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A DYNAMIC CONTACT PROBLEM FOR AN ELECTRO VISCOELASTIC BODY

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ABSTRACT. We consider a dynamic problem which describes a contact between a piezoelectric body and a conductive foundation. The frictionless contact is modelled with the normal compliance, the electric conditions are supposed almost perfect. We prove the existence of a unique weak solution for almost perfect electric contact.

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1. INTRODUCTION

The piezoelectric materials are characterized by the combination of mechanical and electrical proprieties. The mechanical stress is generated when the electrical potential is applied and conversely the electric potential is created when the mechanical stress is present. We consider here an electro viscoelastic material. A general models for electro elastic problems can be found in [5, 9]. The contact problem for the electro viscoelastic material was considered in [3, 10, 11]. For all these references the formulation was assumed to be electrically insulated.

In this paper we study a contact between electro viscoelastic body and a deformable conductive foundation. Our interest is to describe the evolution of the deformation of the body and of the electric potential on the time interval $[0, T]$. This contact is modelled with normal compliance, we suppose that the acceleration of the system is not negligible so that the process is dynamic. Dynamic contact problems with normal compliance were considered in [1, 6, 7] and in the references therein.

Our aim in this paper is to extend some of the results obtained in paper [8], when the electric conditions are almost perfect. The study serves two purposes, the first one is to obtain variational formulation of the problem with regularized condition on the electric field, in a part of the boundary (see [8]) and to prove the existence and uniqueness of weak solutions. The second one is to study the convergence of those solutions to unique solutions of the variational problem with almost perfect electrical contact. This step answers some questions left open in the preceding paper [8]. In this part we make a passage to the limit in a regularized problem, under some a priori estimates and some compactness result for evolutionary problems.Therefore, in Section 2 the piezoelectric problem is stated together with two variational formulations; in Section 3 we state the existence and uniqueness result of the regularized problem $\mathcal{P}_R$ (Theorem 3.1). The proof is based on the theory of evolution equations with monotone operators and a fixed point arguments. In Section 4 we state our main existence and uniqueness result of weak solutions for the piezoelectric problem (Theorem 4.1). The proof is based on the a priori estimates of regularized solutions, followed by a passage to the limit when $\delta \to 0$, this is under a consideration of some compactness results.

2. PROBLEM STATEMENT AND NOTATIONS

We assume that the body occupies the bounded domain Ω, and assume that the boundary Γ of Ω is Lipschitz continuous and partitioned into three disjoint measurable open parts Γ_1, Γ_2, Γ_3, and a partition $\Gamma_1 \sqcup \Gamma_2$ into open parts Γ_a and Γ_b. We assume that $meas\Gamma_2 > 0$ and $meas\Gamma_a > 0$. The body is clamped on Γ_1, therefore the displacement field vanishes there. A volume force of density f_0 acts in $\Omega \times (0, T)$ and surface traction of density f_2 acts in $\Gamma_2 \times (0, T)$. The body may arrive in contact on $\Gamma_3 \times (0, T)$ with an obstacle, we assume that the contact is frictionless and it is modelled with normal compliance. The electric effects leads to the appearance of charges of density q_0. The process is to be assumed electrically static.

We denote by $\mathbb{S}^d$ the space of second order of symmetric tensors on $\mathbb{R}^d$ ($d = 1, 2, 3$) and by $(\cdot)$ and $|\cdot|$ respectively the scalar product and the Euclidean norm in $\mathbb{S}^d$ (resp in $\mathbb{R}^d$).

$$\boldsymbol{u}.\boldsymbol{v} = u_i v_i \qquad |u| = (u.u)^{\frac{1}{2}} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d, i = 1, \cdots d.$$
$$\boldsymbol{\sigma}.\boldsymbol{\tau} = \sigma_{ij}\tau_{ij} \qquad |\tau| = (\tau.\tau)^{\frac{1}{2}} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d, i = 1, \cdots d, j = 1, \cdots d.$$

Here and bellow the indices $i,\ j$ run between 1 and d and the summation convention over repeated indices is adopted. Let $\Omega \subset \mathbb{R}^d$, we shall use the notation

$$\begin{aligned} &H = \{\boldsymbol{u} = (u_i) \mid u_i \in L^2(\Omega)\} = \left(L^2(\Omega)\right)^d, \\ &\mathcal{W} = \{\boldsymbol{D} \in H \mid div\boldsymbol{D} \in L^2(\Omega)\}, \\ &\mathcal{H} = \left\{\boldsymbol{\sigma} = (\sigma)_{ij} \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\}, \\ &H_1 = \{\boldsymbol{u} = (u_i) \mid \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{H}\}, \\ &\mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \mid Div\boldsymbol{\sigma} \in H\}, \end{aligned}$$

with $\boldsymbol{\varepsilon} : H \to \mathcal{H}$ and $Div : \mathcal{H} \to H$ are respectively operators of deformation and divergence defined by :

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\boldsymbol{\varepsilon}_{ij}(\boldsymbol{u})), \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{ij} + u_{ji}) \text{ and } Div\boldsymbol{\sigma} = (\sigma_{ij,j}),$$

The tensors $\mathcal{E} = (e_{ijk})$ and its transpose $\mathcal{E}^* = (e_{kij})$ satisfy the equality

$$\mathcal{E}\boldsymbol{\sigma} \cdot \boldsymbol{v} = \boldsymbol{\sigma} \cdot \mathcal{E}^* \boldsymbol{v},$$

where the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. We assume that the mass density ρ satisfy

$$\rho \in L_\infty(\Omega) \text{ and there exists } \rho_* > 0 \text{ such that } \rho(\boldsymbol{x}) \geq \rho_*, \text{ a.e. in } \Omega,$$

then the space H is Hilbert space endowed with a new inner product,

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega \rho u_i v_i \, dx.$$

The space H, $\mathcal{H}$, H^1 and $\mathcal{H}^1$ are Hilbert spaces endowed with the inner products given by

$$\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} &= \int_\Omega \sigma_{ij}\tau_{ij}\, dx, \\ (\boldsymbol{u}, \boldsymbol{v})_{H_1} &= (\boldsymbol{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div\boldsymbol{\sigma}, Div\boldsymbol{\tau})_H. \end{aligned}$$

the associated norms on these spaces respectively are $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H^1}$ and $|\cdot|_{\mathcal{H}^1}$. Since Γ is assumed be Lipschitz continuous then the unit outward normal vector ν is defined a.e., for every vector $\boldsymbol{v} \in H_1$, we use the notation $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by $\boldsymbol{v}_\nu$ and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ, given by $v_\nu = \boldsymbol{v} \cdot \nu$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. For regular stress field $\boldsymbol{\sigma}$ (say C^1), the application of its trace to $\boldsymbol{\nu}$ is the Cauchy stress vector $\boldsymbol{\sigma\nu}$. We define the normal and tangential components of $\boldsymbol{\sigma}$ by $\sigma_\nu = (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \boldsymbol{\sigma}_\nu \boldsymbol{\nu}$, and recall that the Green's formula holds

(2.1) $$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (Div\boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma\nu} \cdot \boldsymbol{v}, \quad \forall \boldsymbol{v} \in H_1.$$

Let the Hilbert spaces $L^p(0,T;H)$ and $W^{1,p}(0,T;V)$ $1 \leq p \leq +\infty$,

(2.2) $$L^p(0,T;H) = \{\boldsymbol{u} \mid \boldsymbol{u} :]0,T[\to H\},$$

(2.3) $$W^{1,p}(0,T;V) = \left\{\boldsymbol{u} \in L^p(0,T;V),\ \dot{\boldsymbol{u}} = \frac{d\boldsymbol{u}(t)}{dt} \in L^p(0,T;V)\right\}.$$

Here and every where in this paper the dot above the derivative is with respect to the time variable. The spaces $C(0,T;X)$ and $C^1(0,T;X)$ are respectively continuous and differentiable continuous functions from $[0,T]$ into X with a respective norms :

$$|f|_{C(0,T;X)} = \max_{t\in[0,T]} |f|_X \text{ and } |f|_{C^1(0,T;X)} = \max_{t\in[0,T]} |f|_X + \max_{t\in[0,T]} \left|\dot{f}\right|_X.$$

The physical model for the process is as follows :

Problem $\mathcal{P}$. Find a displacement field $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$, an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$, an electric displacement field, such that

