The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 11, Issue 1, Article 8, pp. 1-13, 2014

SOME NEW NONLINEAR INTEGRO-DIFFERENTIAL INEQUALITIES OF GRONWALL-BELLMAN-PACHPATTE TYPE

A. ABDELDAIM

Received 19 August, 2013; accepted 7 February, 2014; published 26 June, 2014.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES,FACULTY OF SCIENCE,PORT SAID UNIVERSITY, PORT SAID, EGYPT.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND HUMANITIES, SHAQRA UNIVERSITY, DAWADMI, SAUDI ARABIA.
URL: http://faculty.ksu.edu.sa/DRABDELDAIM/Pages/Home.aspx
ahassen@su.edu.sa

ABSTRACT. In this paper we establish some new nonlinear integro-differential inequalities of Gronwall-Bellman-Pachpatte type for function of one independent variable. The purpose of this paper is to extend certain results which proved by Pachpatte in [On some fundamental integrodifferential and integral inequalities, An. Sti. Univ. Al. I. Cuza, Iasi, Vol.23 (1977), 77-86]. The inequalities obtained here can be used in the theory of some new classes of nonlinear integro-differential equations. Some applications are also given to illustrate the usefulness of our results.

Key words and phrases: Gronwall-Bellman inequality, Integro-differential inequalities, Integro-differential equations.

2010 *Mathematics Subject Classification.* Primary 26D10, 35A23. Secondary 34A34, 45J05, 47G20.

ISSN (electronic): 1449-5910

1. INTRODUCTION

After the classic word, "Inequalities", which appeared by Hardy et al. [12] in 1934, initiated the discovery of new types of inequalities and its applications in many parts of analysis. In the past several years there has been considerable interest in the application of differential and integral inequalities in many parts of analysis. One of the most used inequalities in theory of ordinary differential equations which due to Gronwall in 1919 [11]. Bellman R. in 1943 introduced the fundamental inequality named Gronwall-Bellman's inequality, this inequality plays a vital role in studying stability and asymptotic behaviour of solutions of differential equations, see [1]-[4],[6]-[9]. Integral inequalities involving functions and their derivatives have played a significant role in the developments of various branches of analysis, see [5, 10]. Pachpatte [13]-[18] has given some integral inequalities of the Gronwall-Bellman type involving functions and their derivatives which are useful in certain applications in theory of differential and integro-differential equations.
The aim of this paper is to extend certain results which proved by Pachpatte in [13] to obtain a new nonlinear integro-differential inequalities which can be used as handy tools to study the qualitative properties as well as the quantitative properties of solutions of some nonlinear integro-differential equations.

2. MAIN RESULTS

In this section, we state and prove some new nonlinear integro-differential inequalities of Gronwall-Bellman-Pachpatte type.

Theorem 2.1. *Let $x(t), \frac{dx(t)}{dt}$ and $k(t)$ be realvalued nonnegative continuous functions defined on $I = [0, \infty)$, for which the inequality*

$$\frac{dx(t)}{dt} \leq x_0 + \int_0^t k(s)[\frac{dx(s)}{ds}]^p[\frac{dx(s)}{ds} + x(s)]ds, \forall t \in I, \tag{2.1}$$

where x_0, p are positive constants. If $p \neq 1$, and $p[2x_0]^p \int_0^t k(s)e^{ps}ds < 1$ for all $t \in I$, then

$$\frac{dx(t)}{dt} \leq [x_0^{(1-p)} + (1-p)\int_0^t k(s)R(s)ds]^{\frac{1}{(1-p)}}, \forall t \in I, \tag{2.2}$$

where

$$R(t) = 2x_0e^t[1 - p[2x_0]^p \int_0^t k(s)e^{ps}ds]^{\frac{-1}{p}}, \forall t \in I. \tag{2.3}$$

Proof. Let $m(t)$ equal the right hand side in 2.1, we have $m(0) = x_0$ and

$$\frac{dx(t)}{dt} \leq m(t), \forall t \in I. \tag{2.4}$$

By taking $t = s$ in 2.4 and integration it from 0 to t, we have

$$x(t) \leq x_0 + \int_0^t m(s)ds, \forall t \in I.$$

Differentiating $m(t)$ with respect to t and using 2.4 and the above inequality we have

$$\frac{dm(t)}{dt} \leq k(t)m^p(t)n(t), \forall t \in I, \tag{2.5}$$

where $n(t) = m(t) + x_0 + \int_0^t m(s)ds, n(0) = 2x_0$. Differentiating $n(t)$ with respect to t and using the fact that $m(t) \leq n(t)$ and 2.5 we obtain

$$\frac{dn(t)}{dt} \leq k(t)n^{(p+1)}(t) + n(t) \Rightarrow n^{-(p+1)}(t)\frac{dn(t)}{dt} - n^{-p}(t) \leq k(t), \forall t \in I.$$

Let $z(t) = n^{-p}(t) \rightarrow [\frac{-1}{p}][\frac{dz(t)}{dt}] = n^{-(p+1)}(t)[\frac{dn(t)}{dt}]; z(0) = [2x_0]^{-p}$, then from the above inequality we have

$$\frac{dz(t)}{dt} + pz(t) \geq -pk(t), \forall t \in I.$$

The above inequality implies the estimation for $z(t)$ such that

$$z(t) \geq e^{-pt}[z(0) - p\int_0^t k(s)e^{ps}ds] \Rightarrow z(t) = n^{-p}(t) \geq R^{-p}(t) \Rightarrow n(t) \leq R(t), \forall t \in I,$$

where $R(t)$ is defined in 2.3. Thus from 2.5 and the above inequality we have

$$\frac{dm(t)}{dt} \leq k(t)R(t)m^p(t) \Rightarrow m^{-p}(t)\frac{dm(t)}{dt} \leq k(t)R(t), \forall t \in I.$$

