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# TO A BANACH *-ALGEBRA IN A SEMIPARTIAL DYNAMICAL SYSTEM

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ABSTRACT. By a partial dynamical system, we mean a triple containing a C*-algebra A, a discrete group G and a partial action of G on A. There are two C*–algebras associated to a given partial dynamical system. These are nothing but the certain C*-completions of a Banach *-algebra. In constructing such a Banach *-algebra, usually, a tedious limit process is used to apply. In this paper, we prove some theorems in this context without any limit process.

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## 1. INTRODUCTION

In the last two decads, the notion of partial crossed product of a $C^*$-algebra by a discrete group has been a very important tool in $C^*$-algebras and dynamical systems. As the name suggests, partial action generalizes the notion of action in a set, in a topological space or in an algebra.

In fact, the idea of partial crossed product of a $C^*$-algebra by a discrete group was introduced by McClanahan [4] as a generalization of Exel's work in [2]. It is well known that the notion of the crossed product of a $C^*$-algebra by an action of a group uses a homomorphism into the automorphism group of the $C^*$-algebra. Since we can not talk about a homomorphism from a group into an inverse semigroup, the notion of partial action covers this flaw. That is, the idea of partial action is to replace the automorphism group by the inverse semigroup of partial automorphism. The general definitions of partial actions and crossed products by partial actions of groups are introuced in [3]. By a partial automorphism of a $C^*$-algebra $A$, we mean a triple $(\alpha, I, J)$ where $I$ and $J$ are closed two-sided ideals of $A$ and

$$\alpha : I \to J$$

is a $C^*$-isomorphism.

This paper is organized as follows:

Section 2 is devoted to multiplier algebras. Partial actions of groups are considered in Section 3, where one of the our main results is proved. In Section 4, actions of inverse semigroups are introduced and two of our main results are proved.

## 2. MULTIPLIER ALGEBRAS

Let $A$ be a unital $C^*$-algebra. We would like to construct a new unital $C^*$-algebra $D$ in which $A$ can be embedded as an ideal. For example, given a $C^*$-algebra, $C_0(X)$, if $X$ is an open subset of a compact Hausdorff space $Y$, then $C_0(X)$ is an ideal in $C(Y)$.

If $I$ is a closed ideal in a $C^*$-algebra $A$, then $I^\perp = \{x \in A : Ix = 0\}$ is the *annihilator* of $I$. Also, $I^\perp$ is a closed ideal of $A$, and if $I^\perp = \{0\}$, then $I$ is *essential* in $A$.

**Theorem 2.1.** *Given a $C^*$-algebra $A$, there is a unital $C^*$-algebra $M(A)$ containing $A$ as an essential ideal. $M(A)$ is* universal *in the sense that whenever $A$ embeds as an ideal in $C^*$-algebra $D$, the identity map on $A$ can be extended uniquely to a $*$-homomorphism from $D$ into $M(A)$ with kernel $A^\perp$. The algebra $M(A)$, which is unique up to isomorphism over $A$, is called the multiplier algebra of $A$.*

*Proof.* [1, II.7.3.1]. ∎

Elements of $M(A)$ are called double centralizer.

By a *double centralizer* of a $C^*$-algebra $A$, we mean a pair $(L, R)$ of bounded linear maps from $A$ to $A$ such that

$$xL(y) = R(x)y,$$

for all $x, y \in A$. More generally, a *left centralizer* of $A$ is a linear map $L : A \to A$ satisfying $L(xy) = L(x)y$, and a *right centralizer* of $A$ is a linear map $R : A \to A$ such that $R(xy) = xR(y)$. If $(L, R)$ is a double centralizer of a $C^*$-algebra $A$, then $\| L \| = \| R \|$, [5, Lemma 2.1.4]. For given $(L, R)$ and $(L', R') \in M(A)$ define their product and norm respectively by

$$(L, R)(L', R') = (LL', R'R),$$

$$\|(L,R)\| = \|L\| = \|R\|.$$

It is easy to check that this product is again a double centralizer of A. That is, $M(A)$ is an algebra under this multiplication.

