x) \in BMO$, i.e. for any $f \in L^{p,\lambda}(R^+)$,*

$$\|[H_\alpha, b]\|_{L^{p,\lambda}(R^+)} \le C\|b\|_{BMO}\|f\|_{L^{p,\lambda}(R^+)}.$$

Proof. This proof is the anology of that of Theorem 2.6. Since

$$|[H_\alpha, b]f(x)| \le \frac{1}{[(1-\alpha)x]^{\frac{1-\lambda}{p}}}\|b\|_{BMO}\|f\|_{L^{p,\lambda}(R^+)},$$

we have

$$\begin{aligned}\|[H_\alpha, b]f\|_{L^{p,\lambda}(R^+)} &\le \|b\|_{BMO}\|f\|_{L^{p,\lambda}(R^+)}\sup_{I\subset R^+}\left(\frac{1}{|I|^\lambda}\int_I \frac{1}{[(1-\alpha)x]^{1-\lambda}}dx\right)\\ &\le C\|b\|_{BMO}\|f\|_{L^{p,\lambda}(R^+)}.\end{aligned}$$

The proof is complete. ∎

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