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ON THE SENDOV CONJECTURE FOR A ROOT CLOSE TO THE UNIT CIRCLE

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ABSTRACT. On Sendov's conjecture, T. Chijiwa quantifies the idea stated by V. Vâjâitu and A. Zaharescu (and M. J. Miller independently), namely that if a polynomial with all roots inside the closed unit disk has a root sufficiently close to the unit circle then there is a critical point at a distance of at most one from that root. Chijiwa provides an estimate of exponential order for the required 'closeness' of the root to the unit circle so that such a critical point may exist. In this paper, we will improve this estimate to polynomial order by making major modifications and strengthening inequalities in Chijiwa's proof.

Key words and phrases: Ilieff, Polynomial, Roots.

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1. INTRODUCTION

Let P be a complex polynomial of degree n with all its roots, $z_1, z_2, ..., z_n$ inside the unit circle. Let its derivative, P' have roots $w_1, w_2, ..., w_{n-1}$. Define,

$$I_P(z_j) := \min_{1\leq k\leq n-1} |z_j - w_k|, \quad I(P) = \max_{1\leq j\leq n} I_P(z_j).$$

Then Sendov's Conjecture (see [1]) states that for $n \geq 2$,

$$I(P) \leq 1.$$

In other words, every root of P has a critical point at a distance of no more than 1 from it. That the distance 1 is best possible is seen by taking, for instance, $P(z) = z^n - 1$.

Sendov's conjecture is true for $2 \leq n \leq 8$, (see [1]). It is also true if $P(0) = 0$ for all $n \geq 8$. Also, Goodman, Rahman and Ratti proved in [3] that if $|z_j| = 1$ for some $j \in \{1, 2, ..., n\}$ then there exists a critical point w_k such that $\left|w_k - \frac{z_j}{2}\right| \leq \frac{1}{2}$, which implies $I(z_j) \leq 1$. The validity of Sendov's conjecture for the roots on the unit circle strongly suggests that the property should hold for roots sufficiently close to the unit circle. V. Vâjâitu and A. Zaharescu proved (see [6]) that this is indeed true, but their proof did not provide numerical estimates of any sort. Chijiwa used their proof outline and provided an estimate for the required closeness of the roots to the unit circle. He specifically proved the following:

Theorem (Chijiwa)**.** *Let P be a complex polynomial of degree $n \geq 4$ with all zeros in the closed unit disk. Let a be a zero of p. If $|a| \geq 1 - \varepsilon_n$, where,*

$$\varepsilon_n = \frac{1}{2n^9 4^n}.$$

Then there exists a critical point w such that,

$$|w - a| \leq 1 - c_n(1 - |a|)$$

where,

$$(\spadesuit) \qquad c_n = \begin{cases} \frac{1}{4} & \textit{if} \quad n \equiv 0 \pmod 4 \\ \frac{n-3}{4(n-1)} & \textit{if} \quad n \equiv 1 \pmod 4 \\ \frac{n-6}{4(n-1)} & \textit{if} \quad n \equiv 2 \pmod 4 \\ \frac{(n-9)}{4(n-1)} & \textit{if} \quad n \equiv 3 \pmod 4. \end{cases}$$

In this paper we will follow a similar line of proof as given by Chijiwa for the above theorem, but we will make certain modification and strengthen inequalities at various places in the proof to significantly improve ε_n.

The following is our result:

Theorem A. *Let P be a complex polynomial of degree $n \geq 8$ with all zeros in the closed unit disk. Let a be a zero of p such that $|a| \geq 1 - \varepsilon_n$, where,*

$$\varepsilon_n = \frac{90}{n^{12} \ln n}.$$

Then there exists a critical point w such that,

$$|w - a| \leq 1 - c_n(1 - |a|)$$

where c_n is given by **(♠)**.

In this paper we assume the contrary that no critical point is close to the root a and then we obtain a contradiction. We will prepare a series of lemmas before proving the main theorem. Lemma 1 is used to prove Lemma 2, which states that the critical points of the polynomial are close to the origin. It is the most crucial lemma in the entire paper. This lemma is first used to prove Lemma 4, which states that the roots of the polynomial are respectively close to the n-th roots of unity. Lemma 4, along with Lemma 5, is used to prove Lemma 6. Lemma 6 is first used to estimate the modulus of the sum of critical points (Lemma 7). Lemma 6 is also used to improve Lemma 4, providing a better estimate for the respective closeness of the roots to the n-th roots of unity (Lemma 8). And then Lemma 7 and Lemma 8 are used to improve Lemma 6 itself (Lemma 9). Then the proofs of Lemmas 7, 8 and 9 are mimicked with new estimates to improve each of these lemmas, giving us Lemmas 10, 11 and 12. Then we move to the main proof where we work with a series of inequalities to get contradictions to the values of c_n.

Note on calculations: In the penultimate step of any calculation, the desired estimate will be presented in the form $[\cdots]n^x \ln(n)^y \varepsilon^z$, where $[\cdots]$ will be an expression involving well-defined constant values. This expression will then be calculated accurately up to five decimal points (taking either floor or ceiling according to the direction of the inequality) and then presented in the final step.

2. LEMMAS

Without loss of generality, let $a := z_1$ and let a be real and positive. Define $\varepsilon := 1-a$. Hence $\varepsilon \leq \varepsilon_n$. Note that for all $n \geq 8$, $0 \leq c_n \leq 0.25$.

Let us assume the contrapositive: Suppose that for all $j = 1, 2, ..., n-1$,

$$(\diamondsuit) \qquad |w_j - a| > 1 - c_n(1-a) = 1 - c_n\varepsilon.$$

Our first lemma provides a lower bound for $\operatorname{Re} \frac{1}{a-w_j}$. For its proof, we need the following Lemma:

Lemma A (Kumar and Shenoy [5]). *Let $p(z)$ be a polynomial with all its n zeros inside the closed unit disk. Let a be one of its zeros. If the disk $|z-a| \leq 2r\sin\frac{\pi}{n}$ contains a zero of p then the disk $|z-a| \leq r$ contains a zero of p'.*

Now we will state and prove Lemma 1. We will closely follow the proof of Lemma 1 given in [2]. This lemma will be used to prove Lemmas 2 and 3.

Lemma 1. *For $j = 1, 2, ..., n-1$:*

$$\operatorname{Re} \frac{1}{a-w_j} > 1 - t_1\varepsilon$$

where,

$$t_1 := 0.05947n^3.$$

Proof. We will use the following identity, (see Equation (2.3) in [1]),

$$\sum_{k=1}^{n-1} \frac{1}{a-w_k} = 2\sum_{k=2}^{n} \frac{1}{a-z_k}.$$

Thus for $j \in \{1, 2, ..., n-1\}$ we have,

$$(1.1) \qquad \operatorname{Re} \frac{1}{a-w_j} = 2\left(\sum_{k=2}^{n} \operatorname{Re} \frac{1}{a-z_k}\right) - \sum_{k\neq j} \operatorname{Re} \frac{1}{a-w_k}.$$