For $L : A \to A$, define $L^* : A \to A$ by $L^*(a) = (L(a^*))^*$. We will see that $L^*$ is linear and the map $L \to L^*$ is an isometric conjugate-linear map from $B(A)$ to itself such that $(L^*)^* = L$ and $(LL')^* = (L')^*L^*$. This shows that if $(L,R)$ is an element of $M(A)$ then so is $(L,R)^* = (R^*,L^*)$. Thus, the map $(L,R) \mapsto (L,R)^*$ is an involution on $M(A)$.

**Theorem 2.2.** *If A is a $C^*$-algebra, then $M(A)$ is a $C^*$-algebra under the multiplication, involution, and norm defined above.*

*Proof.* [1, II. 7.3.4]. ∎

Let I be an ideal of a $C^*$-algebra A. Define the map $\phi : I \to M(I)$ by putting $\phi(x) = (L_x, R_x)$, $x \in I$. This is an algebra homomorphism, and moreover, $L_{xy} = L_x o L_y$, $R_{xy} = R_y o R_x$, which gives

$$\phi(xy) = (L_x o L_y, R_y o R_x) = \phi(x)\phi(y).$$

With the aid of the following Lemma, we can prove our main theorems.

**Lemma 2.3.** *If I and $J$ are ideals in a $C^*$-algebra A, $\pi : I \to J$ is a $C^*$-algebra isomorphism, and $(L,R) \in M(I)$, then*

$$(\pi o L o \pi^{-1}, \pi o R o \pi^{-1}) \in M(J).$$

*Proof.* For arbitrary elements $a, b$ of $J$, there are $x, y$ in I such that $a = \pi(x), b = \pi(y)$ and

$$(\pi o L o \pi^{-1})(ab) = (\pi o L)(xy) = \pi(L(xy)) = \pi(L(x)y)$$

$$= (\pi o L)(x)\pi(y) = (\pi o L o \pi^{-1})(a)b;$$

$$(\pi o R o \pi^{-1})(ab) = (\pi o R)(xy) = \pi(R(xy)) = \pi(xR(y))$$

$$= \pi(x)(\pi o R)(y) = a(\pi o R o \pi^{-1})(b);$$

and

$$(\pi o R o \pi^{-1})(a)b = (\pi o R)(x)b = \pi(R(x))\pi(y) = \pi(R(x)y)$$

$$= \pi(xL(y)) = a(\pi o L)(y) = a(\pi o L o \pi^{-1})(b).$$

Hence, $(\pi o L o \pi^{-1}, \pi o R o \pi^{-1})$ is a double centralizer of $J$. ∎

We know that closed ideals in a $C^*$- algebra are idempotent. Therefore, we can use the following proposition in the proof of our main theorems.

**Proposition 2.4.** *If I is an idempotent algebra, $I^2 = I$, and $(L,R), (L',R') \in M(I)$, then $(R' o L)(a) = (L o R')(a)$ for all $a \in I$.*

*Proof.* Let $a, b$ be arbitrary elements of I. By using the definition of double centralizers we have

$$(R' o L)(ab) = R'(L(ab)) = R'(L(a)b) = L(a)R'(b)$$

$$= L(aR'(b)) = L(R'(ab)) = (L o R')(ab).$$

Since I is an idempotent algebra, it is easy to see that $(R' o L)(a) = (L o R')(a)$ for all $a \in I$. ∎

## 3. PARTIAL ACTIONS OF GROUPS

In this section, we deal with partial crossed products of $C^*$- algebras by partial actions. Therefore, introducing partial actions and their properties seems necessary.

The major new result of this section is the Theorem 3.1 Throughout this section $A$ is a $C^*$-algebra.

**Definition 3.1.** Let $G$ be a discrete group with identity $e$. A *partial action* of $G$ on A is a collection $\{(\theta_g, D_{g^{-1}}, D_g) : g \in G\}$ of partial automorphisms such that
(i) $D_e = A$ and $\theta_e$ is the identity map on A;
(ii) $\theta_h^{-1}(D_h \cap D_{g^{-1}}) \subseteq D_{(gh)^{-1}}$;
(iii) $\theta_g(\theta_h(x)) = \theta_{gh}(x)$ for $x \in \theta_h^{-1}(D_h \cap D_{g^{-1}})$.