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{E}^*\nabla\varphi \text{ in } \Omega \times (0,T), \tag{2.4}$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\gamma}\nabla\varphi \quad \text{in } \Omega \times (0,T), \tag{2.5}$$

$$\rho\ddot{\boldsymbol{u}} = Div\, \boldsymbol{\sigma} + \boldsymbol{f}_0 \quad \text{in } \Omega \times (0,T), \tag{2.6}$$

$$div\, \boldsymbol{D} = \boldsymbol{q}_0 \quad \text{in } \Omega \times \times (0,T), \tag{2.7}$$

$$\boldsymbol{u} = 0, \quad \text{on } \Gamma_1 \times (0,T), \tag{2.8}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{2.9}$$

$$\boldsymbol{\sigma}_\nu = -\,p\,(u_\nu - g)\,,\ \boldsymbol{u}_\tau = 0 \ \text{ on } \Gamma_3 \times (0,T), \tag{2.10}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T), \tag{2.11}$$

$$\boldsymbol{D}\cdot\boldsymbol{\nu} = q_b \quad \text{on } \Gamma_b \times (0,T), \tag{2.12}$$

$$\boldsymbol{D}\cdot\boldsymbol{\nu} = k\chi_{[0,+\infty)}\,(u_\nu - g)\,\phi_L\,(\varphi - \varphi_0) \ \text{ on } \Gamma_3 \times (0,T), \tag{2.13}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \quad \text{in } \Omega. \tag{2.14}$$

$\chi_{[0,+\infty)}$ is the characteristic function of the interval $[0,+\infty)$ defined by

$$\chi_{[0,+\infty)}(r) = \begin{cases} 0 & \text{if } r < 0, \\ 1 & \text{if } r \geq 0. \end{cases}$$

In the equations (2.4)-(2.7) and below and in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $x \in \Omega \cup \Gamma$ and $t \in [0,T]$. The general viscoelastic constitutive law with electric effects is given by (2.4), where $\mathcal{A}$ is a non-linear viscosity function depending on the strain tensor $\boldsymbol{\varepsilon}\,(\dot{\boldsymbol{u}})$, $\mathcal{G}$ is a non-linear elasticity depends $\boldsymbol{\varepsilon}\,(\boldsymbol{u})$. The stress is depends on electric field $-\nabla\varphi$. The relation (2.5) is the electric displacement it is a linear function of strain and electric field. The equations (2.6) and (2.7) are the equilibrium equations, in equation (2.6) we suppose the process is dynamic with a mass density ρ. Here the conditions (2.8) and (2.9) are the displacement and traction boundary conditions, respectively condition (2.10) represents frictionless contact condition with normal compliance. Here p is prescribed function such that $p\,(r) = 0$ when $r \leq 0$, g is the initial gap and the condition $u_\nu - g \geq 0$ represents the penetration of body in the foundation, which is assumed to be conductive. The expressions (2.11) and (2.12) are boundary conditions on electric potential φ and displacement field D on Γ_a and Γ_b. On part of the boundary Γ_3, and during the process of contact the normal of electric displacement field is assumed to be proportional to the difference between the potential of foundation φ_0 and the body's surface potential, given by condition (2.13). The function ϕ_L is introduced to control the boundedness of $\varphi - \varphi_0$, see [8]. Finally, (2.14) is the initial condition on displacement and the velocity field. To present variational formulation of the above problem, we need additional notations. Let us consider the subspaces of H_1 and H^1 defined by

$$\begin{aligned} V &= \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0}, \text{ in } \Gamma_1\}, \\ W &= \left\{\xi \in H^1 \mid \xi = 0, \text{ in } \Gamma_a\right\}, \end{aligned}$$

we recall since $meas\Gamma_1 > 0$ and $meas\Gamma_a > 0$, Korn's and Friederichs-Poincare inequalities hold, thus there exist respectively a constant $C_K > 0$ and $c_F > 0$ which depends respectively only on Γ_1, Γ_a and Ω such that

$$\begin{aligned} |\boldsymbol{\varepsilon}(\boldsymbol{v})|_{\mathcal{H}_1} &\geq c_K |\boldsymbol{v}|_{H_1}, \qquad \forall \boldsymbol{v} \in V, \\ |\nabla\xi|_H &\geq c_F |\xi|_{H^1}, \qquad \forall \xi \in W, \end{aligned}$$

for $\boldsymbol{u}, \boldsymbol{v} \in V$ we have $(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}_1}$ and $\forall \varphi, \xi \in W$, $(\varphi, \xi)_W = (\nabla\varphi, \nabla\xi)_H$, and we have $|\boldsymbol{u}|_V = (\boldsymbol{u}, \boldsymbol{u})_V^{1/2}$, $|\varphi|_V = (\varphi, \varphi)_W^{1/2}$, therefore $(V, |\cdot|_V)$ and $(W, |\cdot|_W)$ are real Hilbert spaces. Moreover, by Sobolev trace theorem, there exist a constants c_0, $\tilde{c}_0$ depending only on Ω and Γ_1, Γ_a such that

$$|\xi|_{L^2(\Gamma_a)} \leq c_0 |\xi|_W, \qquad \forall \xi \in W, \tag{2.15}$$

$$|\boldsymbol{v}|_{L^2(\Gamma_1)^d} \leq \tilde{c}_0 |\boldsymbol{v}|_V, \qquad \forall \boldsymbol{v} \in V. \tag{2.16}$$

Let note V' and W' the dual spaces of V and W, so we have continuous and dense embeddings $V \subset H \subset V'$ rep ($W \subset L^2(\Omega) \subset W'$). To study problem $\mathcal{P}$ we must make some assumptions.The viscosity operator $\mathcal{A}$ and the elasticity one $\mathcal{G}$ satisfy the conditions

$$\left\{ \begin{array}{l} \text{a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d, \\ \text{b) there exists } L_{\mathcal{A}} > 0 \text{ such that} \\ |\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \leq L_{\mathcal{A}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{, a.e. } \boldsymbol{x} \in \Omega, \\ \text{c) there exists } m_{\mathcal{A}} > 0 \text{ such that} \\ (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2, \\ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \text{d) the mapping } \longmapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega, \\ \text{e) the mapping } \boldsymbol{x} \longmapsto \mathcal{A}(\boldsymbol{x}, \mathbf{0}) \text{ belongs to } \mathcal{H}. \end{array} \right. \tag{2.17}$$

$$\left\{ \begin{array}{l} \text{a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d, \\ \text{b) there exists } L_{\mathcal{G}} > 0 \text{ such that} \\ |\mathcal{G}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \leq L_{\mathcal{G}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|, \\ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \text{ such that} \\ \text{c) the mapping } \boldsymbol{x} \longmapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega, \forall \boldsymbol{\varepsilon} \in \mathbb{S}^d, \\ \text{d) the mapping } \boldsymbol{x} \longmapsto \mathcal{G}(\boldsymbol{x}, \mathbf{0}, 0) \text{ belongs to } \mathcal{H}. \end{array} \right. \tag{2.18}$$

The permeability tensor $\boldsymbol{\gamma}$ satisfy

$$\left\{ \begin{array}{l} \text{a) } \mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d, \\ \text{b) } \mathcal{E}(\boldsymbol{x}, \zeta) = \left(e_{ijk}(\boldsymbol{x})\zeta_{jk}\right), \quad \forall \zeta = \left(\zeta_{ij}\right) \in \mathbb{S}^d \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \text{c) } e_{ijk} = e_{ikj} \in L^\infty(\Omega) \end{array} \right. \tag{2.19}$$

$$\left\{ \begin{array}{l} \text{a) } \boldsymbol{\gamma} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \\ \text{b) } \boldsymbol{\gamma}(\boldsymbol{x}, \mathbf{E}) = \left(\gamma_{ij}(\boldsymbol{x})E_j\right), \quad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d \text{a.e. } \boldsymbol{x} \in \Omega, \\ \text{c) } \gamma_{ik} = \gamma_{ji} \in L^\infty(\Omega), \\ \text{d) there exists } m_\gamma > 0 \text{ such that } \gamma_{ij}(\boldsymbol{x})E_iE_j \geq m_\gamma \|\mathbf{E}\|^2, \\ \forall \mathbf{E} = (E_i) \in \mathbb{R}^d \text{a.e. } \boldsymbol{x} \in \Omega. \end{array} \right. \tag{2.20}$$