Hence we need a lower bound for $\mathrm{Re}\,\frac{1}{a-z_k}$ and an upper bound for $\mathrm{Re}\,\frac{1}{a-w_k}$. For the latter, we can use (♢) to get,

$$\begin{aligned}\mathrm{Re}\,\frac{1}{a-w_k} &\leq \left|\frac{1}{a-w_k}\right| < \frac{1}{1-c_n\varepsilon} = 1+\frac{c_n\varepsilon}{1-c_n\varepsilon} \leq 1+\frac{c_n\varepsilon}{1-c_n\varepsilon_n}\\ &\leq 1+\frac{0.25\varepsilon}{1-0.25\varepsilon_8} =\leq 1+\frac{0.25}{1-0.25\cdot\frac{90}{8^{12}\ln 8}}\varepsilon < 1+0.25001\varepsilon.\end{aligned}$$

Now we will estimate $\mathrm{Re}\,\frac{1}{a-z_k}$. We will use the following identity (see (2.5) in [1]),

For any $z\in\mathbb{C}$ and $x\in\mathbb{R}$ such that $x\neq 0$ and $z\neq x$,

$$\mathrm{Re}\,\frac{1}{x-z} = \frac{x-\mathrm{Re}\,z}{|x-z|^2} = \frac{2x^2-2x\mathrm{Re}\,z}{2x|x-z|^2} = \frac{|x-z|^2+x^2-|z|^2}{2x|x-z|^2} = \frac{1}{2x}-\frac{|z|^2-x^2}{2x|x-z|^2}.$$

Thus, for $k=2,...,n$,

$$\begin{aligned}\mathrm{Re}\,\frac{1}{a-z_k} &= \frac{1}{2a}-\frac{|z_k|^2-a^2}{2a|a-z_k|^2} > \frac{1}{2}-\frac{1-a^2}{2a|a-z_k|^2}\\ &> \frac{1}{2}-\frac{1+a}{2a}\cdot\frac{1-a}{|a-z_k|^2} = \frac{1}{2}-\frac{2-\varepsilon}{2(1-\varepsilon)}\cdot\frac{\varepsilon}{|a-z_k|^2}.\end{aligned}$$

Since $|a-w_k| > 1-c_n\varepsilon \geq 1-c_n\varepsilon_n$, using the contrapositive of Lemma A we get $|a-z_k| > 2(1-c_n\varepsilon_n)\sin\frac{\pi}{n}$. We can further estimate $\sin\frac{\pi}{n}$ using its convexity and the fact that $\sin\frac{\pi}{n} < \sin\frac{\pi}{6}$ for $n\geq 8 > 6$ to get,

$$\sin\frac{\pi}{n} > \frac{\sin\frac{\pi}{6}}{\frac{\pi}{6}}\cdot\frac{\pi}{n} = \frac{\frac{1}{2}}{\frac{\pi}{6}}\cdot\frac{\pi}{n} = \frac{6}{2n} = \frac{3}{n}. \tag{1.2}$$

Hence $|a-z_k| > 2(1-c_n\varepsilon_n)\frac{3}{n}$. Futhermore, note that $c_n\leq 0.25$, $n\geq 8$ and $\varepsilon_n\leq\varepsilon_8$ so we get,

$$\begin{aligned}\mathrm{Re}\,\frac{1}{a-z_k} &\geq \frac{1}{2}-\frac{2-\varepsilon_n}{2(1-\varepsilon_n)}\cdot\frac{n^2}{4\cdot 9\cdot(1-c_n\varepsilon_n)^2}\varepsilon\\ &\geq \frac{1}{2}-\frac{2-\varepsilon_8}{2(1-\varepsilon_8)}\cdot\frac{n^2}{36(1-0.25\varepsilon_8)^2}\varepsilon\\ &\geq \frac{1}{2}-\left[\frac{2-\frac{90}{8^{12}\ln 8}}{2(1-\frac{90}{8^{12}\ln 8})}\cdot\frac{1}{36\left(1-0.25\frac{90}{8^{12}\ln 8}\right)^2}\right]n^2\varepsilon\\ &> \frac{1}{2}-0.02778n^2\varepsilon.\end{aligned}$$

Substituting back in (1.1) we obtain,

$$\begin{aligned}\mathrm{Re}\,\frac{1}{a-w_j} &= 2\left(\sum_{k=2}^{n}\mathrm{Re}\,\frac{1}{a-z_k}\right)-\sum_{k\neq j}\mathrm{Re}\,\frac{1}{a-w_k}\\ &> 2(n-1)\left(\frac{1}{2}-0.02778n^2\varepsilon\right)-(n-2)\left(1+0.25001\varepsilon\right)\\ &> (n-1)-2(n-1)0.02778n^2\varepsilon-(n-2)-(n-2)0.25001\varepsilon\\ &= 1-2(n-1)0.02778n^2\varepsilon-(n-2)0.25001\varepsilon\\ &> 1-2\cdot 0.02778n^3\varepsilon-0.25001n\varepsilon\end{aligned}$$

$$
\begin{aligned}
&= 1 - \left[2 \cdot 0.02778 + \frac{0.25001}{n^2}\right] n^3\varepsilon \\
&\geq 1 - \left[2 \cdot 0.02778 + \frac{0.25001}{8^2}\right] n^3\varepsilon \\
&> 1 - 0.05947n^3\varepsilon = 1 - t_1\varepsilon.
\end{aligned}
$$

∎

Equation ($\diamondsuit$) essentially says that no critical point of the polynomial is near the root a. The next lemma states that, in such a case, all the critical points must be close to the origin. This lemma will be used heavily in other lemmas and in the main proof.

We will closely follow the proof of Lemma 2 in [2].

Lemma 2. *For $j = 1, 2, ..., n-1$:*

$$|w_j| < t_4\sqrt{\varepsilon},$$

where,

$$t_4 := t_3 + t_2\sqrt{\varepsilon_n} < 0.34634n\sqrt{n}$$

with,

$$t_2 := 1 + \frac{t_1}{1 - t_1\varepsilon_n} < 0.06143n^3,$$

$$t_3 := \sqrt{\left(\frac{t_1}{1 - t_1\varepsilon_n} + c_n\right)\left(\frac{1}{1 - t_1\varepsilon_n} + 1\right)} < 0.3463n\sqrt{n}.$$

Proof. For $j = 1, 2, ..., n-1$, define $h_j \in \mathbb{R}$ by:

$$\frac{1}{a - h_j} := \operatorname{Re} \frac{1}{a - w_j}.$$

We will first find upper and lower bounds for h_j. By Lemma 1, $\frac{1}{a-h_j} > 1 - t_1\varepsilon$, and since $1 - t_1\varepsilon \geq 1 - t_1\varepsilon_n > 0$ we get,

$$0 < a - h_j < \frac{1}{1 - t_1\varepsilon} \tag{2.1}$$

$$
\begin{aligned}
\implies h_j > a - \frac{1}{1 - t_1\varepsilon} &= (1 - \varepsilon) - \left(1 + \frac{t_1}{1 - t_1\varepsilon}\varepsilon\right) \\
&= -\left(1 + \frac{t_1}{1 - t_1\varepsilon}\right)\varepsilon \geq -\left(1 + \frac{t_1}{1 - t_1\varepsilon_n}\right)\varepsilon = -t_2\varepsilon.
\end{aligned}
$$

On the other hand, for the upper bound on h_j we have,

$$\frac{1}{a - h_j} = \operatorname{Re} \frac{1}{a - w_j} \leq \left|\frac{1}{a - w_j}\right| < \frac{1}{1 - c_n\varepsilon}.$$

Hence $1 - c_n\varepsilon < a - h_j$ which gives,

$$h_j < a - (1 - c_n\varepsilon) = 1 - \varepsilon - 1 + c_n\varepsilon = -(1 - c_n)\varepsilon.$$