It is not hard to see that $\theta_{h^{-1}} = \theta_h^{-1}$, $\theta_h o \theta_h^{-1}$ is the identity map on $D_h$ and conditions (i)-(iii) are equivalent to the following:
$(i)$ $D_e = A$ and $\theta_e$ is the identity map on A;
$(ii)'$ $\theta_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$;
$(iii)'$ $\theta_g(\theta_h(x)) = \theta_{gh}(x)$, for all $x \in D_{h^{-1}} \cap D_{(gh)^{-1}}$.

Let $L = \{a \in \ell^1(G, A) : a(g) \in D_g\}$. Also, let $L$ have the norm, scalar multiplicaiton, and addition inherited from the algebra $\ell^1(G, A)$. Before we define multiplication and involution on $L$, let us agree on the following convention: We shall denote by $a_g\delta_g$ the element of $L$ which is identically zero except for its $g^{th.}$ component which is equal to $a_g$. Any element of $L$, say $x$, is therefore given by

$$x = \sum_{g \in G} a_g \delta_g$$

where the sum has finitely many nonzero terms.

Based on [2] and [4], the definition of multiplication and involution on $L$ is given by

$$(a_g\delta_g)(b_h\delta_h) = \theta_g(\theta_{g^{-1}}(a_g)b_h)\delta_{gh};$$

$$(a_g\delta_g)^* = \theta_{g^{-1}}(a_g^*)\delta_{g^{-1}}.$$

Here, we will prove the main theorem of this section. That is, we will prove that $L$ with the above defined multiplication is associative, without using limit process (compair with [4, prop.2.1]).

**Theorem 3.1.** *L is associative.*

*Proof.* Obviously, L is associative if and only if

$$(a_g\delta_g b_h\delta_h)c_k\delta_k = a_g\delta_g(b_h\delta_h c_k\delta_k)$$

for given elements $g, h, k$ of $G$ and $a_g \in D_g, b_h \in D_h,$ and $c_k \in D_k$. The left hand side of the above equality is nothing but

$$(a_g\delta_g b_h\delta_h)c_k\delta_k = \theta_g((\theta_{g^{-1}})a_g)b_h)\delta_{gh}c_k\delta_k$$

$$= \theta_{gh}\{\theta_{(gh)^{-1}}[\theta_g(\theta_{g^{-1}}(a_g)b_h]c_k\}\delta_{ghk}.$$

Since $a_g \in D_g$ and $\theta_{g^{-1}} : D_g \to D_{g^{-1}}$, we have $\theta_{g^{-1}}(a_g) \in D_{g^{-1}}$, as a consequence, $\theta_{g^{-1}}(a_g)b_h \in D_{g^{-1}}D_h = D_{g^{-1}} \cap D_h$, therefore $\theta_g(\theta_{g^{-1}}(a_g)b_h) \in D_g \cap D_{gh}$, and we can split $\theta_{(gh)^{-1}}$, which gives that

$$\theta_{(gh)^{-1}}[\theta_g(\theta_{g^{-1}}(a_g)b_h)] = \theta_{h^{-1}}(\theta_{g^{-1}}(a_g)b_h).$$

Also, since $\theta_{h^{-1}}(\theta_{g^{-1}}(a_g)b_h)$ is an element of $D_{h^{-1}} \cap D_{h^{-1}g^{-1}}$, we can split $\theta_{gh}$ and consequently

$$\begin{aligned}(a_g\delta_g b_h\delta_h)c_k\delta_k &= \theta_{gh}[\theta_{h^{-1}}(\theta_{g^{-1}}(a_g)b_h)c_k]\delta_{ghk} \\ &= \theta_g\{\theta_h[\theta_{h^{-1}}(\theta_{g^{-1}}(a_g)b_h)c_k]\}\delta_{ghk}.\end{aligned}$$

And, the right hand side is

$$a_g\delta_g(b_h\delta_h c_k\delta_k) = a_g\delta_g[\theta_h(\theta_{h^{-1}}(b_h)c_k)\delta_{hk}]$$

$$= \theta_g\{\theta_{g^{-1}}(a_g)[\theta_h(\theta_{h^{-1}}(b_h)c_k)]\}\delta_{ghk}.$$