We also assume that the normal compliance function p satisfies

$$(2.21)\qquad \begin{cases} \text{a) } p:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+, \\ \text{b) there exists } L_p> \text{ such that } |p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2)|\le L_p|r_1-r_2|, \\ \forall r_1,r_2\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Gamma, \\ \text{c) the mapping } \boldsymbol{x}\longmapsto p(\boldsymbol{x},r) \text{ is Lebesgue mesurable in } \Gamma_3, \forall r\in\mathbb{R}, \\ \text{d) } r\le 0, \quad p(\boldsymbol{x},r)=0, \text{ a.e. } \boldsymbol{x}\in\Gamma_3. \end{cases}$$

As example of normal compliance functions which satisfy (2.21), we may consider $p(\boldsymbol{x},r)=cr_+$, where $c>0$ and $r_+=\max\{0,r\}$. This condition (2.21) means that the reaction of the obstacle is proportional to the penetration $(u_\nu)_+$. The gap function g and the initial potential φ_0 satisfy

$$(2.22)\qquad g\in L^2(\Gamma_3),\qquad g\ge 0 \text{ a.e. on } \Gamma_3,$$

$$(2.23)\qquad \varphi_0\in L^2(\Gamma_3),$$

We suppose that there exist a large positive constant L higher than any peak voltage in system such that $\varphi-\varphi_0$ is bounded by L. This condition do not pose any practical problem for the applicability of system, and allows us to introduced a function ϕ_L defined by

$$(2.24)\qquad \phi_L(s)\begin{cases} -L & \text{if } s<-L, \\ s & \text{if } -L<s<L, \\ L & \text{if } s>L. \end{cases}$$

This truncation is necessary for the solvability of the variational formulation of the problem. Note that ϕ_L is Lipshitz and monotone. We have the following assumptions

$$(2.25)\qquad \boldsymbol{u}_0\in V,\quad \boldsymbol{v}_0\in H.$$

The body forces and surfaces traction and free charges densities have the regularity, with $1\le p\le\infty$

$$(2.26)\qquad \boldsymbol{f}_0\in L^2\left(0,T;L^2(\Omega)^d\right),$$

$$(2.27)\qquad \boldsymbol{f}_2\in L^2\left(0,T;L^2(\Gamma_2)^d\right),$$

$$(2.28)\qquad q_0\in W^{1,p}\left(0,T;L^2(\Omega)\right),$$

$$(2.29)\qquad q_b\in W^{1,p}\left(0,T;L^2(\Gamma_b)\right).$$

We define the element $\boldsymbol{f}(t)\in V'$ by

$$(2.30)\qquad (\boldsymbol{f}(t),\boldsymbol{v})_{V',V}=(\boldsymbol{f}_0(t),\boldsymbol{v})_H+(\boldsymbol{f}_2(t),\boldsymbol{v})_{L^2(\Gamma_2)^d},\quad \forall\boldsymbol{v}\in V, \text{ a.e. } t\in(0,T),$$

and by using Riesz's representation theorem we define un element $q(t)\in W$ by

$$(2.31)\qquad (q(t),\xi)_W=-(q_0(t),\xi)_{L^2(\Omega)}-(q_b(t),\xi)_{L^2(\Gamma_b)},\ \forall\xi\in W, t\in(0,T),$$

we can see that conditions (2.27), (2.26) (2.28) and (2.29) imply that the data $\boldsymbol{f}\in W^{1,p}(0,T;V')$, $q\in W^{1,p}(0,T;W)$. Let $j:V\times V\to\mathbb{R}$ the functional defined by

$$(2.32)\qquad j(\boldsymbol{u},\boldsymbol{v})=\int_{\Gamma_3}p(u_\nu-g)v_\nu\,da, \forall\boldsymbol{u},\boldsymbol{v}\in V,$$

$l : V \times W \to W$ by

$$(2.33) \qquad (l\,(\boldsymbol{u},\varphi)\,,\xi) \;=\; \int_{\Gamma_3} \chi_{(0,+\infty[}\,(u_\nu - g)\,\phi_L\,(\varphi-\varphi_0)\,\xi\,da, \quad \forall \boldsymbol{u}\in V,\ \ \forall \xi,\varphi \in W.$$

By conditions (2.21), (2.22), (2.23) and (2.24), the integrals in (2.32) and in (2.33) are well defined. If $\{\boldsymbol{u},\varphi\}$ are regular functions satisfying (2.4)-(2.14), this imply that $\boldsymbol{u}(t)\in V$, $\varphi(t)\in W$ and keeping in mind the relations (2.1), (2.32), (2.33), we deduce the variational formulation of problem $\mathcal{P}$, noted $\mathcal{P}_V$.

Problem $\mathcal{P}_V$. Find a displacement field $\boldsymbol{u} : \Omega\times[0,T]\to\mathbb{R}^d$, and an electric potential $\varphi : \Omega\times[0,T]\to\mathbb{R}$ such that

$$(2.34) \qquad \begin{array}{l} (\ddot{\boldsymbol{u}}(t),\boldsymbol{v})_{V',V} + (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + \\ (\mathcal{E}^*\nabla\varphi(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + j\,(\boldsymbol{u}(t),\boldsymbol{v}) = (\boldsymbol{f}(t),\boldsymbol{v})_{V',V}\,,\ \forall\boldsymbol{v}\in V, \text{ a.e. } t\in(0,T)\,, \end{array}$$

$$(2.35) \qquad \begin{array}{c} (\boldsymbol{\gamma}\nabla\varphi(t),\nabla\xi)_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t),\nabla\xi)_{\mathcal{H}} + (l\,(\boldsymbol{u}(t),\varphi(t))\,,\xi)_W = (q(t),\xi)_W\,, \\ \forall\xi\in W, \text{ a.e. } t\in(0,T)\,, \end{array}$$

$$(2.36) \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0,\ \ \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0.$$

For the solvability of $\mathcal{P}_V$, we consider the truncation of the function $\chi_{(0,+\infty[}$ noted ψ_δ and its defined by

$$(2.37) \qquad \psi_\delta(r) = \begin{cases} 0 & \text{if } r<0, \\ k\frac{r}{\delta} & \text{if } 0\le r\le\delta, \\ k & \text{if } r\ge\delta, \end{cases}$$

δ is a small parameter which will tend to zero in the sequel. We can see that $\psi_\delta : \Gamma_3\times\mathbb{R}\to\mathbb{R}_+$, an increasing function which satisfies that

$$(2.38) \qquad \begin{array}{c} |\psi_\delta(u_1)-\psi_\delta(u_2)| \le k\,|u_1-u_2| \\ \forall u_1,u_2\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Gamma_3, \end{array}$$

Moreover, we assume that ψ_δ satisfies

a) the mapping $\boldsymbol{x}\longmapsto\psi_\delta(\boldsymbol{x},r)$ is Lebesgue measurable on Γ_3,$\forall r\in\mathbb{R}$,
b) for $r\le 0$, $\quad \psi_\delta(\boldsymbol{x},r)=0$ a.e. $\boldsymbol{x}\in\Gamma_3$.

Replacing $\chi_{(0,+\infty[}$ by the smooth function ψ_δ leads us to replacing the function l in P_V by a function h_δ defined from $V\times W\to W$ and

$$(2.39) \qquad (h_\delta\,(\boldsymbol{u},\varphi)\,,\xi) \;=\; \int_{\Gamma_3} \psi_\delta\,(u_\nu-g)\,\phi_L\,(\varphi-\varphi_0)\,\xi\,da, \quad \forall\boldsymbol{u}\in V, \forall\xi,\varphi\in W, \text{ a.e. } t\in(0,T)\,.$$

We introduce now a regularized problem $\mathcal{P}_R$.