Combining both the upper and lower bounds on h_j we obtain,

$$-t_2\varepsilon < h_j < -(1 - c_n)\varepsilon < 0. \tag{2.2}$$

We now claim that the points w_j, a and h_j form a right-angled triangle in the complex plane, and hence by Pythagoras theorem we should get,

$$|w_j - h_j|^2 + |a - w_j|^2 = |a - h_j|^2.$$

We can prove this by observing the transformation $Tz = \frac{1}{a-z}$ as in Figure 1 of [2], or we can also achieve it through the following algebraic manipulations. Note that,

$$\begin{aligned}|w_j - h_j|^2 + |a - w_j|^2 &= |a - h_j|^2\\ \iff 2|w_j|^2 - 2h_j \text{Re } w_j - 2a\text{Re } w_j &= -2ah_j\\ \iff |w_j|^2 - (a + h_j)\text{Re } w_j + ah_j &= 0\\ \iff (\text{Im } w_j)^2 + (\text{Re } w_j)^2 - (a + h_j)\text{Re } w_j + ah_j &= 0\\ \iff (\text{Im } w_j)^2 + (\text{Re } w_j - a)(\text{Re } w_j - h_j) &= 0.\end{aligned}$$

So proving that w_j, h_j and a form a right-angled triangle is equivalent to showing that,

$$(a - \text{Re } w_j)(\text{Re } w_j - h_j) = (\text{Im } w_j)^2. \tag{2.3}$$

To prove the last equation, note that $\frac{1}{a-h_j} = \text{Re } \frac{1}{a-w_j}$ implies,

$$\begin{aligned}\frac{1}{a-h_j} = \frac{a - \text{Re } w_j}{|a-w_j|^2} &\implies h_j = a - \frac{|a-w_j|^2}{a - \text{Re } w_j}\\ &\implies h_j - \text{Re } w_j = a - \text{Re } w_j - \frac{|a-w_j|^2}{a - \text{Re } w_j}\\ &\implies (h_j - \text{Re } w_j)(a - \text{Re } w_j) = (a - \text{Re } w_j)^2 - |a - w_j|^2\\ &\implies (h_j - \text{Re } w_j)(a - \text{Re } w_j) = (\text{Re } w_j)^2 - |w_j|^2\\ &\implies (h_j - \text{Re } w_j)(a - \text{Re } w_j) = -(\text{Im } w_j)^2.\end{aligned}$$

Hence w_j, h_j and a indeed form a right-angled triangle.

We will estimate $|w_j|$ using the Pythagoras theorem,

$$|w_j - h_j|^2 + |a - w_j|^2 = |a - h_j|^2$$

By the lower bound on $|a - w_j|$ due to (◇) and the upper bound on $|a - h_j|$ by (2.1) we get,

$$|w_j - h_j|^2 + (1 - c_n\varepsilon)^2 < \frac{1}{(1 - t_1\varepsilon)^2}.$$

Thus we have,

$$\begin{aligned}|w_j - h_j|^2 &< \left(\frac{1}{1 - t_1\varepsilon}\right)^2 - (1 - c_n\varepsilon)^2\\ &= \left(\frac{1}{1 - t_1\varepsilon} + 1 - c_n\varepsilon\right)\left(\frac{t_1}{1 - t_1\varepsilon} + c_n\right)\varepsilon\\ &< \left(\frac{1}{1 - t_1\varepsilon} + 1\right)\left(\frac{t_1}{1 - t_1\varepsilon} + c_n\right)\varepsilon = t_3^2\varepsilon.\end{aligned}$$

Using the above and the bound on h_j given by (2.2) we finally obtain,

$$|w_j| \le |h_j| + |w_j - h_j| < |-t_2\varepsilon| + t_3\sqrt{\varepsilon} = t_2\varepsilon + t_3\sqrt{\varepsilon} \le (t_2\sqrt{\varepsilon_n} + t_3)\sqrt{\varepsilon} = t_4\sqrt{\varepsilon}.$$

(Note that the above estimate holds even in the degenerate case of $w_j = h_j$.)

Let us estimate the constants using Lemma 1 and basic inequalities,

$$t_2 = 1 + \frac{t_1}{1 - t_1\varepsilon_n} < 1 + \frac{0.05947n^3}{1 - 0.05947n^3 \cdot \frac{90}{n^{12}\ln n}} \le n^3\left[\frac{1}{n^3} + \frac{0.05947}{1 - 0.05947 \cdot \frac{90}{n^9 \ln n}}\right]$$

$$\leq n^3 \left[\frac{1}{8^3} + \frac{0.05947}{1 - 0.05947 \cdot \frac{90}{8^9 \ln 8}}\right] < 0.06143n^3.$$

Also,

$$\begin{aligned} t_3^2 &= \left(\frac{1}{1 - t_1\varepsilon_n} + 1\right)\left(\frac{t_1}{1 - t_1\varepsilon_n} + c_n\right) \\ &< \left(\frac{1}{1 - \frac{0.05947 \cdot 90}{n^9 \ln n}} + 1\right)\left(\frac{0.05947n^3}{1 - \frac{0.05947 \cdot 90}{n^9 \ln n}} + 0.25\right) \\ &= \left(\frac{1}{1 - \frac{0.05947 \cdot 90}{n^9 \ln n}} + 1\right)\left(\frac{0.05947}{1 - \frac{0.05947 \cdot 90}{n^9 \ln n}} + \frac{0.25}{n^3}\right) n^3 \\ &\leq \left(\frac{1}{1 - \frac{0.05947 \cdot 90}{8^9 \ln 8}} + 1\right)\left(\frac{0.05947}{1 - \frac{0.05947 \cdot 90}{8^9 \ln 8}} + \frac{0.25}{8^3}\right) n^3 < 0.11992n^3. \end{aligned}$$

Taking square-root,

$$t_3 < 0.3463n\sqrt{n}.$$

Last,

$$\begin{aligned} t_4 = t_3 + t_2\sqrt{\varepsilon_n} &< 0.3463n\sqrt{n} + 0.06143n^3 \cdot \frac{\sqrt{90}}{n^6\sqrt{ln(n)}} \\ &= \left[0.3463 + \frac{0.06143\sqrt{90}}{n^4\sqrt{n \ln n}}\right] n\sqrt{n} \\ &< \left[0.3463 + \frac{0.06143\sqrt{90}}{8^4\sqrt{8 \ln 8}}\right] n\sqrt{n} < 0.34634n\sqrt{n}. \end{aligned}$$

∎

Lemma 2 provides an estimate for $|w_j|$ with a factor of $\sqrt{\varepsilon}$. However, if we consider the real parts of the critical points rather than the moduli then we can find an estimate with a factor of ε. The next lemma provides this improved estimate. For the proof we will closely follow the proof of Lemma 3 in [2].