By taking $t = s$ in the above inequality and integration it from 0 to t, we have

$$m(t) \leq [x_0^{(1-p)} + (1-p)\int_0^t k(s)R(s)ds]^{\frac{1}{(1-p)}}, \forall t \in I.$$

Using the above inequality in 2.4 we get the desired inequality in 2.2. This completes the proof. ∎

Theorem 2.2. *Let $x(t), \frac{dx(t)}{dt}$ and $k(t)$ be realvalued nonnegative continuous functions defined on $I = [0, \infty)$, for which the inequality*

$$\frac{dx(t)}{dt} \leq x_0 + \int_0^t k(s)[\frac{dx(s)}{ds}]^p[[\frac{dx(s)}{ds}]^2 + x(s)]ds, \forall t \in I, \tag{2.6}$$

where $x_0 > 1, p > 0$ are constants. If
$2(p+1)[x_0^2 + x_0]^{(p+1)}\int_0^t k(s)e^{(p+1)s}ds < 1, \forall t \in I$ *and* $p \neq 1$*, then*

$$\frac{dx(t)}{dt} \leq [x_0^{(1-p)} + (1-p)\int_0^t k(s)R_1(s)ds]^{\frac{1}{(1-p)}}, \forall t \in I, \tag{2.7}$$

where

$$R_1(t) = [x_0^2 + x_0]e^t[1 - 2(p+1)[x_0^2 + x_0]^{(p+1)}\int_0^t k(s)e^{(p+1)s}ds]^{\frac{-1}{(p+1)}}, \forall t \in I. \tag{2.8}$$

Proof. Let $m_1(t)$ equal the right hand side in 2.6, we have $m_1(0) = x_0$ and

$$\frac{dx(t)}{dt} \leq m_1(t), \forall t \in I. \tag{2.9}$$

By taking $t = s$ in 2.9 and integration it from 0 to t, we have

$$x(t) \leq x_0 + \int_0^t m_1(s)ds, \forall t \in I.$$

Differentiating $m_1(t)$ with respect to t and using 2.9 and the above inequality we have

$$\frac{dm_1(t)}{dt} \leq k(t)m_1^p(t)n_1(t), \forall t \in I, \tag{2.10}$$

where $n_1(t) = m_1^2(t) + x_0 + \int_0^t m_1(s)ds, n_1(0) = x_0^2 + x_0$, but $x_0 > 1 \rightarrow m_1(t) > 1$, thus we have

$$m_1(t) \leq m_1^2(t) \leq n_1(t), \forall t \in I.$$

Differentiating $n_1(t)$ with respect to t and using 2.10 and the above inequality we obtain

$$\frac{dn_1(t)}{dt} \leq 2k(t)n_1^{(p+2)}(t) + n_1(t) \Rightarrow n_1^{-(p+2)}(t)\frac{dn_1(t)}{dt} - n_1^{-(p+1)}(t) \leq 2k(t), \forall t \in I.$$

Let $z_1(t) = n_1^{-(p+1)}(t) \rightarrow [\frac{-1}{(p+1)}][\frac{dz_1(t)}{dt}] = n_1^{-(p+2)}(t)[\frac{dn_1(t)}{dt}]; z_1(0) = [x_0^2 + x_0]^{-(p+1)}$, then from the above inequality we have

$$\frac{dz_1(t)}{dt} + (p+1)z_1(t) \geq -2(p+1)k(t), \forall t \in I.$$

The above inequality implies the estimation for $z_1(t)$ such that

$$z_1(t) \geq e^{-(p+1)t}[z_1(0) - 2(p+1)\int_0^t k(s)e^{(p+1)s}ds], \forall t \in I \Rightarrow$$

$$z_1(t) = n_1^{-(p+1)}(t) \geq R_1^{-(p+1)}(t) \Rightarrow n_1(t) \leq R_1(t), \forall t \in I,$$

where $R_1(t)$ is defined in 2.8. Thus from 2.10 and the above inequality we have

$$\frac{dm_1(t)}{dt} \leq k(t)R_1(t)m_1^p(t) \Rightarrow m_1^{-p}(t)\frac{dm_1(t)}{dt} \leq k(t)R_1(t), \forall t \in I.$$

By taking $t = s$ in the above inequality and integration it from 0 to t, we have

$$m_1(t) \leq [x_0^{(1-p)} + (1-p)\int_0^t k(s)R_1(s)ds]^{\frac{1}{(1-p)}}, \forall t \in I.$$

Using the above inequality in 2.9 we get the desired inequality in 2.7. This completes the proof. ∎

Theorem 2.3. *Let $x(t)$, $\frac{dx(t)}{dt}$ and $k(t)$ be realvalued nonnegative continuous functions defined on $I = [0, \infty)$, for which the inequality*

$$\frac{dx(t)}{dt} \leq x_0 + \int_0^t k(s)[\frac{dx(s)}{ds}]^p[[\frac{dx(s)}{ds}]^p + x(s)]ds, \forall t \in I, \tag{2.11}$$

where $x_0 > 1$ and $p > 1$ are constants, if $(p-1)x_0^{(p-1)}\int_0^t k(s)R_2(s)ds < 1$, and $h(t) = p(2p-1)[x_0^p + x_0]^{(2p-1)}\int_0^t k(s)\exp((2p-1)s)ds < 1, \forall t \in I$, then

$$\frac{dx(t)}{dt} \leq x_0[1 - (p-1)x_0^{(p-1)}\int_0^t k(s)R_2(s)ds]^{\frac{-1}{(p-1)}}, \forall t \in I, \tag{2.12}$$

where

$$R_2(t) = [x_0^p + x_0]e^t[1 - h(t)]^{\frac{-1}{(2p-1)}}, \forall t \in I. \tag{2.13}$$