If we apply $\theta_{g^{-1}}$ on the last equalities of the both sides we see that the equality holds if and only if

$$\theta_h[\theta_{h^{-1}}(\theta_{g^{-1}}(a_g)b_h)c_k] = \theta_{g^{-1}}(a_g)[\theta_h(\theta_{h^{-1}}(b_h)c_k)]$$

is verified for all $a_g \in D_g, b_h \in D_h$, and $c_k \in D_k$. Since

$$\theta_{g^{-1}} : D_g \to D_{g^{-1}}$$

is an isomorphism, we see that $\theta_{g^{-1}}(a_g)$ runs over $D_{g^{-1}}$, and as a consequence the equality holds if and only if

$$\theta_h[\theta_{h^{-1}}(a_g b_h)c_k] = a_g[\theta_h(\theta_{h^{-1}}(b_h)c_k)]$$

for every $a_g \in D_{g^{-1}}, b_h \in D_h$, and $c_k \in D_k$. If $g = k = e$, then $D_g = D_k = A$, and $L$ is associative if and only if the above equality holds for arbitrary $h \in G$, $a_g, c_k \in A$, and $b_h \in D_h$. Obviuosly, this equality is equivalent to say that

$$(\theta_h o R_{c_k} o \theta_{h^{-1}}) o L_{a_g} = L_{a_g} o(\theta_h o R_{c_k} o \theta_{h^{-1}})$$

is valid on $D_h$ for every $h \in G$ and all $a_g, c_k \in A$.

Let $R_{c_k}$ be a right multiplier on $D_{h^{-1}}$ and $L_{a_g}$ be a left multiplier on $D_h$. By Lemma 2.3, we see that $\theta_h o R_{c_k} o \theta_{h^{-1}}$ is a right multiplier on $D_h$. Since all ideals $D_h (h \in G)$ are idempotent, Proposition 2.4 implies that

$$(\theta_h o R_{c_k} o \theta_{h^{-1}}) o L_{a_g} = L_{a_g} o(\theta_h o R_{c_k} o \theta_{h^{-1}}).$$

∎

## 4. ACTION OF AN INVERSE SEMIGROUP

In this section, we will present the definition of an *action* of an inverse semigroup on a $C^*$-algebra. Then we will consider a Banach $*$-algebra, whose $C^*$-completion is nothing but the $C^*$-crossed product $A \times_\theta S$.

If $A$ is any set, by $I(A)$ we mean the inverse semigroup of all bijections between subsets of $A$. The major new results of this section are Proposition 4.1 and Theorem 4.2

Here, we present the definition of an action of an inverse semigroup on $C^*$-algebra.

**Definition 4.1.** An action of an inverse semigroup $S$ on a $C^*$-algebra $A$ is a semigroup homomorphism

$$\theta : S \to I(A)$$

such that

(i) for every $s \in S$, the domain (and hence also the range) of $\theta_s$ is a closed two sided ideal of A, and $\theta_s$ is a $*$-isomorphism.

(ii) the linear span of the union of the domains of all $\theta_s$ is dense in A.

If $e$ is an idempotent element of $S$, we denote by $E_e$ the domain of $\theta_e$. Also, for each $s \in S$, we have that $\theta_s$ is a $*$-isomorphism from $E_{s^*}$ to $E_s$. See also [6, Sec 3.3]. In fact, $E_{s^*}$ and $E_s$ are closed two-sided ideals in A and $(\theta_s, E_{s^*}, E_s)$ is a partial automorphism of A. Note that, if $s \in S$, then from the definition of inverse semigroup we have $ss^*s = s$ and $s^*ss^* = s^*$. By applying $\theta$ on both sides of the above relations, we have

$$\theta_{s^*} = \theta_s^{-1}.$$

Throughout this section, by a *semipartial dynamical system*, we mean a triple $(A, S, \theta)$ in which A is a $C^*$-algebra, $S$ is a (not necessairly unital) inverse semigroup, and $\theta$ is an action of $S$ on A.