Problem $\mathcal{P}_R$. Find a displacement field $\boldsymbol{u}_\delta : \Omega\times[0,T]\to\mathbb{R}^d$ and an electric potential $\varphi_\delta : \Omega\times[0,T]\to\mathbb{R}$, such that

$$(2.40) \qquad \begin{array}{l} (\ddot{\boldsymbol{u}}_\delta(t),\boldsymbol{v})_{V',V} + (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\delta(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_\delta(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \\ +\,(\mathcal{E}^*\nabla\varphi_\delta(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + j\,(\boldsymbol{u}_\delta(t),\boldsymbol{v}) = (\boldsymbol{f}(t),\boldsymbol{v})_{V',V}\,, \quad \forall\boldsymbol{v}\in V, \text{ a.e. } t\in(0,T)\,, \end{array}$$

$$(2.41) \qquad \begin{array}{c} (\boldsymbol{\gamma}\nabla\varphi_\delta(t),\nabla\xi)_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\delta(t),\nabla\xi)_{\mathcal{H}} + (h_\delta\,(\boldsymbol{u}_\delta(t),\varphi_\delta(t))\,,\xi)_W = (q(t),\xi)_W \\ \forall\xi\in W, \text{ a.e. } t\in(0,T)\,, \end{array}$$

$$(2.42) \qquad \boldsymbol{u}_\delta(0) = \boldsymbol{u}_0,\ \ \dot{\boldsymbol{u}}_\delta(0) = \boldsymbol{v}_0,$$

To simplify the notation we take in the first study $\boldsymbol{u}_\delta = \boldsymbol{u}$ and $\varphi_\delta = \varphi$, and we prove that are weak solutions of problem $\mathcal{P}_R$.

3. AN EXISTENCE AND UNIQUENESS RESULT FOR THE REGULARIZED PROBLEM

Theorem 3.1. *Assume that the conditions (2.17), (2.39) hold. Then there exists a unique solution of the problem $\mathcal{P}_R$. Moreover the solution satisfies*

$$\boldsymbol{u} \in W^{1,2}(0,T;V) \cap C^1([0,T];H), \quad \ddot{\boldsymbol{u}} \in L^2(0,T;V'), \quad \varphi \in W^{1,2}(0,T;W). \tag{3.1}$$

The proof of Theorem 3.1 will be carried on in several steps. It is based on results of evolution equations with monotone operators, Banach's fixed point theorem and the two following classical results on parabolic equations, see [2].

Theorem 3.2. *Let V and H be a real Hilbert spaces satisfying $V \subset H \subset V'$, with continuous and dense injection, and let $A : V \to V'$ be a hemicontinuous monotone operator which satisfies*

$$\exists \alpha_0 > 0, \alpha_1 \in \mathbb{R} \quad \textit{such that } (Au,u)_{V',V} \geq \alpha_0 \, |u|_V^2 + \alpha_1, \forall u \in V,$$

Then a given $u_0 \in H$ and $f \in L^2(0,T;V')$, there exist a unique function u which satisfies

$$u \in L^2(0,T;V) \cap C\,(0,T;H)\,, \quad \dot{u} \in L^2(0,T;V'),$$

$$\dot{u}(t) + Au(t) = f(t), \qquad \textit{a.e. } t \in (0,T)\,, \tag{3.2}$$

$$u(0) = u_0. \tag{3.3}$$

We assume first, that assumptions (2.17)-(2.29) hold and let $\eta \in L^2(0,T;V')$, we consider the following problems.

Problem $\mathcal{P}_\eta^1$. Find a displacement field $\boldsymbol{u}_\eta : \Omega \times [0,T] \to \mathbb{R}^d$ such that

$$\begin{gathered}(\ddot{\boldsymbol{u}}_\eta(t), \boldsymbol{v})_H + (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} = (\boldsymbol{f}(t), \boldsymbol{v})\,, \\ \forall \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T)\,,\end{gathered} \tag{3.4}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0, \ \dot{\boldsymbol{u}}_\eta(0) = \boldsymbol{v}_0, \tag{3.5}$$

Problem $\mathcal{P}_\eta^2$ Find an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{gathered}\left(\boldsymbol{\gamma}\nabla\varphi_\eta(t), \nabla\xi\right)_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t), \nabla\xi)_{\mathcal{H}} + \left(h_\delta\left(\boldsymbol{u}_\eta(t), \varphi_\eta(t)\right), \xi\right)_W = (q(t), \xi)_W\,, \\ \forall \xi \in W, \text{ a.e. on } (0,T)\,,\end{gathered} \tag{3.6}$$

Lemma 3.3. *there exists a unique solution to the problem $\mathcal{P}_1^\eta$. Moreover it satisfies*

$$\boldsymbol{u}_\eta \in W^{1,2}(0,T;V) \cap C^1([0,T];H), \qquad \ddot{\boldsymbol{u}}_\eta \in L^2(0,T;V'). \tag{3.7}$$

If $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ are two solutions of the problem $\mathcal{P}_1^\eta$ corresponding to the data $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in L^2(0,T;V')$, then there exist a constant $c > 0$ such that

$$|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|_V^2 \leq \frac{1}{m_{\mathcal{A}}} \left[\int_0^t |\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)|_H^2 \, ds\right]. \tag{3.8}$$

Proof. It is easy to see that the operator $A : V \to V$ defined by

$$(A\boldsymbol{u}, \boldsymbol{u})_{V,V} = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}\,, \tag{3.9}$$

is monotone and continuous (by conditions (2.17)) We recall that $\boldsymbol{f} - \boldsymbol{\eta} \in L^2(0,T;V')$ and $\boldsymbol{v}_0 \in H$, (see the conditions (2.30), (3.5)). We recall now Theorem 3.2, there exist a unique function $\boldsymbol{v}_\eta$ which satisfies

$$\boldsymbol{v}_\eta \in L^2(0,T;V) \cap C([0,T];H), \quad \dot{\boldsymbol{v}}_\eta \in L^2(0,T;V'), \tag{3.10}$$

(3.11) $$\dot{\boldsymbol{v}}_\eta(t) + A\boldsymbol{v}_\eta(t) = f(t), \qquad \text{a.e. } t \in (0,T)\,,$$

(3.12) $$\boldsymbol{v}_\eta(0) = \boldsymbol{v}_0.$$

Let $\boldsymbol{u}_\eta : [0,T] \to V$ be a function satisfying

(3.13) $$\boldsymbol{u}_\eta(t) = \int_0^t \boldsymbol{v}_\eta(s)\, ds + u_0.$$

Since $\boldsymbol{v}_\eta \in C([0,T]\,; H)$, then $\boldsymbol{u}_\eta$ is well defined, it is clear that using (3.9), (3.10), (3.11), (3.12), and (3.13), we deduce that $\boldsymbol{u}_\eta$ is a unique solution of problem $\mathcal{P}^1_\eta$, with the regularity (3.7). Let $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in L^2(0,T;V')$ for which we have respectively $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, are solutions of problems $\mathcal{P}^1_{\eta_i}$, $i = 1,2$. Keeping in mind that (2.17) and that $\boldsymbol{u}_1$, $\boldsymbol{u}_2 \in W^{1,2}(0,T;V)$ and we note $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ defined by (3.13). So we deduce that

$$m_{\mathcal{A}} \int_0^t |\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)|_V^2 \;\; ds \le \int_0^t |\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)|_{H'}^2 \;\; ds,$$

which implies

$$|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|_V^2 \le \frac{1}{m_{\mathcal{A}}} \int_0^t |\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)|_{H'}^2 \;\; ds.$$

∎

For existence and uniqueness of solution of the problem $\mathcal{P}^2_\eta$, it is based on monotonicity of the operator $\mathcal{A}$, the bounds $|\psi_\delta(u_\nu - g)| \le k$ and $|\phi_L(\varphi - \varphi_0)| \le L$ and the trace inequality (2.15).

Lemma 3.4. *There exists a unique solution*

(3.14) $$\varphi_\eta \in W^{1,2}(0,T;W),$$

of problem $\mathcal{P}^2_\eta$. If φ_1, φ_2 are two solutions of the problem $\mathcal{P}^2_\eta$ corresponding to the data $\boldsymbol{\eta}_1$and $\boldsymbol{\eta}_2 \in L^2(0,T;V')$, then there exist a constant $c > 0$ such that

(3.15) $$|\varphi_1(t) - \varphi_2(t)|_W \le c\, |\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|_V\,.$$

Proof. Let define the operator $A(t) : W \to W$, for $t \in [0,T]$, by

$$\begin{aligned} A_\eta(t)\varphi(t), \xi_W = {} & (\boldsymbol{\gamma}\nabla\varphi(t), \nabla\xi)_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t), \nabla\xi)_{\mathcal{H}} \\ & + \left(h_\delta\left(\boldsymbol{u}_\eta(t), \varphi(t)\right), \xi\right)_W, \qquad \forall \xi \in W. \end{aligned}$$

Let φ_1, $\varphi_2 \in W$, since $\boldsymbol{\gamma}$ satisfies (2.20), the function ϕ_L is monotone and $\psi_\delta \ge 0$, this implies

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \varphi_1 - \varphi_2)_W \ge m_\gamma\, |\varphi_1 - \varphi_2|_W^2\,.$$

Thus the operator $A_\eta(t)$ is strongly monotone. Now by Conditions (2.20) and (2.15),we have

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \xi)_W \le c\, |\varphi_1 - \varphi_2|_W\, |\xi|_W\,,$$

this implies that $A_\eta(t)$ is Lipshitz continuous. The equation $A_\eta(t)\varphi(t) = q(t)$, has a unique solution $\varphi_\eta(t) \in W$, for $q(t) \in W$. The function $\varphi_\eta(t)$ is then the unique solution of the problem $\mathcal{P}^2_\eta$. It is a classical result of evolutionary elliptic problems see for example [4].