Proof. Let $m_2(t)$ equal the right hand side in 2.11, we have $m_2(0) = x_0$ and

$$\frac{dx(t)}{dt} \leq m_2(t), \forall t \in I. \tag{2.14}$$

By taking $t = s$ in 2.14 and integration it from 0 to t, we have

$$x(t) \leq x_0 + \int_0^t m_2(s)ds, \forall t \in I.$$

Differentiating $m_2(t)$ with respect to t and using 2.14 and the above inequality we have

$$\frac{dm_2(t)}{dt} \leq k(t)m_2^p(t)n_2(t), \forall t \in I, \tag{2.15}$$

where $n_2(t) = m_2^p(t) + x_0 + \int_0^t m_2(s)ds; n_2(0) = x_0^p + x_0$, but $x_0 > 1$ and $p > 1$, thus we have

$$m_2(t) > 1 \Rightarrow m_2(t) \leq m_2^p(t) \leq n_2(t), \forall t \in I.$$

Differentiating $n_2(t)$ with respect to t and using 2.15 and the above inequality we obtain

$$\frac{dn_2(t)}{dt} \leq pk(t)n_2^{2p}(t) + n_2(t) \Rightarrow n_2^{-2p}(t)\frac{dn_2(t)}{dt} - n_2^{-(2p-1)}(t) \leq pk(t), \forall t \in I.$$

Let $z_2(t) = n_2^{-(2p-1)}(t) \to [\frac{-1}{(2p-1)}][\frac{dz_2(t)}{dt}] = n_2^{-2p}(t)[\frac{dn_2(t)}{dt}]; z_2(0) = [x_0^p + x_0]^{-(2p-1)}$, then from the above inequality and the fact that $(2p-1) > 0$ we have

$$\frac{dz_2(t)}{dt} + (2p-1)z_2(t) \geq -p(2p-1)k(t), \forall t \in I.$$

The above inequality implies the estimation for $z_2(t)$ such that

$$z_2(t) \geq e^{-(2p-1)t}[z_2(0) - p(2p-1)\int_0^t k(s)e^{(2p-1)s}ds], \forall t \in I \Rightarrow$$

$$z_2(t) = n_2^{-(2p-1)}(t) \geq R_2^{-(2p-1)}(t) \Rightarrow n_2(t) \leq R_2(t), \forall t \in I,$$

where $R_2(t)$ is defined in 2.13. Thus from 2.15 and the above inequality we have

$$\frac{dm_2(t)}{dt} \leq k(t)m_2^p(t)R_2(t) \Rightarrow m_2^{-p}(t)\frac{dm_2(t)}{dt} \leq k(t)R_2(t), \forall t \in I.$$

By taking $t = s$ in the above inequality and integration it from 0 to t and using the fact that $p > 1 \to (p-1) > 0$, we have

$$m_2(t) \leq x_0[1 - (p-1)x_0^{(p-1)}\int_0^t k(s)R_2(s)ds]^{\frac{-1}{(p-1)}}, \forall t \in I.$$

Using the above inequality in 2.14 we get the desired inequality in 2.12. This completes the proof. ∎

Theorem 2.4. *Let $x(t), \frac{dx(t)}{dt}$ and $k(t)$ be realvalued nonnegative continuous functions defined on $I = [0, \infty)$, for which the inequality*

$$\frac{dx(t)}{dt} \leq x_0 + \int_0^t k(s)[\frac{dx(s)}{ds}]^p[[\frac{dx(s)}{ds}] + x^2(s)]ds, \forall t \in I \tag{2.16}$$

where $x_0 > 1$ and $p > 1$ are constants, if $(p-1)x_0^{(p-1)}\int_0^t k(s)R_3(s)ds < 1$,

$$g(t) = p(2p-1)[[x_0^2 + x_0]^p + x_0]^{(2p-1)}\int_0^t k(s)\exp((2p-1)s)ds < 1,$$

$$h_2(t) = p[x_0^2 + x_0]^p\int_0^t [k(s) - 2]\exp(2p\int_0^s Q(\lambda)d\lambda)ds < 1, \forall t \in I,$$

then

$$\frac{dx(t)}{dt} \leq x_0[1 - (p-1)x_0^{(p-1)}\int_0^t k(s)R_3(s)ds]^{\frac{-1}{(p-1)}}, \forall t \in I, \tag{2.17}$$

where

$$R_3(t) = [x_0^2 + x_0][1 - h_2(t)]^{\frac{-1}{p}}\exp(2\int_0^t Q(s)ds), \forall t \in I, \tag{2.18}$$

$$Q(t) = [[x_0^2 + x_0]^p + x_0]e^t[1 - g(t)]^{\frac{-1}{(2p-1)}}, \forall t \in I. \tag{2.19}$$

Proof. Let $m_3(t)$ equal the right hand side in 2.16 we have $m_3(0) = x_0$ and

$$\frac{dx(t)}{dt} \le m_3(t), \forall t \in I. \tag{2.20}$$

By taking $t = s$ in 2.20 and integration it from 0 to t, we have

$$x(t) \le x_0 + \int_0^t m_3(s)ds, \forall t \in I.$$

Differentiating $m_3(t)$ with respect to t and using the above inequality and 2.20 we obtain

$$\frac{dm_3(t)}{dt} \le k(t)m_3^p(t)n_3(t), \forall t \in I, \tag{2.21}$$

where $n_3(t) = m_3(t) + x_0^2 + 2x_0 \int_0^t m_3(s)ds + [\int_0^t m_3(s)ds]^2; n_3(0) = x_0^2 + x_0 \forall t \in I$. Differentiating $n_3(t)$ with respect to t and using the fact that $m_3(t) \le n_3(t)$ and 2.21 we obtain