Let $(A, S, \theta)$ be a semipartial dynamical system. Take

$$L = \{x \in \ell^1(S, A) : x(s) \in E_s\}.$$

It is not hard to see that

$$L = \bigoplus_{s \in S} E_s.$$

If $x \in L$, then $x = (a_s)_{s \in S}$ in which $a_s \in E_s$ and $a_s = 0$ for all but finitely many $s$. For given $s \in S$ and $a_s \in E_s$, let $a_s\delta_s$ be the element of $L$ which is identically zero except for its $s^{th.}$ component which is equal to $a_s$. Therefore any element of $L$, say $x = (a_s)_{s \in S}$, is given by $x = \sum\limits_{s \in S} a_s\delta_s$, where the sum has finitely many nonzero terms. Let $L$ have the norm, scalar multiplication and, addition inherited from $\ell^1(S, A)$. Based on [2] and [4] multiplication and involution is given by

$$(a_s\delta_s).(b_t\delta_t) = \theta_s(\theta_{s^*}(a_s)b_t)\delta_{st} \text{ , and } (a_s\delta_s)^* = \theta_{s^*}(a_s^*)\delta_{s^*}$$

for every $s, t$ in $S, a_s \in E_s$, and $b_t \in E_t$. By using [7, Prop.4.1] it is easy to see that $L$ is a normed $*$-algebra.

Here, let us present an auxiliary double centralizer which will be used in the proof of the main theorem of this section.

**Proposition 4.1.** *Given an inverse semigroup $S$, a $C^*$-algebra $A$, and a partial automorphism $(\theta_s, E_{s^*}, E_s)$ of $A$, if $L$ and $R$ are respectively, a left and a right centralizer of $E_{s^*}$, then*

$$(\theta_s o L o \theta_{s^*}, \theta_s o R o \theta_{s^*}) \in M(E_s).$$

*Proof.* It is obvious that $\theta_s o L o \theta_{s^*}$ and $\theta_s o R o \theta_{s^*}$ are linear transformations on $E_s$. To prove that $(\theta_s o L o \theta_{s^*}, \theta_s o R o \theta_{s^*})$ is a multiplier of $E_s$, let $\theta_s o L o \theta_{s^*} = L'$ and $\theta_s o R o \theta_{s^*} = R'$. For $x \in A$ and $a, b \in E_s$, we have

$$\begin{aligned} L'_x(ab) &= \theta_s o L_x(\theta_{s^*}(ab)) = \theta_s(x\theta_{s^*}(ab)) \\ &= \theta_s(x)\theta_s(\theta_{s^*}(ab)) = \theta_s(x)(ab), \end{aligned}$$

and

$$\begin{aligned} L'(a)b &= [\theta_s o L_x(\theta_{s^*}(a))]b] = [\theta_s(x\theta_{s^*}(a))]b \\ &= [\theta_s(x)a]b = \theta_s(x)(ab). \end{aligned}$$

That is, $L'(ab) = L'(a)b$.
Also,

$$\begin{aligned} R'(ab) &= (\theta_s o R o \theta_{s^*})(ab) = \theta_s o R_x(\theta_{s^*}(ab)) \\ &= \theta_s(\theta_{s^*}(ab)x) = (ab)\theta_s(x), \end{aligned}$$

and

$$aR'(b) = a(\theta_s o R_x o \theta_{s^*})(b) = a(\theta_s o R_x(\theta_{s^*}(b))$$

$$= a\theta_s(\theta_{s^*}(b)x) = a(b\theta_s(x)) = (ab)\theta_s(x),$$

therefore,

$$R'(ab) = aR'(b).$$

Finally,

$$R'(a)b = [(\theta_s o R_x o \theta_{s^*})(a)]b = [\theta_s o R_x(\theta_{s^*}(a))]b$$
$$= [\theta_s(\theta_{s^*}(a)x)]b = [a\theta_s(x)]b.$$

Also,

$$aL'(b) = a[(\theta_s o L_x o \theta_{s^*})b] = a[\theta_s o L_x(\theta_{s^*}(b))]$$
$$= a\theta_s(x\theta_{s^*}(b)) = a\theta_s(x)b.$$

That is, $R'(a)b = aL'(b)$, and these facts show that $(\theta_s o L o \theta_{s^*}, \theta_s o R o \theta_{s^*})$ is a multiplier of $E_s$. ∎

Now, we would like to show that the normed $*$-algebra $L$ is associative with the multiplication defined following the Definition 4.1. Note that the following result is proved in Proposition 4.1 of [7], and not only we are not assuming that our inverse semigroup is unital, but we will prove the result without the approximate identity.