Let now prove that $\varphi_\eta \in W^{1,2}(0,T;W)$, let t_1, $t_2 \in [0,T]$ and $\varphi_\eta(t_1) = \varphi_1$, $\varphi_\eta(t_2) = \varphi_2$, $\boldsymbol{u}_\eta(t_1) = \boldsymbol{u}_1$, $\boldsymbol{u}_\eta(t_2) = \boldsymbol{u}_2$, $q(t_1) = q_1$, $q(t_2) = q_2$, recall that the function ψ_δ is positive and

ϕ_L is monotone. Then we deduce from conditions (2.20), (2.15), (2.16), (2.19), (2.38), and the bounds of the functions ψ_δ, ϕ_L, that

$$\begin{aligned}
& m_\gamma \left|\varphi_1-\varphi_2\right|_W^2+\int_{\Gamma_3} \psi_\delta\left(u_{2\nu}-g\right)\left[\phi_L\left(\varphi_1-\varphi_0\right)-\phi_L\left(\varphi_2-\varphi_0\right)\right]\left(\varphi_1-\varphi_2\right) \\
& \leq\left[c_{\mathcal{E}}\left|\boldsymbol{u}_1-\boldsymbol{u}_2\right|_V+\left|q_1-q_2\right|_W\right]\left|\varphi_1-\varphi_2\right|_W \\
& +\int_{\Gamma_3}\left|\psi_\delta\left(u_{1\nu}-g\right)-\psi_\delta\left(u_{2\nu}-g\right)\right|\left|\phi_L\left(\varphi_1-\varphi_0\right)\right|\left|\varphi_1-\varphi_2\right| da, \\
& \leq\left[\left(c_{\mathcal{E}}+L k c_0 \bar{c}_0\right)\left|\boldsymbol{u}_1-\boldsymbol{u}_2\right|_V+\left|q_1-q_2\right|_W\right]\left|\varphi_1-\varphi_2\right|_W,
\end{aligned}$$

which implies that

$$\left|\varphi_1-\varphi_2\right|_W \leq c\left[\left|\boldsymbol{u}_1-\boldsymbol{u}_2\right|_V+\left|q_1-q_2\right|_W\right]. \tag{3.16}$$

Since $q_i \in W^{1,2}\left(0,T;W\right)$ and $u_i \in W^{1,2}(0,T;V)$, $i=1,2$, then we have $\varphi_\eta \in W^{1,2}(0,T;W)$. We show next that we have the estimation (3.15). Let $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in L^2(0,T;V')$ and φ_1, φ_2, are respectively solutions of problems $\mathcal{P}^2_{\eta_i}$, $i=1,2$, and $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, are solutions of problems $\mathcal{P}^1_{\eta_i}$, $i=1,2$. With similar arguments we deduce that

$$\left|\varphi_1-\varphi_2\right|_W \leq \frac{1}{m_\gamma}\left(c_{\mathcal{E}}+k L c_0 \bar{c}_0\right)\left|\boldsymbol{u}_1-\boldsymbol{u}_2\right|_V.$$

∎

Let now define the operator $\Lambda : L^2\left(0,T;V'\right) \rightarrow L^2\left(0,T;V'\right)$ by

$$(\Lambda \eta(t), v)_{V',V}=\left(\mathcal{G}\left(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta), \boldsymbol{\varepsilon}(\boldsymbol{v})\right)_{\mathcal{H}}+j(\boldsymbol{u}_\eta, \boldsymbol{v})+\left(\mathcal{E}^* \nabla \varphi_\eta, \boldsymbol{\varepsilon}(\boldsymbol{v})\right)\right)_{\mathcal{H}}, \tag{3.17}$$

where $\boldsymbol{u}_\eta$, φ_η are respectively the unique solutions of problems $\mathcal{P}^1_\eta$ and $\mathcal{P}^2_\eta$. We have the following result.

Theorem 3.5. *The operator Λ has a unique fixed point*

$$\boldsymbol{\eta}^* \in L^2\left(0,T;V'\right).$$

Proof. Let $t\in[0,T]$, $\eta_i \in L^2\left(0,T;V'\right)$, $i=1,2$, and use the notation $\boldsymbol{u}_{\eta_i}=\boldsymbol{u}_i$ and $\varphi_{\eta_i}=\varphi_i$, $i=1,2$. We apply proprieties (2.18 (b)), (2.19), (2.21) we deduce that

$$\left|\Lambda \boldsymbol{\eta}_1,-\Lambda \boldsymbol{\eta}_2\right|_{V'} \leq c\left[\left|\boldsymbol{u}_1-\boldsymbol{u}_2\right|_V+\left|\varphi_1-\varphi_2\right|_W\right].$$

This yields to

$$\left|\Lambda\left(\boldsymbol{\eta}_1\right)-\Lambda\left(\boldsymbol{\eta}_2\right)\right|_{L^2(0,T;V')} \leq c\left[\left|\boldsymbol{u}_1-\boldsymbol{u}_2\right|_{L^2(0,T,V)}+\left|\varphi_1-\varphi_2\right|_{L^2(0,T;W)}\right].$$

Apply now the inequalities (3.8), (3.15), we deduce

$$\left|\Lambda \boldsymbol{\eta}_1-\Lambda \boldsymbol{\eta}_2\right|_{L^2(0,T;V')} \leq T c\left[\left|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2\right|_{L^2(0,T;V')}\right],$$

reiterating the estimation n times yields

$$\left|\Lambda^n \boldsymbol{\eta}_1-\Lambda^n \boldsymbol{\eta}_2\right|_{L^2(0,T;V')} \leq \frac{(T c)^n}{n!}\left|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2\right|_{L^2(0,T;V')} \tag{3.18}$$

this implies that the operator Λ is a contraction on $L^2(0,T;V')$. By Banach's fixed point theorem, Λ has a unique fixed point $\boldsymbol{\eta}^* \in L^2\left(0,T;V'\right)$. ∎

We come back to the proof of Theorem 3.1

Proof. Existence. Let $\boldsymbol{\eta}^* \in L^2(0,T;V')$ a fixed point of Λ and $(\boldsymbol{u}_{\eta^*};\varphi_{\eta^*})$ be the solution of problems $\mathcal{P}^1_{\eta^*}$ and $\mathcal{P}^2_{\eta^*}$. Then using (2.40), (2.41) and (3.17), keeping in mind that $\Lambda(\boldsymbol{\eta}^*) = \boldsymbol{\eta}^*$, we deduce that $(\boldsymbol{u}_{\eta^*};\varphi_{\eta^*})$ is the solution of the regularized problem $\mathcal{P}_R$. The regularity (3.1) is provided by Lemma 3.3 and Lemma 3.4.

Uniqueness. the uniqueness of the Theorem 3.1 is the consequence of the uniqueness of the fixed point of the operator Λ given by (3.17). For more details see [11]. ∎

4. AN EXISTENCE AND UNIQUENESS RESULT OF THE PROBLEM $\mathcal{P}_V$

Theorem 4.1. *Assume that the conditions (2.17), (2.39) hold. Then there exists a unique solution of the problem $\mathcal{P}_V$. Moreover the solution satisfies*

$$\boldsymbol{u} \in W^{1,2}(0,T;V) \cap C^1([0,T];H),\ \ddot{\boldsymbol{u}} \in L^2(0,T;V),\ \varphi \in W^{1,2}(0,T;W). \tag{4.1}$$

Recall that the unique solution of the regularized problem $\mathcal{P}_R$, means that there exists a unique sequences $\boldsymbol{u}_\delta$ and φ_δ solutions for (2.40) and (2.41), with the initial conditions (2.42) and with regularity (3.1). To deduce that there exists a unique solution noted $\boldsymbol{u}$ and φ for equations (2.34), (2.35) with (2.36), we pass to the limit when $\delta \to 0$, in the problem $\mathcal{P}_R$, taking in consideration some a priori estimations on the sequences $\boldsymbol{u}_\delta$ and φ_δ, the proprieties of $\mathcal{A}$, $\mathcal{G}$, j, ϕ, h and some compactness results of evolutionary problems.