$$\frac{dn_3(t)}{dt} \le k(t)n_3^{(p+1)}(t) + 2[x_0 + \int_0^t n_3(s)ds]n_3(t), \forall t \in I,$$

thus

$$\frac{dn_3(t)}{dt} \le k(t)n_3^{(p+1)}(t) - 2n_3^{(p+1)}(t) + 2n_3^{(p+1)}(t) + 2[x_0 + \int_0^t n_3(s)ds]n_3(t) \Rightarrow$$

$$\frac{dn_3(t)}{dt} \le n_3^{(p+1)}(t)[k(t) - 2] + 2n_3(t)z_3(t), \forall t \in I, \tag{2.22}$$

where $z_3(t) = n_3^p(t) + x_0 + \int_0^t n_3(s)ds$, where $z_3(0) = [x_0^2 + x_0]^p + x_0$ for all $t \in I$, but $x_0 > 1$ and $p > 1$ thus we have

$$n_3(t) > 1 \Rightarrow n_3(t) \le n_3^p(t) \le z_3(t), \forall t \in I.$$

Differentiating $z_3(t)$ with respect to t and using the above inequality and 2.22 we obtain

$$\frac{dz_3(t)}{dt} \le pz_3^{2p}(t)[k(t) - 2] + 2pz_3^{(p+1)}(t) + z_3(t), \forall t \in I,$$

but $x_0 > 1 \to z_3(t) > 1$, and $1 < p \to p + 1 < 2p \to z_3^{(p+1)}(t) \le z_3^{2p}(t)$, thus from the above inequality we have

$$\frac{dz_3(t)}{dt} \le pz_3^{2p}(t)k(t) + z_3(t), \Rightarrow z_3^{-2p}(t)\frac{dz_3(t)}{dt} - z_3^{-(2p-1)}(t) \le pk(t), \forall t \in I.$$

Define $w(t) = z_3^{-(2p-1)}(t) \to w(0) = z_3^{-(2p-1)}(0) = [[x_0^2+x_0]^p+x_0]^{-(2p-1)}$, and $[\frac{-1}{(2p-1)}][\frac{dw(t)}{dt}] = z_3^{-2p}(t)[\frac{dz_3(t)}{dt}], \forall t \in I$. Thus from the above inequality we have

$$\frac{dw(t)}{dt} + (2p - 1)w(t) \ge -p(2p - 1)k(t), \forall t \in I.$$

The above inequality implies the estimation for $w(t)$ such that

$$w(t) \ge \exp(-(2p - 1)t)[w(0) - p(2p - 1) \int_0^t k(s)exp((2p - 1)s)ds] \Rightarrow$$

$$w(t) = z_3^{-(2p-1)}(t) \ge Q^{-(2p-1)}(t) \Rightarrow z_3(t) \le Q(t), \forall t \in I,$$

where $Q(t)$ is defined in 2.19. From the above inequality and 2.22 we obtain

$$\frac{dn_3(t)}{dt} \le n_3^{(p+1)}(t)[k(t) - 2] + 2n_3(t)Q(t) \Rightarrow n_3^{-(p+1)}(t)[\frac{dn_3(t)}{dt}] - 2n_3^{-p}(t)Q(t) \le [k(t) - 2],$$

for all $t \in I$. Define $u(t) = n_3^{-p}(t) \rightarrow [\frac{-1}{p}][\frac{du(t)}{dt}] = n_3^{-(p+1)}(t)[\frac{dn_3(t)}{dt}], \forall t \in I$, and $u(0) = [x_0^2 + x_0]^{-p}$. Thus from the above inequality we have

$$\frac{du(t)}{dt} + 2pu(t)Q(t) \geq -p[k(t) - 2], \forall t \in I.$$

The above inequality implies the estimation for $u(t)$ such that

$$u(t) \geq \exp(-2p \int_0^t Q(s)ds)[u(0) - p \int_0^t [k(s) - 2] \exp(2p \int_0^s Q(\lambda)d\lambda))ds] \Rightarrow$$

$$u(t) = n_3^{-p}(t) \geq R_3^{-p}(t) \Rightarrow n_3(t) \leq R_3(t), \forall t \in I,$$

where $R_3(t)$ is defined in 2.18. From the above inequality and 2.21 we obtain

$$\frac{dm_3(t)}{dt} \leq k(t)m_3^p(t)R_3(t) \Rightarrow m_3^{-p}(t)\frac{dm_3(t)}{dt} \leq k(t)R_3(t), \forall t \in I.$$

By taking $t = s$ in the above inequality and integration it from 0 to t and using the fact that $p > 1 \rightarrow (p-1) > 0$, we have

$$m_3(t) \leq x_0[1 - (p-1)x_0^{(p-1)} \int_0^t k(s)R_3(s)ds]^{\frac{-1}{(p-1)}}, \forall t \in I.$$

Using the above inequality in 2.20 we get the desired inequality in 2.17. This completes the proof. ∎