**Theorem 4.2.** *The normed $*$-algebra $L$ is associative. (compare with* [7, Prop.4.1.]

*Proof.* Let $x = \sum_{r\in S} a_r\delta_r, y = \sum_{s\in S} a_s\delta_s$, and $z = \sum_{t\in S} a_t\delta_t$ are arbitrary elements of $L$. We want to show that

$$(x.y).z = x.(y.z).$$

It suffices to show this for $x = a_r\delta_r, y = a_s\delta_s$, and $z = a_t\delta_t$.

$$\begin{aligned}(a_r\delta_r.a_s\delta_s).a_t\delta_t &= \theta_r(\theta_{r^*}(a_r)a_s)\delta_{rs}.a_t\delta_t \\ &= \theta_{rs}\{\theta_{(rs)^*}[\theta_r(\theta_{r^*}(a_r)a_s)]a_t\}\delta_{rst} \\ &= \theta_{rs}\{\theta_{s^*r^*}[\theta_r(\theta_{r^*}(a_r)a_s)]a_t\}\delta_{rst} \\ &= \theta_{rs}\{\theta_{s^*}[\theta_{r^*}(a_r)a_s]a_t\}\delta_{rst} \\ &= \theta_r\{\theta_s(\theta_{s^*}[\theta_{r^*}(a_r)a_s]a_t)\}\delta_{rst}.\end{aligned}$$

Note that, since $a_r \in E_r$ and $\theta_{r^*} : E_r \to E_{r^*}$ is a $C^*$-algebra isomorphism, we have $\theta_{r^*}(a_r) \in E_{r^*}$ and $\theta_{r^*}(a_r)a_s \in E_{r^*}E_s = E_{r^*} \cap E_s \subset E_{r^*}$. We used the fact that $\theta_{r^*r} = \theta_{r^*}\theta_r$ is the identity map on $E_{r^*}$.

On the other hand,

$$a_r\delta_r.(a_s\delta_s.a_t\delta_t) = a_r\delta_r.[\theta_s(\theta_{s^*}(a_s)a_t)]\delta_{st}$$
$$= \theta_r\{\theta_{r^*}(a_r)[\theta_s(\theta_{s^*}(a_s)a_t)]\}\delta_{rst}.$$

Therefore, $L$ is associative if and only if

$$\theta_r\{\theta_s(\theta_{s^*}[\theta_{r^*}(a_r)a_s]a_t)\}\delta_{rst} = \theta_r\{\theta_{r^*}(a_r)[\theta_s(\theta_{s^*}(a_s)a_t)]\}\delta_{rst}.$$

If we apply $\theta_{r^*}$ to the both sides of the above equality, we see that equality holds if and only if

$$\theta_s(\theta_{s^*}[\theta_{r^*}(a_r)a_s]a_t) = \theta_{r^*}(a_r)[\theta_s(\theta_{s^*}(a_s)a_t)].$$

Since $\theta_{r^*} : E_r \to E_{r^*}$ is an isomorphism, $\theta_{r^*}(a_r)$ runs over $E_{r^*}$ and consequently, the last equality is equivalent to

$$\theta_s(\theta_{s^*}[a\ a_s]a_t) = a[\theta_s(\theta_{s^*}(a_s)a_t)]$$

for arbitrary elements $a$ in $E_{r^*}, a_s \in E_s$, and $a_t \in E_t$. If $r = t$, we have $E_r = E_t$. Therefore, it suffices to prove that

$$\theta_s(\theta_{s^*}[a\ a_s]a_t) = a[\theta_s(\theta_{s^*}(a_s)a_t)]$$

for $a_r, a_t$ in $E_r, s \in S$, and $a_s \in E_s$.

Let $R_{a_t}$ be a right multiplier of $E_{s^*}$ and $L_a$ be a left multiplier of $E_s$. By Lemma 2.3, $\theta_s o R_{a_t} o \theta_{s^*}$ is a right multiplier of $E_s$. Since A is a $C^*$-algebra and all closed ideals of A are idempotent, by Proposition 2.4 we have

$$[(\theta_s o R_{a_t} o \theta_{s^*}) o L_a](a_s) = [L_a o(\theta_s o R_{a_t} o \theta_{s^*})](a_s),$$

which is nothing but the desired equality. ∎

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