4.1. A priori estimates.

4.1.1. *Estimates on a sequence φ_δ.* Let replace $\xi = \varphi_\delta(t)$ in (2.41)

$$\begin{aligned}&(\boldsymbol{\gamma}\nabla\varphi_\delta(t), \nabla\varphi_\delta(t))_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\delta(t), \nabla\varphi_\delta(t))_{\mathcal{H}}\\ &+ \textstyle\int_{\Gamma_3} \psi_\delta\,(u_\nu(t) - g)\,\phi_L\,(\varphi_\delta(t) - \varphi_0)\,\varphi_\delta(t)\,da = (q(t), \varphi_\delta(t))_W\,,\end{aligned}$$

take into account (2.20), (2.19), (2.24), (2.31) and (2.37), then we have this estimate

$$|\varphi_\delta(t)|^2_W \le [c\,|\boldsymbol{u}_\delta(t)|_V + |q(t)|_W]\,|\varphi_\delta(t)|_W\,,$$

then

$$|\varphi_\delta|_{L^2(0,T;W)} \le c\,|\boldsymbol{u}_\delta|_{L^2(0,T;V)} + |q|_{L^2(0,T;W)}\,. \tag{4.2}$$

Here and above c denotes a generic positive constant which may depend on $\mathcal{A}$, $\mathcal{G}$, j, ϕ, h , Ω, Γ_1, Γ_2, Γ_a, Γ_b, Γ_3 and T and whose value may changes from line to line.

4.1.2. *Estimates on a sequence $\boldsymbol{u}_\delta$.* Keeping in mind that $\boldsymbol{u}_\delta \in W^{1,2}(0,T;V)$ and that $\mathcal{A}(\boldsymbol{x},\boldsymbol{0}) \in \mathcal{H}$, $\mathcal{G}(\boldsymbol{x},\boldsymbol{0},0) \in \mathcal{H}$ then replace $\boldsymbol{v}$=$\boldsymbol{v}_\delta(t) = \dot{u}_\delta(t)$ in (2.40)

$$\begin{aligned}&\tfrac{d}{dt}\,|\boldsymbol{v}_\delta(t)|^2_H + (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_\delta(t)) - \mathcal{A}(\boldsymbol{x},\boldsymbol{0}), \boldsymbol{\varepsilon}(\boldsymbol{v}_\delta(t)))_{\mathcal{H}}\\ &+ (\mathcal{A}(\boldsymbol{x},\boldsymbol{0}), \boldsymbol{\varepsilon}(\boldsymbol{v}_\delta(t)))_{\mathcal{H}} + (\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_\delta(t)) - \mathcal{G}(\boldsymbol{x},\boldsymbol{0}), \boldsymbol{\varepsilon}(\boldsymbol{v}_\delta(t)))_{\mathcal{H}}\\ &+ (\mathcal{G}(\boldsymbol{x},\boldsymbol{0},0), \boldsymbol{\varepsilon}(\boldsymbol{v}_\delta(t)))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_\delta(t), \boldsymbol{\varepsilon}(\boldsymbol{v}_\delta(t)))_{\mathcal{H}} + j(\boldsymbol{u}_\delta(t), \boldsymbol{v}_\delta(t))\\ &= (\boldsymbol{f}, \boldsymbol{v}_\delta(t))_{V'}\,,\end{aligned} \tag{4.3}$$

For convenience call $\mathcal{A}(\boldsymbol{x},\boldsymbol{0}) = \mathcal{A}_0$, $\mathcal{G}(\boldsymbol{x},\boldsymbol{0}) = \mathcal{G}_0$, Recall properties (2.17), (2.18), (2.21), and (2.16) then

$$\begin{aligned}&\frac{d}{dt}\,|\boldsymbol{v}_\delta(t)|^2_H + m_{\mathcal{A}}\,|\boldsymbol{v}_\delta(t)|^2_V\\ \le\ &(L_{\mathcal{G}} + L_p\tilde{c}_0)\,|\boldsymbol{u}_\delta(t)|_V\,|\boldsymbol{v}_\delta(t)|_V\\ &+ \left[c\,|\varphi_\delta(t)|_{W\ L^2(\Omega)} + |\boldsymbol{f}(\mathbf{t})|_{V'} + |\mathcal{G}_0|_{\mathcal{H}} + |\mathcal{A}_0|_{\mathcal{H}}\right]|\boldsymbol{v}_\delta(t)|_V\,,\end{aligned} \tag{4.4}$$

let $\alpha \in \mathbb{R}_+^*$, since

$$\begin{aligned} & \frac{d}{dt}\left|\boldsymbol{v}_\delta(t)\right|_H^2 + m_{\mathcal{A}}\left|\boldsymbol{v}_\delta(t)\right|_V^2 \\ \leq\ & \left(L_{\mathcal{G}} + L_p\tilde{c}_0\right)\frac{\alpha}{2}\left|\boldsymbol{u}_\delta(t)\right|_V^2 + \frac{1}{2\alpha}\left(L_{\mathcal{G}} + L_p\tilde{c}_0 + 1\right)\left|\boldsymbol{v}_\delta(t)\right|_V^2 \\ & + \frac{c\alpha}{2}\left[\left|\varphi_\delta(t)\right|_W^2 + \left|\boldsymbol{f}(t)\right|_{V'}^2 + \left|\mathcal{G}_0\right|_{\mathcal{H}}^2 + \left|\mathcal{A}_0\right|_{\mathcal{H}}^2\right], \end{aligned} \tag{4.5}$$

choose $\alpha < \frac{2m_{\mathcal{A}}}{(L_{\mathcal{G}}+L_p\tilde{c}_0+1)}$, and recall that

$$\left|\varphi_\delta(t)\right|_W \leq c\left|\boldsymbol{u}_\delta(t)\right|_V + \left|q(t)\right|_W .$$

We integrate from 0 to t in (4.5), we have

$$\begin{aligned} \left|\boldsymbol{v}_\delta(t)\right|_H^2 + c\left|\boldsymbol{v}_\delta\right|_{L^2(0,t;V)}^2 \leq & \left[\frac{1}{2\alpha}\left(L_{\mathcal{G}} + L_p\tilde{c}_0\right) + c\right]\left|\boldsymbol{u}_\delta\right|_{L^2(0,t;V)}^2 \\ & + \left|\boldsymbol{f}\right|_{L^2(0,T;V')}^2 + T\left|\mathcal{G}_0\right|_{\mathcal{H}}^2 + T\left|\mathcal{A}_0\right|_{\mathcal{H}}^2 + \left|\boldsymbol{v}_0\right|_H^2 . \end{aligned} \tag{4.6}$$

Let $\tilde{f} = \left|\boldsymbol{f}\right|_{L^2(0,T;V')}^2 + T\left|\mathcal{G}_0\right|_{\mathcal{H}}^2 + T\left|\mathcal{A}_0\right|_{\mathcal{H}}^2 + \left|\boldsymbol{v}_0\right|_H^2$, we deduce from (4.6)

$$\left|\boldsymbol{v}_\delta\right|_{L^2(0,t;V)}^2 \leq c\left|\boldsymbol{u}_\delta\right|_{L^2(0,t;V)}^2 + \tilde{f}. \tag{4.7}$$

From (3.1), we have that $u_\delta \in C^1(0, T.V)$, then

$$\boldsymbol{u}_\delta(t) = \int_0^t \boldsymbol{v}_\delta(s)ds + u_0,$$

this implies that

$$\left|\boldsymbol{u}_\delta(t)\right|_V^2 \leq c\left[\left|\boldsymbol{v}_\delta\right|_{L^2(0,t;V)}^2 + \left|u_0\right|_{L^2(0,T;V)}^2\right]. \tag{4.8}$$