Lemma 3. *For* $j = 1, 2, ..., n-1$*:*

$$-t_2\varepsilon < \operatorname{Re} w_j < t_5\varepsilon,$$

where,

$$t_5 := 0.11998n^3.$$

Proof. We first claim that $\operatorname{Re} w_j \geq h_j$. Indeed, from (2.3) we recall that,

$$(a - \operatorname{Re} w_j)(\operatorname{Re} w_j - h_j) = (\operatorname{Im} w_j)^2.$$

From Lemma 2 we get,

$$\begin{aligned} a - \operatorname{Re} w_j \geq (1-\varepsilon) - |w_j| &> 1 - \varepsilon - t_4\sqrt{\varepsilon} \geq 1 - (\varepsilon_n + t_4\sqrt{\varepsilon_n}) \\ &> 1 - \left(\frac{90}{n^{12}\ln n} + (0.34634n\sqrt{n})\frac{\sqrt{90}}{n^6\sqrt{\ln n}}\right) \\ &\geq 1 - \left(\frac{90}{8^{12}\ln 8} + (0.34634)\frac{\sqrt{90}}{8^4\sqrt{8}\sqrt{\ln 8}}\right) \\ &> 0.9998 > 0. \end{aligned} \tag{3.1}$$

Since $a - \text{Re } w_j > 0$ and $(\text{Im } w_j)^2 \geq 0$, it follows that $\text{Re } w_j - h_j \geq 0$. Thus the bound on h_j as per (2.2) gives us the following bound on $\text{Re } w_j$,

$$\text{Re } w_j > h_j > -t_2\varepsilon.$$

On the other hand, for the upper bound on $\text{Re } w_j$, we will use (2.3) again,

$$(\text{Re } w_j - h_j)(a - \text{Re } w_j) = (\text{Im } w_j)^2 \leq |w_j|^2 < t_4^2\varepsilon.$$

Using the estimate for $a - \text{Re } w_j$ from (3.1),

$$(\text{Re } w_j - h_j)(0.9998) < t_4^2\varepsilon.$$

From (2.2), we note that $h_j < 0$,

$$\begin{aligned}\text{Re } w_j &< h_j + \frac{t_4^2\varepsilon}{0.9998} < 0 + \frac{(0.34634n\sqrt{n})^2\varepsilon}{0.9998} \\ &= \frac{0.34634^2}{0.9998}n^3\varepsilon < 0.11998n^3\varepsilon = t_5\varepsilon.\end{aligned}$$

∎

Remark 3.1. For $j = 1, 2, ..., n-1$, we have,

$$|\text{Re } w_j| < t_5\varepsilon \tag{3.2}$$

Proof. Since $-t_2\varepsilon < \text{Re } < t_5\varepsilon$ and $-t_2 < 0 < t_5$, we have,

$$|\text{Re } w_j| < \max\{|-t_2\varepsilon|, |t_5\varepsilon|\}.$$

(3.2) then follows from the observation that,

$$t_2 < 0.06143n^3 < 0.11998n^3 = t_5.$$

∎

Lemma 2 dictates that the critical points are close to the origin. This suggests that the roots of the polynomial should be respectively close to the n-th roots of unity. Lemma 4 quantifies this idea. Moreover, in this lemma we will make key improvements in some of the estimations given by Chijiwa in [2]. But to prove it we first we need the following lemma:

Lemma B (Miller [4])**.** *Let f be a polynomial of degree m. Let $p \in \mathbb{C}$. If $f'(p) \neq 0$ then there exists a zero z_0 of f such that $|p - z_0| \leq m\left|\frac{f(p)}{f'(p)}\right|$.*

For the next lemma, we need additional notation. Define $\xi_r := \exp\left(\frac{2(r-1)i\pi}{n}\right)$ for $r = 1, 2, ..., n$.

Lemma 4. *For $r = 1, 2, ..., n$ there exists $k_r \in \{1, 2, ..., n\}$ such that,*

$$|\xi_r - z_{k_r}| < t_6\sqrt{\varepsilon},$$

where,

$$t_6 := 0.69434n^2\sqrt{n}.$$

Proof. Write:

$$P'(z) = nz^{n-1} + \sum_{j=0}^{n-2} n(-1)^j S_j z^{n-1-j}$$

where,

$$S_j := \sum_{1\leq v_1 < v_2 < \ldots < v_j \leq n-1} w_{v_1}w_{v_2}\cdots w_{v_j}. \tag{4.1}$$

Then we have,

$$P(z) = P(0) + z^n + \sum_{j=1}^{n-1} (-1)^j \frac{n}{n-j} S_j z^{n-j}.$$

We need to estimate $P(\xi_r)$ and $P'(\xi_r)$ to use Lemma B.

$$\begin{aligned} P(\xi_r) = P(\xi_r) - P(a) &= \xi_r^n - a^n + \sum_{j=1}^{n-1} (-1)^j \frac{n}{n-j} S_j(\xi_r^{n-j} - a^{n-j}) \\ &= 1 - (1-\varepsilon)^n + \sum_{j=1}^{n-1} (-1)^j \frac{n}{n-j} S_j(\xi_r^{n-j} - (1-\varepsilon)^{n-j}) \qquad (4.2) \\ \implies |P(\xi_r)| &\le |1-(1-\varepsilon)^n| + \sum_{j=1}^{n-1} \frac{n}{n-j} |S_j| \cdot \left(|\xi_r|^{n-j} + |1-\varepsilon|^{n-j}\right). \end{aligned}$$

Using Bernoulli's Inequality we get $1 - n\varepsilon < (1-\varepsilon)^n$. We also note that $|\xi| = 1$ and $|1-\varepsilon| < 1$ hence we have,

$$|P(\xi_r)| \le |n\varepsilon| + \sum_{j=1}^{n-1} \frac{n}{n-j} |S_j| \cdot (1+1) = n\varepsilon + 2\sum_{j=1}^{n-1} \frac{n}{n-j} |S_j|. \qquad (4.3)$$

By (4.1) and the estimate on $|w_j|$ by Lemma 2 we get the following,

$$|S_j| = \sum_{1 \le v_1 < v_2 < \ldots < v_j \le n-1} |w_{v_1} w_{v_2} \cdots w_{v_j}| < \binom{n-1}{j} (t_4\sqrt{\varepsilon})^j. \qquad (4.4)$$

Substituting back in (4.3),

$$\begin{aligned} |P(\xi_r)| &\le n\varepsilon + 2\sum_{j=1}^{n-1} \frac{n}{n-j} \binom{n-1}{j} (t_4\sqrt{\varepsilon})^j = n\varepsilon + 2\sum_{j=1}^{n-1} \binom{n}{j} (t_4\sqrt{\varepsilon})^j \\ &= n\varepsilon + 2\left[(1+t_4\sqrt{\varepsilon})^n - 1 - (t_4\sqrt{\varepsilon})^n\right] < n\varepsilon + 2\left[(1+t_4\sqrt{\varepsilon})^n - 1\right]. \end{aligned}$$

Since $1 + x < \exp(x)$ we get,

$$|P(\xi_r)| < n\varepsilon + 2\left[\exp(nt_4\sqrt{\varepsilon}) - 1\right].$$

We will estimate $\exp(nt_4\sqrt{\varepsilon}) - 1$. Note that,

$$t_4\sqrt{\varepsilon} \le t_4\sqrt{\varepsilon_n} < 0.34634n\sqrt{n} \cdot \frac{\sqrt{90}}{n^6\sqrt{\ln n}} \le \frac{0.34634\sqrt{90}}{8^3\sqrt{8}\sqrt{\ln 8}} \cdot \frac{1}{n} < \frac{0.00158}{n}. \qquad (4.5)$$

The function $g(x) = \frac{\exp(nx)-1}{nx}$ is positive and increasing for $x > 0$. Hence we have,

$$g(t_4\sqrt{\varepsilon}) < g\left(\frac{0.00158}{n}\right) = \frac{\exp(0.00158) - 1}{0.00158} < 1.0008.$$