Theorem 2.5. *Let $x(t)$, $\frac{dx(t)}{dt}$ and $k(t)$ be realvalued nonnegative continuous functions defined on $I = [0, \infty)$, for which the inequality*

$$\frac{dx(t)}{dt} \leq x_0 + \int_0^t k(s)[\frac{dx(s)}{ds}]^p[[\frac{dx(s)}{ds}]^2 + x^2(s)]ds, \forall t \in I, \tag{2.23}$$

where $x_0 > 1$, $p > 1$ are constants, if $(p-1)x_0^{(p-1)} \int_0^t k(s)R_4(s)ds < 1$, and

$$g_1(t) = 2(p+1)[x_0 + [2x_0^2]^{(p+1)}] \int_0^t k(s)e^s ds < 1,$$

$$h_3(t) = 2(p+1)[2x_0^2]^{(p+1)} \int_0^t [k(s) - 1] \exp(2(p+1) \int_0^s Q_1(\lambda)d\lambda)ds < 1,$$

for all $t \in I$, then

$$\frac{dx(t)}{dt} \leq x_0[1 - (p-1)x_0^{(p-1)} \int_0^t k(s)R_4(s)ds]^{\frac{-1}{(p-1)}}, \forall t \in I, \tag{2.24}$$

where

$$R_4(t) = 2x_0^2[1 - h_3(t)]^{\frac{-1}{(p+1)}} \exp(2 \int_0^t Q_1(s)ds), \forall t \in I, \tag{2.25}$$

$$Q_1(t) = [x_0 + [2x_0^2]^{(p+1)}]e^t[1 - g_1(t)]^{-1}, \forall t \in I. \tag{2.26}$$

Proof. Let $m_4(t)$ equal the right hand side in 2.23 we have $m_4(0) = x_0$ and

$$\frac{dx(t)}{dt} \leq m_4(t), \forall t \in I. \tag{2.27}$$

By taking $t = s$ in 2.27 and integration it from 0 to t, we have

$$x(t) \leq x_0 + \int_0^t m_4(s)ds, \forall t \in I.$$

Differentiating $m_4(t)$ with respect to t and using the above inequality and 2.27 we obtain

$$\frac{dm_4(t)}{dt} \le k(t)m_4^p(t)n_4(t), \forall t \in I, \tag{2.28}$$

where $n_4(t) = m_4^2(t) + x_0^2 + 2x_0 \int_0^t m_4(s)ds + [\int_0^t m_4(s)ds]^2, n_4(0) = 2x_0^2, \forall t \in I$ but, $x_0 > 1 \to m_4(t) > 1$, thus we have

$$m_4(t) \le m_4^2(t) \le n_4(t), \forall t \in I.$$

Differentiating $n_4(t)$ with respect to t and using 2.28 and the above inequality we obtain

$$\frac{dn_4(t)}{dt} \le 2k(t)n_4^{(p+2)}(t) + 2[x_0 + \int_0^t n_4(s)ds]n_4(t), \forall t \in I,$$

thus

$$\frac{dn_4(t)}{dt} \le 2k(t)n_4^{(p+2)}(t) - 2n_4^{(p+2)}(t) + 2n_4^{(p+2)}(t) + 2[x_0 + \int_0^t n_4(s)ds]n_4(t) \Rightarrow$$

$$\frac{dn_4(t)}{dt} \le 2[k(t)-1]n_4^{(p+2)}(t) + 2n_4(t)z_4(t), \forall t \in I, \tag{2.29}$$

where $z_4(t) = n_4^{(p+1)}(t) + x_0 + \int_0^t n_4(s)ds$, where $z_4(0) = [x_0 + [2x_0^2]^{(p+1)}]$ for all $t \in I$. Differentiating $z_4(t)$ with respect to t and using 2.29 we obtain

$$\frac{dz_4(t)}{dt} \le 2(p+1)[k(t)-1]n_4^{2(p+1)}(t) + 2(p+1)n_4^{(p+1)}(t)z_4(t) + n_4(t), \forall t \in I,$$

but, $n_4^{(p+1)}(t) \le z_4(t)$, thus from the above inequality we have

$$\frac{dz_4(t)}{dt} \le 2(p+1)k(t)z_4^2(t) + z_4^{\frac{1}{(p+1)}}(t), \forall t \in I,$$

but, $x_0 > 1; p > 1 \to z_4(t) > 1 \to z_4^{\frac{1}{(p+1)}}(t) \le z_4(t)$, for all $t \in I$, thus from the above inequality we have

$$\frac{dz_4(t)}{dt} \le 2(p+1)k(t)z_4^2(t) + z_4(t) \Rightarrow z_4^{-2}(t)\frac{dz_4(t)}{dt} - z_4^{-1}(t) \le 2(p+1)k(t), \forall t \in I.$$

Define $w_1(t) = z_4^{-1}(t), w_1(0) = [x_0 + [2x_0^2]^{(p+1)}]^{-1}$, and $-\frac{dw_1(t)}{dt} = z_4^{-2}(t)[\frac{dz_4(t)}{dt}]$, for all $t \in I$. Thus from the above inequality we have

$$\frac{dw_1(t)}{dt} + w_1(t) \ge -2(p+1)k(t), \forall t \in I.$$

The above inequality implies the estimation for $w_1(t)$ such that

$$w_1(t) \ge e^{-t}[w_1(0) - 2(p+1)\int_0^t k(s)e^s ds] \Rightarrow$$

$$w_1(t) = z_4^{-1}(t) \ge Q_1^{-1}(t) \Rightarrow z_4(t) \le Q_1(t), \forall t \in I,$$

where $Q_1(t)$ is defined in 2.26. From the above inequality and 2.29 we obtain

$$\frac{dn_4(t)}{dt} \le 2[k(t)-1]n_4^{(p+2)}(t) + 2n_4(t)Q_1(t) \Rightarrow n_4^{-(p+2)}(t)[\frac{dn_4(t)}{dt}] - 2n_4^{-(p+1)}(t)Q_1(t) \le 2[k(t)-1],$$

for all $t \in I$. Define $u_1(t) = n_4^{-(p+1)}(t) \to [\frac{-1}{(p+1)}][\frac{du_1(t)}{dt}] = n_4^{-(p+2)}(t)[\frac{dn_4(t)}{dt}]$, and $u_1(0) = [2x_0^2]^{-(p+1)}$. Thus from the above inequality we have

$$\frac{du_1(t)}{dt} + 2(p+1)u_1(t)Q_1(t) \ge -2(p+1)[k(t)-1], \forall t \in I.$$