From (4.7) and (4.8) we have

$$\left|\boldsymbol{u}_\delta(t)\right|_V^2 \leq \left|\boldsymbol{u}_\delta\right|_{L^2(0,t;V)}^2 + c \tag{4.9}$$

Apply Gronwall's lemma to the function $\left|\boldsymbol{u}_\delta(t)\right|_V^2$ we have

$$\left|\boldsymbol{u}_\delta\right|_{L^2(0,T;V)} \leq c. \tag{4.10}$$

From the estimate (4.10) of $\boldsymbol{u}_\delta$, we deduce those of φ_δ given respectively by (4.2),

$$\left|\varphi_\delta\right|_{L^2(0,T;W)} \leq c, \tag{4.11}$$

we also deduce from the boundedness of $(\boldsymbol{u}_\delta)$ in $L^2(0,T;V)$ and (4.7) that $\dot{\boldsymbol{u}}_\delta$ is bounded on $L^2(0,T;V)$,

$$\left|\dot{\boldsymbol{u}}_\delta\right|_{L^2(0,T;V)}^2 \leq c. \tag{4.12}$$

Now for the estimation of $\ddot{\boldsymbol{u}}_\delta$ in $L^2(0,T;V')$, recall the equation

$$\begin{aligned} & \left(\ddot{\boldsymbol{u}}_\delta(t), \boldsymbol{v}\right)_{V',V} = \mathcal{A}\left(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\delta(t)), \boldsymbol{\varepsilon}(\boldsymbol{v})\right)_{\mathcal{H}} - \left(\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_\delta(t)), \boldsymbol{\varepsilon}(\boldsymbol{v})\right)_{\mathcal{H}} \\ & - \left(\mathcal{E}^*\nabla\varphi_\delta(t), \boldsymbol{\varepsilon}(\boldsymbol{v})\right)_{\mathcal{H}} - j\left(\boldsymbol{u}_\delta(t), \boldsymbol{v}\right) + \left(\boldsymbol{f}(t), \boldsymbol{v}\right)_{V',V}, \quad \forall \boldsymbol{v} \in V, \end{aligned}$$

then

$$\begin{aligned} \left|\left(\ddot{\boldsymbol{u}}_\delta(t), \boldsymbol{v}\right)\right|_{V',V} \leq\ & \left[L_{\mathcal{A}}\left|\dot{\boldsymbol{u}}_\delta(t)\right|_V + L_{\mathcal{G}}\left|\boldsymbol{u}_\delta(t)\right|_V + \right. \\ & \left. c_{\mathcal{E}}\left|\nabla\varphi_\delta(t)\right|_H + L_p\left|\boldsymbol{u}_\delta(t)\right|_V + \left|\boldsymbol{f}(t)\right|_V\right]\left|\boldsymbol{v}\right|_V, \end{aligned}$$

so we have that

$$\left|\ddot{\boldsymbol{u}}_\delta\right|_{L^2(0;T,V')} \leq c. \tag{4.13}$$

4.2. **Passage to the limit ($\delta \to 0$).** Before going to the limit, we give the following result of compactness for evolutionary problems.

Lemma 4.2. *Let X, Y and H a three Banach's spaces such that $X \subseteq H \subseteq Y$, the injection $X \hookrightarrow H$ is compact. Let $\mathcal{F}$ a subset of $L^p(0,T;H)$, $1 \le p < \infty$ which satisfy*

a) $\mathcal{F}$ is bounded in $L^p(0,T;X)$,
b) $\frac{d\mathcal{F}}{dt} = \left\{\frac{df}{dt} \mid f \in \mathcal{F}\right\}$ is bounded in $L^1(0,T;Y)$,

then $\mathcal{F}$ is relatively compact in $L^p(0,T;H)$ and $\gamma\mathcal{F}$ the trace of $\mathcal{F}$ is relatively compact in $L^p(0,T;H)$ on the boundary Γ.

For the proof see for example [12].

A convergence of the sequence $(\boldsymbol{u}_\delta)$.

To apply a Lemma 4.2, let take $X = Y = H = V$ and $\mathcal{F} = \{\boldsymbol{u}_\delta\}$ then $X = V$, $H = H$, and $Y = V'$ for $\mathcal{F} = \{\boldsymbol{u}_\delta\}$, the conditions a) and b) are satisfied for $p = 2$. From (4.10), (4.12) $\mathcal{F} = \{\boldsymbol{u}_\delta\}$ is relatively compact in $L^2(0,T;V)$, and $\mathcal{F} = \{\boldsymbol{u}_\delta\}$ is relatively compact in $L^2(0,T;H)$. this implies that there exist a subsequence noted $(\boldsymbol{u}_\delta)$ such that

$$\boldsymbol{u}_\delta \to \boldsymbol{u}, \quad \text{strongly in } L^2(0,T;V), \tag{4.14}$$

the convergence in $L^2(0,T;V)$ imply that

$$\boldsymbol{u}_\delta \to \boldsymbol{u}, \quad \text{strongly in } L^2(0,T;H), \tag{4.15}$$

Therefore from (4.10), the relation (2.16) given on the trace of $\boldsymbol{u}_\delta$ and Lemma 4.2 there exist a subsequence $\{\gamma\boldsymbol{u}_\delta\}$ such that

$$\gamma\boldsymbol{u}_\delta \to \gamma\boldsymbol{u}, \quad \text{strongly in } L^2(0,T;L^2(\Gamma_3)), \tag{4.16}$$

and there exist a subsequence noted $\{\boldsymbol{u}_\delta\}$ such that

$$\boldsymbol{u}_\delta \to \boldsymbol{u}, \quad \text{strongly in } L^2(0,T;H). \tag{4.17}$$

Now the estimate (4.13) implies that

$$\ddot{\boldsymbol{u}}_\delta \rightharpoonup \ddot{\boldsymbol{u}}, \quad \text{weakly in } L^2(0,T;V'). \tag{4.18}$$

A convergence of the sequence (φ_δ).

From the estimation (4.11) we have

$$\varphi_\delta \rightharpoonup \varphi, \text{ weakly in } L^2(0,T;W), \tag{4.19}$$

and from the embedding $W \subset L^2(\Omega)$ which is compact, we conclude that there exist a subsequence (φ_δ) which satisfies

$$\varphi_\delta \to \varphi, \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{4.20}$$

Proof of theorem 4.1. All the convergences above (4.14) allow us to pass to the limit in the equations of the system $\mathcal{P}_R$. From the weak convergence (4.18) in $L^2(0,T;V')$ we have

$$(\ddot{\boldsymbol{u}}_\delta, \boldsymbol{v})_{V',V} \to (\ddot{\boldsymbol{u}}, \boldsymbol{v})_{V',V}\ , \quad \forall \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T)\,.$$

The strong convergence of first derivative $\dot{\boldsymbol{u}}_\delta$ in $L^2(0,T;H)$ with the assumption (2.17)(b) on $\mathcal{A}$ leads to

$$(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\delta(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \to (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}\,, \quad \forall \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T)\,.$$

The strong convergence of $\boldsymbol{u}_\delta$ respectively in the spaces $L^2(0,T;V)$ and $L^2(0,T;H)$ (see (4.15) with the property (2.18)(b) imply that

$$(\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}_\delta(t)), \boldsymbol{\varepsilon}(\boldsymbol{v})_{\mathcal{H}}) \to (\mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}\,, \tag{4.21}$$

$$\forall \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T)\,. \tag{4.22}$$

From the convergence (4.19) and the strong convergence of $\boldsymbol{u}_\delta$ in $L^2(0,T;H)$ with the assumption on p (2.21)(b) we have

$$(\mathcal{E}^*\nabla\varphi_\delta(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \to (\mathcal{E}^*\nabla\varphi(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}\ ,\quad \forall \boldsymbol{v}\in V,\text{ a.e. } t\in(0,T)\,,$$

and

$$j\left(\boldsymbol{u}_\delta(t),\boldsymbol{v}\right)\to j\left(\boldsymbol{u}(t),\boldsymbol{v}\right)\ ,\quad \forall \boldsymbol{v}\in V,\text{ a.e. } t\in(0,T)\,.$$