Thus we get,

$$\begin{aligned} |P(\xi_r)| &< n\varepsilon + 2\left[\exp(nt_4\sqrt{\varepsilon}) - 1\right] = n\varepsilon + 2\left[g(t_4\sqrt{\varepsilon})nt_4\sqrt{\varepsilon}\right] \\ &< n\varepsilon + 2 \cdot 1.0008 \cdot nt_4\sqrt{\varepsilon} \le n\sqrt{\varepsilon_n}\sqrt{\varepsilon} + 2.0016nt_4\sqrt{\varepsilon} \\ &< \frac{\sqrt{90}}{n^6\sqrt{\ln n}} \cdot n\sqrt{\varepsilon} + 2.0016 \cdot 0.34634n^2\sqrt{n}\sqrt{\varepsilon} \end{aligned}$$

$$= \left[\frac{\sqrt{90}}{n^7\sqrt{n}\sqrt{\ln n}} + 2.0016 \cdot 0.34634 \right] n^2\sqrt{n}\sqrt{\varepsilon}$$

$$\leq \left[\frac{\sqrt{90}}{8^7\sqrt{8}\sqrt{\ln 8}} + 2.0016 \cdot 0.34634 \right] n^2\sqrt{n}\sqrt{\varepsilon}$$

$$< 0.69324 n^2\sqrt{n}\sqrt{\varepsilon}. \tag{4.6}$$

Now let us estimate $P'(\xi_r)$. Lemma 2 gives us $|\xi_r - w_j| \geq 1 - |w_j| > 1 - t_4\sqrt{\varepsilon}$ for $j = 1, ..., n-1$. So we get,

$$|P'(\xi_r)| = n \prod_{j=1}^{n-1} |\xi_r - w_j| \geq n \prod_{j=1}^{n-1} (|\xi_r| - |w_j|) > n \prod_{j=1}^{n-1} \left(1 - t_4\sqrt{\varepsilon}\right) = n(1 - t_4\sqrt{\varepsilon})^{n-1}.$$

Since $1 - t_4\sqrt{\varepsilon} \geq 1 - t_4\sqrt{\varepsilon_n} > 0$, we can use Bernoulli's inequality to get,

$$|P'(\xi_r)| > n(1 - (n-1)t_4\sqrt{\varepsilon}) > n(1 - nt_4\sqrt{\varepsilon_n})$$

$$\geq n \left(1 - n \cdot 0.34634 n\sqrt{n} \cdot \frac{\sqrt{90}}{n^6\sqrt{\ln n}} \right)$$

$$\geq n \left[1 - 0.34634 \cdot \frac{\sqrt{90}}{8^3\sqrt{8}\sqrt{\ln 8}} \right] > 0.99842n.$$

Thus, by Lemma B there exists a zero z_{k_r} of P such that,

$$|z_{k_r} - \xi_r| \leq n \left| \frac{P(\xi_r)}{P'(\xi_r)} \right| < n \cdot \frac{0.69324 n^2\sqrt{n}\sqrt{\varepsilon}}{0.99842n}$$

$$= \frac{0.69324}{0.99842} \cdot n^2\sqrt{n}\sqrt{\varepsilon} < 0.69434 n^2\sqrt{n}\sqrt{\varepsilon} = t_6\sqrt{\varepsilon}.$$

∎

Remark 4.1. We can relabel the roots of P so that for $r = 1, 2, ..., n$,

$$|z_r - \xi_r| < t_6\sqrt{\varepsilon}.$$

Proof. We want to show that no root z_j can be close to two different roots of unity. This is equivalent to saying that all the roots $z_{k_r}, r = 1, ..., n$ are distinct. To see this we note that,

$$|z_{k_r} - \xi_r| < 0.69434 n^2\sqrt{n}\sqrt{\varepsilon_n} = 0.69434 \cdot \frac{\sqrt{90}}{n^3\sqrt{n}\sqrt{\ln n}}$$

$$\leq 0.69434 \cdot \frac{\sqrt{90}}{8^2\sqrt{8}\sqrt{\ln 8}} \cdot \frac{1}{n} < \frac{3}{n}.$$

We recall the lower bound on $\sin \frac{\pi}{n}$ given by (1.2), namely that for $n \geq 8 > 6$, $\sin \frac{\pi}{n} > \frac{3}{n}$. We thus have $|z_{k_r} - \xi_r| < \sin \frac{\pi}{n}$. Note that the distance between any two roots of unity, ξ_j and ξ_k is given by,

$$|\xi_j - \xi_k| = \left| 2 \sin \frac{(j-k)\pi}{n} \right| \geq 2 \sin \frac{\pi}{n}.$$

Thus the n disks $|z - \xi_r| < \sin \frac{\pi}{n}, r = 1, 2, ..., n$ are disjoint and hence there must be n distinct roots $z_{k_1}, z_{k_2}, ..., z_{k_n}$, one in each disk.

∎

The next lemma will be used in some of the succeeding lemmas. We will considerably improve upon Lemma 5 of [2].

Lemma 5. *For* $n \geq 8$,

$$\sum_{j=1}^{n-1} \frac{1}{\sin \frac{j\pi}{n}} < 1.01734 n \ln n\,.$$

Proof. We make two cases based on the parity of n. Suppose first that n is odd, i.e. $n = 2m+1, m \in \mathbb{N}$. Then we have,

$$\sum_{j=1}^{n-1} \csc \frac{j\pi}{n} = 2\sum_{j=1}^{m} \csc \frac{j\pi}{n} = 2\csc\frac{\pi}{n} + 2\sum_{j=2}^{m} \csc \frac{j\pi}{n}.$$

Since $\csc x$ is positive and decreasing for $x \in \left(0, \frac{\pi}{2}\right]$, we can approximate the sum by an integral,

$$\begin{aligned}\sum_{j=1}^{n-1} \csc \frac{j\pi}{n} &< 2\csc\frac{\pi}{n} + 2\int_1^m \csc\frac{x\pi}{n} dx \\ &< 2\csc\frac{\pi}{n} + 2\int_1^{\frac{n}{2}} \csc\frac{x\pi}{n} dx \\ &< 1 + 2\csc\frac{\pi}{n} + 2\int_1^{\frac{n}{2}} \csc\frac{x\pi}{n} dx.\end{aligned}$$

If $n = 2m, m \in \mathbb{N}$ then we have,

$$\begin{aligned}\sum_{j=1}^{n-1} \csc \frac{j\pi}{n} &= \csc\frac{\pi}{2} + 2\csc\frac{\pi}{n} + 2\sum_{j=2}^{m-1} \csc \frac{j\pi}{n} \\ &< 1 + 2\csc\frac{\pi}{n} + 2\int_1^{m-1} \csc\frac{x\pi}{n} dx \\ &< 1 + 2\csc\frac{\pi}{n} + 2\int_1^{\frac{n}{2}} \csc\frac{x\pi}{n} dx.\end{aligned}$$