The above inequality implies the estimation for $u_1(t)$ such that

$$u_1(t) \geq \exp(-2(p+1)\int_0^t Q_1(s)ds)[u_1(0)-2(p+1)\int_0^t [k(s)-1]\exp(2(p+1)\int_0^s Q_1(\lambda)d\lambda)ds] \Rightarrow$$

$$u_1(t) = n_4^{-(p+1)}(t) \geq R_4^{-(p+1)}(t) \Rightarrow n_4(t) \leq R_4(t), \forall t \in I,$$

where $R_4(t)$ is defined in 2.25. From the above inequality and 2.28 we obtain

$$\frac{dm_4(t)}{dt} \leq k(t)m_4^p(t)R_4(t) \Rightarrow m_4^{-p}(t)\frac{dm_4(t)}{dt} \leq k(t)R_4(t), \forall t \in I.$$

By taking $t = s$ in the above inequality and integration it from 0 to t and using the fact that $p > 1 \rightarrow (p-1) > 0$, we have

$$m_4(t) \leq x_0[1-(p-1)x_0^{(p-1)}\int_0^t k(s)R_4(s)ds]^{\frac{-1}{(p-1)}}, \forall t \in I.$$

Using the above inequality in 2.27 we get the desired inequality in 2.24. This completes the proof. ∎

Theorem 2.6. *Let $x(t)$, $\frac{dx(t)}{dt}$ and $k(t)$ be realvalued nonnegative continuous functions defined on $I = [0,\infty)$, for which the inequality*

$$\frac{dx(t)}{dt} \leq x_0 + \int_0^t k(s)[\frac{dx(s)}{ds}]^p[[\frac{dx(s)}{ds}]^p + x^2(s)]ds, \forall t \in I, \tag{2.30}$$

where $x_0 > 1$ and $p > 1$ are constants, if $(p-1)x_0^{(p-1)}\int_0^t k(s)R_5(s)ds < 1$,

$$g_2(t) = p(2p-1)[x_0 + [x_0^p + x_0^2]^{(2p-1)}]\int_0^t k(s)e^s ds < 1,$$

$$h_4(t) = (2p-1)\int_0^t [pk(s)-2][[x_0^p + x_0^2]\exp(2\int_0^s Q_2(\lambda)d\lambda)]^{(2p-1)}ds < 1,$$

for all $t \in I$, then

$$\frac{dx(t)}{dt} \leq x_0[1-(p-1)x_0^{(p-1)}\int_0^t k(s)R_5(s)ds]^{\frac{-1}{(p-1)}}, \forall t \in I, \tag{2.31}$$

where

$$R_5(t) = [[x_0^p + x_0^2]\exp(2\int_0^t Q_2(s)ds)][1-h_4(t)]^{\frac{-1}{(2p-1)}}, \tag{2.32}$$

$$Q_2(t) = [x_0 + [x_0^p + x_0^2]^{(2p-1)}]e^t[1-g_2(t)]^{-1}, \forall t \in I. \tag{2.33}$$

Proof. Let $m_5(t)$ equal the right hand side in 2.30 we have $m_5(0) = x_0$ and

$$\frac{dx(t)}{dt} \leq m_5(t), \forall t \in I. \tag{2.34}$$

By taking $t = s$ in 2.34 and integration it from 0 to t, we have

$$x(t) \leq x_0 + \int_0^t m_5(s)ds, \forall t \in I.$$

Differentiating $m_5(t)$ with respect to t and using 2.34 and the above inequality we obtain

$$\frac{dm_5(t)}{dt} \leq k(t)m_5^p(t)n_5(t), \forall t \in I, \tag{2.35}$$

where $n_5(t) = m_5^p(t) + x_0^2 + 2x_0 \int_0^t m_5(s)ds + [\int_0^t m_5(s)ds]^2$; $n_5(0) = x_0^p + x_0^2$, but, $x_0 > 1, p > 1 \rightarrow m_5(t) > 1$, thus we have

$$m_5(t) \leq m_5^p(t) \leq n_5(t), \forall t \in I.$$

Differentiating $n_5(t)$ with respect to t and using 2.35 and the above inequality we obtain

$$\frac{dn_5(t)}{dt} \leq pk(t)n_5^{2p}(t) + 2n_5(t)[x_0 + \int_0^t n_5(s)ds], \forall t \in I,$$

hence

$$\frac{dn_5(t)}{dt} \leq pk(t)n_5^{2p}(t) - 2n_5^{2p}(t) + 2n_5^{2p}(t) + 2n_5(t)[x_0 + \int_0^t n_5(s)ds], \forall t \in I, \Rightarrow$$

$$\frac{dn_5(t)}{dt} \leq n_5^{2p}(t)[pk(t) - 2] + 2n_5(t)z_5(t), \forall t \in I, \tag{2.36}$$

where $z_5(t) = n_5^{(2p-1)}(t) + x_0 + \int_0^t n_5(s)ds, z_5(0) = x_0 + [x_0^p + x_0^2]^{(2p-1)}$, for all $t \in I$. Differentiation $z_5(t)$ with respect to t and using 2.36 and the fact that $n_5^{(2p-1)}(t) \leq z_5(t)$ we have

$$\frac{dz_5(t)}{dt} \leq p(2p-1)k(t)z_5^2(t) + n_5(t), \forall t \in I,$$

but $x_0 > 1 \rightarrow n_5(t) > 1$ and $p > 1 \rightarrow (2p-1) > 1 \rightarrow n_5(t) \leq n_5^{(2p-1)}(t) \leq z_5(t)$, then from the above inequality we have

$$\frac{dz_5(t)}{dt} \leq p(2p-1)k(t)z_5^2(t) + z_5(t) \Rightarrow z_5^{-2}(t)\frac{dz_5(t)}{dt} - z_5^{-1}(t) \leq p(2p-1)k(t), \forall t \in I.$$

