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ON A CONJECTURE OF A LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTION

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ABSTRACT. In this short note we prove a conjecture, related to a logarithmically completely monotonic function, presented in [5]. Then, we extend by proving a more generalized theorem. At the end we pose an open problem on a logarithmically completely monotonic function involving q-Digamma function.

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1. INTRODUCTION

Recall from [13, Chapter XIII], [17, Chapter 1] and [18, Chapter IV] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and satisfies

$$0 \leq (-1)^n f^{(n)}(x) < \infty \tag{1.1}$$

for $x \in I$ and $n \geq 0$. The celebrated Bernstein-Widder's Theorem (see [17, p. 3, Theorem 1.4] or [18, p. 161, Theorem 12b]) characterizes that a necessary and sufficient condition that f should be completely monotonic for $0 < x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t), \tag{1.2}$$

where $\alpha(t)$ is non-decreasing and the integral converges for $0 < x < \infty$. This expresses that a completely monotonic function f on $[0, \infty)$ is a Laplace transform of the measure α.

It is common knowledge that the classical Euler's gamma function $\Gamma(x)$ may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{1.3}$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called psi function or digamma function.

An alternative definition of the gamma function $\Gamma(x)$ is

$$\Gamma(x) = \lim_{p\to\infty} \Gamma_p(x), \tag{1.4}$$

where

$$\Gamma_p(x) = \frac{p! p^x}{x(x+1)\cdots(x+p)} = \frac{p^x}{x(1+x/1)\cdots(1+x/p)} \tag{1.5}$$

for $x > 0$ and $p \in \mathbb{N}$. See [3, p. 250]. The p-analogue of the psi function $\psi(x)$ is defined as the logarithmic derivative of the Γ_p function, that is,

$$\psi_p(x) = \frac{d}{dx} \ln \Gamma_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}. \tag{1.6}$$

The function ψ_p has the following properties (see [9, p. 374, Lemma 5] and [11, p. 29, Lemma 2.3]).

(1) It has the following representations

$$\psi_p(x) = \ln p - \sum_{k=0}^{p} \frac{1}{x+k} = \ln p - \int_0^\infty \frac{1 - e^{-(p+1)t}}{1 - e^{-t}} e^{-xt} dt. \tag{1.7}$$

(2) It is increasing on $(0, \infty)$ and ψ_p' is completely monotonic on $(0, \infty)$.

In [2, pp. 374–375, Theorem 1], it was proved that the function

$$\theta_\alpha(x) = x^\alpha [\ln x - \psi(x)] \tag{1.8}$$

is completely monotonic on $(0, \infty)$ if and only if $\alpha \leq 1$.

For the history, backgrounds, applications and alternative proofs of this conclusion, please refer to [4], [14, p. 8, Section 1.6.6] and closely-related references therein.

A positive function f is said to be *logarithmically completely monotonic* [9] on an open interval I, if f satisfies

$$(-1)^n [\ln f(x)]^{(n)} \geq 0, (x \in I, n = 1, 2, \ldots). \tag{1.9}$$

If the inequality (1.2) is strict, then f is said to be *strictly logarithmically completely monotonic*. Let C and L denote the set of completely monotonic functions and the set of logarithmically completely monotonic functions, respectively. The relationship between completely monotonic functions and logarithmically completely monotonic functions can be presented [9] by $L \subset C$.

2. Main results

In [5] has been posed the following conjecture.

Conjecture 2.1. *The function*

$$q(t) := t^{t(\psi(t)-\log t)-\gamma} \tag{2.1}$$

is logarithmically completely monotonic on $(0, \infty)$.

Proof. One easily finds that

$$\log q(t) = -t \cdot (\log t - \psi(t)) \log t - \gamma \cdot \log t \tag{2.2}$$

Let $h(t) = -\gamma \cdot \log t, g(t) = -\log t; f(t) = t \cdot (\log t - \psi(t))$. Alzer [2] proved that the function $f(t) = t \cdot (\log t - \psi(t))$ is strictly completely monotonic on $(0, \infty)$. The functions $g(t) = -\log t$ and $h(t) = -\gamma \cdot \log t$ are also strictly completely monotonic on $(0, \infty)$.

We complete the proof by recalling the results from [18].

1) The product of two completely monotone functions is completely monotonic function.

2) A non-negative finite linear combination of completely monotone functions is completely monotonic function. ∎

We extend the previous result to the following theorem.

Theorem 2.2. *The function*

$$q_p(t) := t^{t \cdot (\psi_p(t) - \log \frac{pt}{t+p+1}) - \gamma} \tag{2.3}$$

is logarithmically completely monotonic on $(0, \infty)$.

Proof. One easily finds that

$$\log q_p(t) = -t(\log t - \psi_p(t)) \log t - \gamma \cdot \log t \tag{2.4}$$

Let $h(t) = -\gamma \cdot \log t, g(t) = -\log t; f_p(t) = t \cdot (\log \dfrac{pt}{t+p+1} - \psi_p(t))$.

Krasniqi and Qi [10] proved that the function $f_p(t) = t \cdot (\log \frac{pt}{t+p+1} - \psi_p(t))$ is strictly completely monotonic on $(0, \infty)$. The functions $g(t) = -\log t$ and $h(t) = -\gamma \cdot \log t$ are also strictly completely monotonic on $(0, \infty)$.

By refering the same results from [18] as in previous proof, we complete the proof. ∎

Remark 2.1. Letting $p \to \infty$ in Theorem 2.2 , we obtain Conjecture 2.1.

At the end we pose the following open problem:

Problem 1. *Let* $\psi_q(t)$ *be* q*-Digamma function. Find the family of functions* $\theta(t)$ *such that*

$$q(t) := t^{t \cdot (\psi_q(t) - \log \theta(t)) - \gamma} \tag{2.5}$$

is logarithmically completely monotonic on $(0, \infty)$.

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