So in both cases we have,

$$\begin{aligned}\sum_{j=1}^{n-1} \csc \frac{j\pi}{n} &< 1 + 2\csc\frac{\pi}{n} + 2\int_1^{\frac{n}{2}} \csc\frac{x\pi}{n} dx \\ &= 1 + 2\csc\frac{\pi}{n} - \frac{2n}{\pi}\ln\left|\frac{\csc\frac{\frac{n}{2}\pi}{n} + \cot\frac{\frac{n}{2}\pi}{n}}{\csc\frac{\pi}{n} + \cot\frac{\pi}{n}}\right| \\ &= 1 + 2\csc\frac{\pi}{n} - \frac{2n}{\pi}\ln\left|\frac{1+0}{\csc\frac{\pi}{n} + \cot\frac{\pi}{n}}\right| \\ &= 1 + 2\csc\frac{\pi}{n} + \frac{2n}{\pi}\ln\left(\csc\frac{\pi}{n} + \cot\frac{\pi}{n}\right).\end{aligned}$$

Let us estimate the terms involved. Note that by (1.2), $\csc\frac{\pi}{n} = \frac{1}{\sin\frac{\pi}{n}} < \frac{n}{3}$ for $n \geq 8 > 6$. On the other hand,

$$\csc\frac{\pi}{n} + \cot\frac{\pi}{n} = \frac{1}{\sin\frac{\pi}{n}}\left[1 + \cos\frac{\pi}{n}\right] < \frac{n}{3}\left[1+1\right] = \frac{2n}{3} < n.$$

Since $\ln(x)$ is monotonically increasing for $x > 0$ we have,

$$\ln\left(\csc\frac{\pi}{n} + \cot\frac{\pi}{n}\right) < \ln n.$$

Thus,

$$\sum_{j=1}^{n-1} \csc\frac{j\pi}{n} < 1 + \frac{2n}{3} + \frac{2n}{\pi}\ln n = \left[\frac{1}{n\ln n} + \frac{2}{3\ln n} + \frac{2}{\pi}\right] n\ln n$$
$$\leq \left[\frac{1}{8\ln 8} + \frac{2}{3\ln 8} + \frac{2}{\pi}\right] n\ln n < 1.01734 n\ln n.$$

∎

The next lemma will be used to improve some of our previous estimates.
We will strengthen various inequalities of Lemma 6 from [2].

Lemma 6. *For* $r = 1, 2, ..., n$,

$$\left|\xi_r \prod_{j\neq r}(\xi_j - z_j) - n\right| < t_7\sqrt{\varepsilon},$$

where,

$$t_7 = 0.35872 n^4 \sqrt{n}\ln n.$$

Proof. Let $r \in \{1, ..., n\}$. Let $H(z) := z^n - 1 = (z - \xi_r)H_r(z)$. Then $H'(z) = H_r(z) + (z - \xi_r)H_r'(z)$, which implies that $H'(\xi_r) = n\xi_r^{n-1} = H_r(\xi_r)$. Let $P(z) = (z - z_r)Q_r(z)$. Define,

$$K := |Q_r(\xi_r) - H_r(\xi_r)| = \left|\prod_{j\neq r}(\xi_r - z_j) - n\xi_r^{n-1}\right|.$$

Since $|\xi_r| = 1$, we have $K = |\xi_r \prod_{j\neq r}(\xi_r - z_j) - n\xi_r^n| = |\xi_r \prod_{j\neq r}(\xi_r - z_j) - n|$, which is in fact the expression we want to bound.

Note that,

$$Q_r(\xi_r) = \prod_{j\neq r}(\xi_r - z_j) = H_r(\xi_r)\prod_{j\neq r}\frac{(\xi_r - z_j)}{(\xi_r - \xi_j)} = H_r(\xi_r)\prod_{j\neq r}\left[1 + \frac{\xi_j - z_j}{\xi_r - \xi_j}\right].$$

Let $\Delta_j := \frac{\xi_j - z_j}{\xi_r - \xi_j}$ for $j \neq r$. Then we can write,

$$Q_r(\xi_r) = H_r(\xi_r)\prod_{j\neq r}(1 + \Delta_j)$$
$$\implies Q_r(\xi_r) - H_r(\xi_r) = H_r(\xi_r)\left[\prod_{j\neq r}(1 + \Delta_j) - 1\right]$$
$$\implies K = |Q_r(\xi_r) - H_r(\xi_r)| = |H_r(\xi_r)|\left|\prod_{j\neq r}(1 + \Delta_j) - 1\right|$$
$$= n\left|\prod_{j\neq r} e^{\ln(1+\Delta_j)} - 1\right| = n\left|e^{\sum_{j\neq r}\ln(1+\Delta_j)} - 1\right|.$$

Since $|e^z - 1| \leq e^{|z|} - 1$ (consider the power series) we get,

$$\implies K \leq n\left[e^{\left|\sum_{j\neq r}\ln(1+\Delta_j)\right|} - 1\right] \leq n\left[e^{\sum_{j\neq r}|\ln(1+\Delta_j)|} - 1\right]. \tag{6.1}$$

Let us estimate the term above. By Lemma 4 and (1.2) we obtain,

$$|\Delta_j| = \left|\frac{\xi_j - z_j}{\xi_r - \xi_j}\right| < \frac{t_6\sqrt{\varepsilon}}{\left|2\sin\frac{(r-j)\pi}{n}\right|} \tag{6.2}$$

$$\le \frac{t_6\sqrt{\varepsilon}}{2\sin\frac{\pi}{n}} < \frac{nt_6\sqrt{\varepsilon}}{6} \tag{6.3}$$

$$\le \frac{n\cdot 0.69434n^2\sqrt{n}\cdot\sqrt{\varepsilon_n}}{6} \le \frac{n}{6}\cdot 0.69434n^2\sqrt{n}\cdot\frac{\sqrt{90}}{n^6\sqrt{\ln n}}$$

$$\le \frac{0.69434}{6}\cdot\frac{\sqrt{90}}{n^2\sqrt{n}\sqrt{\ln n}} \le \frac{0.69434}{6}\cdot\frac{\sqrt{90}}{8^2\sqrt{8}\sqrt{\ln 8}} < 0.00421.$$

From the power series of logarithm, we observe that $|\ln(1+\Delta_j)| \le -\ln(1-|\Delta_j|)$. Consider the function $u(x) := \frac{-\ln(1-x)}{x}$. This function is positive and increasing on $x \in (0,1)$. Hence we have,

$$u(|\Delta_j|) \le u(0.00421) < 1.00212.$$

Thus we get,

$$|\ln(1+\Delta_j)| \le -\ln(1-|\Delta_j|) < 1.00212|\Delta_j|.$$

Let $\gamma := \sum_{j\neq r} |\ln(1+\Delta_j)|$. Then summing up the above over $j = 1, 2, ..., n$ and using (6.2) we get,

$$\gamma < 1.00212\sum_{j\neq r}|\Delta_j| < \frac{1.00212t_6\sqrt{\varepsilon}}{2}\sum_{j\neq r}\frac{1}{\left|\sin\frac{(r-j)\pi}{n}\right|}.$$

We already have an estimate for the summation due to Lemma 5. Furthermore, using the estimate for t_6 from Lemma 4 we get,

$$\gamma < \frac{1.00212\cdot 0.69434n^2\sqrt{n}\sqrt{\varepsilon}}{2}\cdot 1.01734n\ln n < 0.35394n^3\sqrt{n}\ln n\sqrt{\varepsilon} \tag{6.4}$$

$$\le 0.35394n^3\sqrt{n}\ln n\sqrt{\varepsilon_n} = 0.35394\cdot\frac{\sqrt{\ln n}\sqrt{90}}{n^2\sqrt{n}} \le 0.35394\cdot\frac{\sqrt{\ln 8}\sqrt{90}}{8^2\sqrt{8}}$$

$$< 0.02675.$$

In the above chain of inequalities, we used the fact that $\frac{\ln n}{n}$ is monotonically decreasing for $n \ge 8 > e$, hence $\frac{\ln n}{n} \le \frac{\ln 8}{8}$.