Let $w_2(t) = z_5^{-1}(t) \rightarrow w_2(0) = [x_0 + [x_0^p + x_0^2]^{(2p-1)}]^{-1}$; and, $\frac{dw_2(t)}{dt} = z_5^{-2}(t)[\frac{dz_5(t)}{dt}]$, then from the above inequality we obtain

$$\frac{dw_2(t)}{dt} + w_2(t) \geq -p(2p-1)k(t), \forall t \in I.$$

The above inequality implies the following estimation for $w_2(t)$ such that

$$w_2(t) \geq e^{-t}[w_2(0) - p(2p-1)\int_0^t k(s)e^s ds] \Rightarrow w_2(t) = z_5^{-1}(t) \geq Q_2^{-1}(t) \Rightarrow z_5(t) \leq Q_2(t), \forall t \in I,$$

where $Q_2(t)$ is defined in 2.33. From 2.36 and the above inequality we have

$$\frac{dn_5(t)}{dt} \leq n_5^{2p}(t)[pk(t) - 2] + 2n_5(t)Q_2(t), \forall t \in I \Rightarrow$$

$$n_5^{-2p}(t)\frac{dn_5(t)}{dt} - 2n_5^{-(2p-1)}(t)Q_2(t) \leq [pk(t) - 2], \forall t \in I.$$

Let $u_2(t) = n_5^{-(2P-1)}(t) \rightarrow [\frac{-1}{(2p-1)}][\frac{du_2(t)}{dt}] = n_5^{-2p}(t)[\frac{dn_5(t)}{dt}]$ for all $t \in I$ and $u_2(0) = [x_0^p + x_0^2]^{-(2p-1)}$, then from the above inequality we obtain

$$\frac{du_2(t)}{dt} + 2(2p-1)Q_2(t)u_2(t) \geq -(2p-1)[pk(t) - 2], \forall t \in I.$$

The above inequality implies the following estimation for $u_2(t)$ such that

$$u_2(t) \geq \exp(-2(2p-1)\int_0^t Q_2(s)ds)[u_2(0)$$

$$-(2p-1)\int_0^t [pk(s) - 2]\exp(2(2p-1)\int_0^s Q_2(\lambda)d\lambda)ds], \forall t \in I \Rightarrow$$

$$u_2(t) = n_5^{-(2p-1)}(t) \geq R_5^{-(2p-1)}(t) \Rightarrow n_5(t) \leq R_5(t), \forall t \in I,$$

where $R_5(t)$is defined in 2.32. From 2.35 and the above inequality we obtain

$$\frac{dm_5(t)}{dt} \leq k(t)m_5^p(t)R_5(t) \Rightarrow m_5^{-p}(t)\frac{dm_5(t)}{dt} \leq k(t)R_5(t), \forall t \in I.$$

By taking $t = s$ in the above inequality and integration it from 0 to t and using the fact that $p > 1 \to (p-1) > 0$, we have

$$m_5(t) \leq x_0[1-(p-1)x_0^{(p-1)}\int_0^t k(s)R_5(s)ds]^{\frac{-1}{(p-1)}}, \forall t \in I.$$

Then from the above inequality in 2.34 we obtain the desired 2.31. This completes the proof. ∎

3. SOME APPLICATIONS

In this section, we use some inequalities obtained in Section2 to obtain the bound for the solutions of some integro-differential equations of the form

$$\frac{dx(t)}{dt} = f(t) + \int_0^t F(t,s,x(s),\frac{dx(s)}{ds})ds, \forall t \in I, \tag{3.1}$$

where $x(t), \frac{dx(t)}{dt}$ be realvalued nonegative continuous functions defined on I, and $f(t), F(t,s,x(s),\frac{dx(s)}{ds})$ are the elements of R^n, the set of real numbers, and continuous on the respective domains of their definitions.
As an applications of the Theorem 2.1 we present the following example:

Example 3.1. *Let $f(t)$ and $F(t,s,x(s),\frac{dx(s)}{ds})$ in 3.1 satisfy*

$$| f(t) | \leq x_0, \forall t \in I, \tag{3.2}$$

$$| F(t,s,x(s),\frac{dx(s)}{ds}) | \leq k(t)[| \frac{dx(t)}{dt} |]^p[| \frac{dx(t)}{dt} | + | x(t) |], \forall t \in I, \tag{3.3}$$

where x_0, p and $k(t)$ as defined in the Theorem 2.1, then

$$| \frac{dx(t)}{dt} | \leq [x_0^{(1-p)} + (1-p)\int_0^t k(s)R(s)ds]^{\frac{1}{(1-p)}}, \forall t \in I, \tag{3.4}$$

where $R(t)$ is defined in 2.3.

Proof. Using 3.2 and 3.3 in 3.1 we have

$$| \frac{dx(t)}{dt} | \leq x_0 + \int_0^t k(s)[| \frac{dx(s)}{ds} |]^p[| \frac{dx(s)}{ds} | + | x(s) |]ds, \forall t \in I.$$

Now an applications of Theorem 2.1 the above inequality yields the desired bound in 3.4. This completes the proof. ∎

As an applications of the Theorem 2.3 we present the following example:

Example 3.2. *Let $f(t)$ and $F(t,s,x(s),\frac{dx(s)}{ds})$ in 3.1 satisfy 3.2 and*

$$| F(t,s,x(s),\frac{dx(s)}{ds}) | \leq k(t)[| \frac{dx(t)}{dt} |]^p[[| \frac{dx(t)}{dt} |]^p + | x(t) |], \forall t \in I, \tag{3.5}$$

where x_0, p and $k(t)$ as defined in the Theorem 2.3, then

$$| \frac{dx(t)}{dt} | \leq x_0[1-(p-1)x_0^{(p-1)}\int_0^t k(s)R_2(s)ds]^{\frac{-1}{(p-1)}}, \forall t \in I, \tag{3.6}$$

where $R_2(t)$ is defined in 2.13.