Therefore, we obtain the equation of displacement field of system $\mathcal{P}_V$. Recall now the regularized equation of the electric potential

$$(\boldsymbol{\gamma}\nabla\varphi_\delta(t),\nabla\xi)_{\mathcal{H}}-(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\delta(t),\nabla\xi)_{\mathcal{H}}+\left(h_\delta\left(\boldsymbol{u}_\delta(t),\varphi_\delta(t)\right),\xi\right)_W=(q(t),\xi)_W$$
$$\forall\xi\in W,\text{ a.e. on }(0,T)\,,$$

the convergence (4.19), (4.20), (4.14) allow us to pass to the limit on terms $(\boldsymbol{\gamma}\nabla\varphi_\delta(t),\nabla\xi)_{\mathcal{H}}$ and $(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\delta(t),\nabla\xi)_{\mathcal{H}}$. Remind that $h_\delta\left(\boldsymbol{u}_\delta(t),\varphi_\delta(t)\right)$ is

$$\left(h_\delta\left(\boldsymbol{u}_\delta(t),\varphi_\delta(t)\right),\xi\right)=\int_{\Gamma_3}\psi_\delta\left(u_{\delta\nu}(t)-g\right)\phi_L\left(\varphi_\delta(t)-\varphi_0\right)\xi\,da.$$

First we have the strong convergence (4.16)

$$u_{\delta\nu}(t)\to u_\nu(t),\ \text{strongly in } L^2(\Gamma_3),\text{ a.e. on }(0,T)\,,\tag{4.23}$$

secondly, we have for a pointwise value $r\in\mathbb{R}^+$,

$$\psi_\delta\left(r\right)\to k\chi_{[0,+\infty[}(r)\text{ when }\delta\to 0,$$

since ψ_δ is Lipshitz continuous with $\psi_\delta\left(0\right)=0$ we have

$$\left|\psi_\delta\left(u_\nu-g\right)\right|_{L^2(\Gamma_3)}\leq\left|u_\nu-g\right|_{L^2(\Gamma_3)}\,,$$

now with the dominated convergence theorem, we deduce that

$$\psi_\delta\left(u_\nu-g\right)\to k\chi_{[0,+\infty[}\left(u_\nu-g\right),\ \text{strongly in } L^2(\Gamma_3).\tag{4.24}$$

We also have

$$\begin{aligned}&\left|\psi_\delta\left(u_{\delta\nu}-g\right)-k\chi_{[0,+\infty[}\left(u_\nu-g\right)\right|_{L^2(\Gamma_3)}\\&=\left|\psi_\delta\left(u_{\delta\nu}-g\right)-\psi_\delta\left(u_\nu-g\right)+\psi_\delta\left(u_\nu-g\right)-k\chi_{[0,+\infty[}\left(u_\nu-g\right)\right|_{L^2(\Gamma_3)}\\&\leq\left|\psi_\delta\left(u_{\delta\nu}-g\right)-\psi_\delta\left(u_\nu-g\right)\right|_{L^2(\Gamma_3)}\\&\quad+\left|\psi_\delta\left(u_\nu-g\right)-k\chi_{[0,+\infty[}\left(u_\nu-g\right)\right|_{L^2(\Gamma_3)}\\&\leq k\left|u_{\delta\nu}-u_\nu\right|_{L^2(\Gamma_3)}+\left|\psi_\delta\left(u_\nu-g\right)-k\chi_{[0,+\infty[}\left(u_\nu-g\right)\right|_{L^2(\Gamma_3)}\,,\end{aligned}$$

using (4.23), (4.24) and the convergences above, we obtain the strong convergence

$$\psi_\delta\left(u_{\delta\nu}-g\right)\to k\chi_{[0,+\infty[}\left(u_\nu-g\right),\ \text{strongly in } L^2(\Gamma_3),\text{ a.e. in }(0,T)\,.$$

In another hand, since

$$\left|\phi_L\left(\varphi_\delta-\varphi_0\right)-\phi_L\left(\varphi-\varphi_0\right)\right|_{L^2(\Gamma_3)}\leq cL_\phi\left|\varphi_\delta-\varphi\right|_{L^2(\Gamma_3)}\,,$$

and from the strong convergence of φ_δ in $L^2(0,T;W)$, and trace Theorem (2.15), we have

$$\phi_L\left(\varphi_\delta-\varphi_0\right)\to\phi_L\left(\varphi-\varphi_0\right),\ \ \text{strongly in } L^2(\Gamma_3),\text{ a.e. on }(0,T)\,.$$

We get now

$$\psi_\delta\left(u_{\delta\nu}-g\right)\phi_L\left(\varphi_\delta-\varphi_0\right)\to k\chi_{[0,+\infty[}\left(u_\nu-g\right)\phi_L\left(\varphi-\varphi_0\right),\text{ a.e. in }\Gamma_3,$$

and because of the boundedness of the functions ϕ_L, ψ_δ and the dominated convergence theorem, we get

$$\psi_\delta\left(u_{\delta\nu}-g\right)\phi_L\left(\varphi_\delta-\varphi_0\right)\to k\chi_{[0,+\infty[}\left(u_\nu-g\right)\phi_L\left(\varphi-\varphi_0\right),\text{ strongly in } L^2(\Gamma_3),$$

consequently, we have that

$$h_\delta\left(\boldsymbol{u}_\delta, \varphi_\delta\right) \rightarrow l\left(\boldsymbol{u}, \varphi\right) \text{ , a.e. on } (0, T) .$$

We conclude now, that φ is a solution of the electric potential equation (2.35) of the system $\mathcal{P}_V$.

Before ending the existence of the solution to the problem $\mathcal{P}_V$, recall the strong convergence of $\boldsymbol{u}_\delta$ to $\boldsymbol{u}$ in $L^2(0, T; V)$ and of $\dot{\boldsymbol{u}}_\delta$ to $\dot{\boldsymbol{u}}$ in $L^2(0, T; H)$, allow us to obtain the initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0$. It is clear that the uniqueness of solutions $\boldsymbol{u}$ and φ is a consequence of the uniqueness of the limit. From the limit process, the solutions $\boldsymbol{u}$ and φ of the problem $\mathcal{P}_V$ have the same regularity of the sequences $\boldsymbol{u}_\delta$ and φ_δ. The proof of the Theorem 4.1 is complete. ∎

REFERENCES

[1] K. T. ANDREWS, A. KLARBRING, M. SHILLOR, and S. WRIGHT, A dynamic contact problem with friction and wear, *Int. J. Eng. Sci.*, **35** (1997), (14), pp. 1291–1309.

[2] V. BARBU, *Semi Groups and Differential Equations in Banach Spaces*, Noordhoff International, Leyden, The Nerthelands, (1976).

[3] P. BIZONGA, F. LEBON and F. MACERI, The unilateral frictional contact of piezoelectric body with a rigid support, in *Contact Mechanics*, J.A.C, Martin and Manuel D.P. Monteiro Marques (ed), Kluwer Dordrecht, (2002), pp. 347–354.

[4] H. BRESIS, Equations et inéquations non linéaires dans les espaces vectoriels en dualité, *Ann. Inst. Fourier, (Grenoble)* **18** (1968), pp. 115–175.

[5] T. IKEDA, *Fundamentals of Piezoelectricity*, Oxford University Press, Oxford, (1990).

[6] K. KUTTLER, M. SHILLOR and M. SOFONEA, Set-valued pseudomonotone maps and degenerate evolution inclusions, *Comm. Contemp. Math.*, **1** (1999), (1), pp. 87–123.

[7] K. L. KUTTLER, Quasistatic evolution of damage in an elastic-viscoplastic material, *Electron. J. Diff. Eqns.*, 2005, No. 147, pp. 1–25.

[8] Z. LARGUET, M. SHILLOR and M. SOFONEA, A frictional contact problem for an electro-viscoelastic body, *Electronic Journal of Differential Equations*, 2007, No. 170, pp. 1–17.

[9] R. D. MINDLIN, Polorisation gradient in elastic dielectrics, *Inst. J. Solids Structures*, **4** (1968), pp. 637–663.

[10] V. Z. PATRON and B. A. KUDRYAVTSEV, *Electromagnetoelasticity, Piezoelectrics and Electrically Conductive Solids*, Gordon & Breach, London,(1988).

[11] M. ROCHDI, M. SHILLOR and M. SOFONEA, Quasistatic viscoelastic contact compliance and friction, *Journal of Elasticity*, **51** (1998), pp. 105–126

[12] J. SIMON, Compact sets in the space $L^p(0, T; B)$, *Ann. Mat. Pur. Appl.*, **146** (1987), pp. 65–96.