The function $g(x) := \frac{\exp(x)-1}{x}$ is positive and increasing for $x > 0$. So we get,

$$g(\gamma) < g(0.02675) = 1.0135$$
$$\implies \exp(\gamma) - 1 < 1.0135\gamma.$$

Substituting back in (6.1), then using the estimate for γ given by (6.4) we obtain,

$$K \le n[\exp(\gamma)-1] < n\cdot 1.0135\gamma < n\cdot 1.0135\cdot 0.35394n^3\sqrt{n}\ln n\sqrt{\varepsilon}$$
$$\implies K = \left|\xi_r\prod_{j\neq r}(\xi_j - z_j) - n\right| < 0.35872n^4\sqrt{n}\ln n\sqrt{\varepsilon} < t_7\sqrt{\varepsilon}.$$

∎

Corollary 6.1. By taking the real and imaginary parts of $\xi_r\prod_{j\neq r}(\xi_j - z_j) - n$ respectively we get,

$$n - t_7\sqrt{\varepsilon} < \mathrm{Re}\left(\xi_r\prod_{j\neq r}(\xi_j - z_j)\right) < n + t_7\sqrt{\varepsilon},$$

$$-t_7\sqrt{\varepsilon} < \operatorname{Im}\left(\xi_r \prod_{j\neq r}(\xi_j - z_j)\right) < t_7\sqrt{\varepsilon}.$$

Also,

$$n - t_7\sqrt{\varepsilon} < \left|\xi_r \prod_{j\neq r}(\xi_j - z_j)\right| < n + t_7\sqrt{\varepsilon}.$$

Remark 6.1. Some estimations:

$$\begin{aligned} n - t_7\sqrt{\varepsilon} &\geq n - t_7\sqrt{\varepsilon_n} > n - 0.35872 n^4\sqrt{n}\ln n \cdot \frac{\sqrt{90}}{n^6\sqrt{\ln n}} \\ &\geq \left[1 - \frac{0.35872\sqrt{90}\sqrt{\ln n}}{n^2\sqrt{n}}\right] n \geq \left[1 - \frac{0.35872\sqrt{90}\sqrt{\ln 8}}{8^2\sqrt{8}}\right] n > 0.97289n. \end{aligned}$$

$$\begin{aligned} n - t_7\sqrt{\varepsilon} &\leq n + t_7\sqrt{\varepsilon_n} < n + 0.35872 n^4\sqrt{n}\ln n \cdot \frac{\sqrt{90}}{n^6\sqrt{\ln n}} \\ &\leq \left[1 + \frac{0.35872\sqrt{90}\sqrt{\ln n}}{n^2\sqrt{n}}\right] n \leq \left[1 + \frac{0.35872\sqrt{90}\sqrt{\ln 8}}{8^2\sqrt{8}}\right] n < 1.02711n. \end{aligned}$$

The next lemma provides an estimate for $|S_1|$ (see equation (4.1)). This lemma will be used to improve many of our previous estimates.

We will closely follow the proof of Lemma 7 in [2].

Lemma 7. *We have,*

$$|S_1| < t_9\varepsilon,$$

where,

$$t_9 = (n-1)\sqrt{t_5^2 + \frac{(t_8 + 2nt_5)^2}{32}} < 0.04682(n-1)n^6\ln n\,,$$

with,

$$t_8 = 0.26304 n^6 \ln n\,.$$

Proof. Note that for $r \in \{1, 2, ..., n\}$:

$$\frac{\xi_r - z_r}{\xi_r} = \frac{P(\xi_r)}{\xi_r \prod_{j\neq r}(\xi_r - z_j)}.$$

Using (4.2) to express $P(\xi_r)$,

$$\begin{aligned} \frac{\xi_r - z_r}{\xi_r} =& \frac{1}{\xi_r \prod_{j\neq r}(\xi_r - z_j)}\left[1 - (1-\varepsilon)^n - \frac{nS_1}{n-1}(\xi_r^{n-1} - (1-\varepsilon)^{n-1})\right. \\ &\left. + \sum_{j=2}^{n-1}(-1)^j \frac{nS_j}{n-j}(\xi_r^{n-j} - (1-\varepsilon)^{n-j})\right]. \end{aligned} \tag{7.1}$$

Thus we have,

$$\frac{\xi_r - z_r}{\xi_r} = \frac{T_1}{\lambda_r} + \frac{T_2}{\lambda_r}, \tag{7.2}$$

where,

$$T_1 := -\frac{nS_1}{n-1}(\xi_r^{n-1} - (1-\varepsilon)^{n-1}),$$

$$T_2 := 1 - (1-\varepsilon)^n + \sum_{j=2}^{n-1} (-1)^j \frac{nS_j}{n-j} (\xi_r^{n-j} - (1-\varepsilon)^{n-j}),$$
$$\lambda_r := \xi_r \prod_{j \neq r} (\xi_r - z_j), \quad r = 1, 2, ..., n.$$

Taking real parts we have,

$$\text{Re}\, \frac{\xi_r - z_r}{\xi_r} = \text{Re}\, \frac{T_1}{\lambda_r} + \text{Re}\, \frac{T_2}{\lambda_r}.$$

Since $\text{Re}\, \frac{\xi_r - z_r}{\xi_r} = 1 - \text{Re}\, (z_r \overline{\xi_r}) \geq 1 - |z_r \overline{\xi_r}| \geq 0$,

$$\text{Re}\, \frac{T_1}{\lambda_r} + \text{Re}\, \frac{T_2}{\lambda_r} \geq 0 \implies \text{Re}\, \frac{T_1}{\lambda_r} \geq -\text{Re}\, \frac{T_2}{\lambda_r}$$
$$\implies \text{Re}\, \frac{T_1}{\lambda_r} \geq -\frac{|T_2|}{|\lambda_r|}$$
$$\implies \frac{\text{Re}\, T_1 \cdot \text{Re}\, \lambda_r + \text{Im}\, T_1 \cdot \text{Im}\, \lambda_r}{|\lambda_r|^2} \geq -\frac{|T_2|}{|\lambda_r|}.$$

The Corollary and Remark to Lemma 6 guarantee us that $\text{Re}\, \lambda_r > 0$, so we can write,

$$\text{Re}\, T_1 \geq -\left[\frac{|T_2| \cdot |\lambda_r| + \text{Im}\, T_1 \cdot \text{Im}\, \lambda_r}{\text{Re}\, \lambda_r} \right]. \tag{7.3}$$

Let us estimate $\text{Im}\, T_1$,

$$|\text{Im}\, T_1| \leq |T_1| = \frac{n}{n-1} |S_1| \cdot |\xi_r^{n-1} - (1-\varepsilon)^{n-1}| < \frac{n}{n-1} |S_1| \cdot 2. \tag{7.4}$$

Using Lemma 2 for $S_1 = \sum_{j=1}^{n-1} w_j$ we get,

$$|\text{Im}\, T_1| < \frac{2n|S_1|}{n-1} < \frac{2n}{n-1}(n-1) t_4 \sqrt{\varepsilon} = 2nt_4\sqrt{\varepsilon} < 0.69268 n^2 \sqrt{n} \sqrt{\varepsilon}. \tag{7.5}$$