Proof. Using 3.2 and 3.5 in 3.1 we have

$$\mid \frac{dx(t)}{dt} \mid \leq x_0 + \int_0^t k(s)[\mid \frac{dx(s)}{ds} \mid]^p[[\mid \frac{dx(s)}{ds} \mid]^p + \mid x(s) \mid]ds, \forall t \in I.$$

Now an applications of Theorem 2.3 the above inequality yields the desired bound in 3.6. This completes the proof. ∎

As an applications of the Theorem 2.5 we present the following example:

Example 3.3. *Let $f(t)$ and $F(t,s,x(s),\frac{dx(s)}{ds})$ in 3.1 satisfy 3.2 and*

$$\mid F(t,s,x(s),\frac{dx(s)}{ds}) \mid \leq k(t)[\mid \frac{dx(t)}{dt} \mid]^p[[\mid \frac{dx(t)}{dt} \mid]^2 + [\mid x(t) \mid]^2], \forall t \in I, \tag{3.7}$$

where x_0, p and $k(t)$ as defined in the Theorem 2.5, then

$$\mid \frac{dx(t)}{dt} \mid \leq x_0[1-(p-1)x_0^{(p-1)} \int_0^t k(s)R_4(s)ds]^{\frac{-1}{(p-1)}}, \forall t \in I, \tag{3.8}$$

where $R_4(t)$ is defined in 2.25.

Proof. Using 3.2 and 3.7 in 3.1 we have

$$\mid \frac{dx(t)}{dt} \mid \leq x_0 + \int_0^t k(s)[\mid \frac{dx(s)}{ds} \mid]^p[[\mid \frac{dx(s)}{ds} \mid]^2 + [\mid x(s) \mid]^2]ds, \forall t \in I.$$

Now an applications of Theorem 2.5 the above inequality yields the desired bound in 3.8. This completes the proof. ∎

Remark 3.1. Finally, we note that the integro-differential inequalities established in Section 2 have many possible applications in the theory of integro-differential equations.

REFERENCES

[1] A. ABDELDAIM, On some new Gronwall-Bellman-Ou-Iang type integral inequalities to study certain epidemic models, *J. Integral Equations. Appl.*, **Vol.24,** No.2 (2012), pp. 149-166.

[2] A. ABDELDAIM and M. YAKOUT, On some new integral inequalities of Gronwall-Bellman-Pachpatte type, *Appl. Math. Comput.*, **217(20)** (2011), pp. 7895-7907.

[3] R. P. AGARWALL and E. THANDAPANI, Remarks on generalizations of Gronwall's inequality, *Chinese J. Math.*,**Vol.9** (1981), pp. 1-22.

[4] R. P. AGARWALL, S. DENG and W. ZHANG, Generalizations of retarded Gronwall-like inequality and its applications, *Appl. Math. Comput.*,**Vol.165** (2005), pp. 599-612.

[5] AL-JASER and K. M. FURATI, Singular fractional integro-differential inequalities and applications, *J. Inequal. Appl.* (2011), pp. 1-18.

[6] D. BAINOV and P. SIMEONOV, *Integral Inequalities and Applications*, New York, Kluwer Academic Publishers, 1992.

[7] R. BELLMAN, The stability of solutions of linear differential equations, *Duke Math. J.*,**Vol.10** (1943), pp. 643-647.

[8] S. S. DRAGOMIR, Some Gronwall type inequalities and applications, *RGMIA Monographs*, Victoria University, Australlia, 2000, `Online: http://rgmia.org/monographs/Gronwall.html.`

[9] H. EL-OWAIDY, A. RAGAB and A. ABDELDAIM, On some new integral inequalities of Gronwall-Bellman type, *Appl. Math. Comput.*, **106** (1999), pp. 289-303.

[10] A. FOUAD MEGAHED, G. HAMAD and SH. R. SALEM, On some integrodifferential inequalities in one variable defined on an infinite interval, *Arch. Math.*,**Vol.23** (1987), pp. 23-27.

[11] T. H. GRONWALL, Note on the derivatives with respect to a paramater of the solutions of a system of differential equations, *Ann. Math.*,**Vol.20** (1919), pp. 292-296.

[12] G. H. HARDY, J. E. LITTLEWOOD and G. POLYA, *Inequalities*, Cambridge, Cambridge University Press, 1934.

[13] B. G. PACHPATTE, On some fundamental integrodifferential and integral inequalities, *An. Sti. Univ. Al. I. Cuza" Iasi*, **Vol.23** (1977), pp. 77-86.

[14] B. G. PACHPATTE, On some fundamental integrodifferential and their discrete analogues, *Proc. Indian Acad. Sci.*, **Vol.87A** (1978), pp. 201-207.

[15] B. G. PACHPATTE, A note on some fundamental integrodifferential inequalities, *Tamkang J. Math.*, **Vol.13** (1982), pp. 63-67.

[16] B. G. PACHPATTE, On some integral inequalities and their applications to higher order differential equations, *Soochow, J. Math.*, **Vol.22** (1988), pp. 221-230.

[17] B. G. PACHPATTE, On some inequalities useful in the theory of certain higher order differential and difference equations, *Studia Univ. Babes-Bolyai, Mathematica*, **Vol.40** (1995), pp. 15-34.

[18] B. G. PACHPATTE, *Inequalities for Differential and Integral Equations*, Academic Press, New York, 1998.