Now let us estimate $|T_2|$,

$$|T_2| \leq 1 - (1-\varepsilon)^n + \sum_{j=2}^{n-1} |(-1)^j| \frac{n}{n-j} |S_j| \cdot \left| \xi_r^{n-v} - (1-\varepsilon)^{n-v} \right|$$
$$< n\varepsilon + \sum_{j=2}^{n-1} \frac{n}{n-j} |S_j| \cdot 2 < n\varepsilon + 2 \sum_{j=2}^{n-1} \frac{n}{n-j} \binom{n-1}{j} (t_4\sqrt{\varepsilon})^j$$
$$= n\varepsilon + 2 \sum_{j=2}^{n-1} \binom{n}{j} (t_4\sqrt{\varepsilon})^j = n\varepsilon + 2 \left[(1 + t_4\sqrt{\varepsilon})^n - 1 - nt_4\sqrt{\varepsilon} - (t_4\sqrt{\varepsilon})^n \right]$$
$$< n\varepsilon + 2 \left[(1 + t_4\sqrt{\varepsilon})^n - 1 - nt_4\sqrt{\varepsilon} \right] < n\varepsilon + 2 \left[\exp(nt_4\sqrt{\varepsilon}) - 1 - nt_4\sqrt{\varepsilon} \right].$$

The function $f(x) = \frac{\exp(nx) - 1 - nx}{n^2 x^2}$ is positive and increasing on $x > 0$. Thus by (4.5) we get,

$$f(t_4\sqrt{\varepsilon}) < f\left(\frac{0.00158}{n} \right) < 0.50027$$
$$\implies \exp(nt_4\sqrt{\varepsilon}) - 1 - nt_4\sqrt{\varepsilon} < 0.50027 n^2 t_4^2 \varepsilon.$$

So we have,

$$|T_2| < n\varepsilon + 2 \cdot 0.50027 n^2 t_4^2 \varepsilon < n\varepsilon + 2 \cdot 0.50027 n^2 (0.34634 n\sqrt{n})^2 \varepsilon$$

$$= \left[\frac{1}{n^4} + 2 \cdot 0.50027 \cdot 0.34634^2\right] n^5\varepsilon$$

$$\le \left[\frac{1}{8^4} + 2 \cdot 0.50027 \cdot 0.34634^2\right] n^5\varepsilon < 0.12027 n^5\varepsilon. \tag{7.6}$$

Using (7.5), the above inequality, and the Corollary and Remark of 6 to estimate the terms in equation (7.3) we get,

$$\begin{aligned}
\mathrm{Re}\, T_1 &\ge -\left[\frac{|T_2|\cdot|\lambda_r| + \mathrm{Im}\, T_1 \cdot \mathrm{Im}\,\lambda_r}{\mathrm{Re}\,\lambda_r}\right] > -\left[\frac{|T_2|(n + t_7\sqrt{\varepsilon}) + \mathrm{Im}\, T_1 \cdot t_7\sqrt{\varepsilon}}{n - t_7\sqrt{\varepsilon}}\right] \\
&\ge -\left[\frac{0.12027 n^5\varepsilon \cdot 1.02711 n + 0.69268 n^2\sqrt{n}\sqrt{\varepsilon} \cdot 0.35872 n^4\sqrt{n}\ln n\sqrt{\varepsilon}}{0.97289 n}\right] \\
&= -\left[\frac{0.12027 \cdot 1.02711 n^5\varepsilon + 0.69268 \cdot 0.35872 n^6 \ln n\,\varepsilon}{0.97289}\right] \\
&= -\left[\frac{\frac{0.12027\cdot 1.02711}{n\ln n} + 0.69268 \cdot 0.35872}{0.97289}\right] n^6 \ln n\,\varepsilon \\
&\ge -\left[\frac{\frac{0.12027\cdot 1.02711}{8\ln 8} + 0.69268 \cdot 0.35872}{0.97289}\right] n^6 \ln n\,\varepsilon \\
&> -0.26304 n^6 \ln n\,\varepsilon = -t_8\varepsilon.
\end{aligned}$$

Note that,

$$\begin{aligned}
0 &< t_8\varepsilon + \mathrm{Re}\, T_1 \\
&= t_8\varepsilon + \mathrm{Re}\,\left[-\frac{nS_1}{n-1}(\xi_r^{n-1} - (1-\varepsilon)^{n-1})\right] \\
&= t_8\varepsilon + \frac{n(1-\varepsilon)^{n-1}}{n-1}\mathrm{Re}\, S_1 + \frac{n}{n-1}\mathrm{Re}\,\left[-S_1\xi_r^{n-1}\right] \\
&= t_8\varepsilon + \frac{n(1-\varepsilon)^{n-1}}{n-1}\mathrm{Re}\, S_1 - \frac{n}{n-1}\mathrm{Re}\, S_1 \cdot \mathrm{Re}\,\xi_r^{n-1} + \frac{n}{n-1}\mathrm{Im}\, S_1 \cdot \mathrm{Im}\,\xi_r^{n-1}.
\end{aligned} \tag{7.7}$$

Since $\xi_r^{n-1} = \overline{\xi_r}$ we can write,

$$\begin{aligned}
0 &< t_8\varepsilon + \frac{n(1-\varepsilon)^{n-1}}{n-1}\mathrm{Re}\, S_1 - \frac{n}{n-1}\mathrm{Re}\, S_1 \cdot \mathrm{Re}\,\xi_r - \frac{n}{n-1}\mathrm{Im}\, S_1 \cdot \mathrm{Im}\,\xi_r \\
&\le t_8\varepsilon + \frac{n(1-\varepsilon)^{n-1}}{n-1}|\mathrm{Re}\, S_1| + \frac{n}{n-1}|\mathrm{Re}\, S_1| \cdot |\mathrm{Re}\,\xi_r| - \frac{n}{n-1}\mathrm{Im}\, S_1 \cdot \mathrm{Im}\,\xi_r.
\end{aligned}$$

Since $(1-e)^n < 1$ and $|\mathrm{Re}\,\xi_r| \le 1$,

$$0 < t_8\varepsilon + \frac{n}{n-1}|\mathrm{Re}\, S_1| + \frac{n}{n-1}|\mathrm{Re}\, S_1| - \frac{n}{n-1}\mathrm{Im}\, S_1 \cdot \mathrm{Im}\,\xi_r.$$

Note that $\mathrm{Re}\, S_1 = \sum_{j=1}^{n-1} \mathrm{Re}\, w_j$. From Remark 3.1 we get,

$$|\mathrm{Re}\, S_1| < (n-1)t_5\varepsilon \tag{7.8}$$

$$\begin{aligned}
\implies 0 &< t_8\varepsilon + 2\frac{n}{n-1}(n-1)t_5\varepsilon - \frac{n}{n-1}\mathrm{Im}\, S_1 \cdot \mathrm{Im}\,\xi_r \\
&< t_8\varepsilon + 2nt_5\varepsilon - \frac{n}{n-1}\mathrm{Im}\, S_1 \cdot \mathrm{Im}\,\xi_r.
\end{aligned}$$

Thus,

$$\text{Im } S_1 \cdot \text{Im } \xi_r < \frac{n-1}{n}(t_8 + 2nt_5)\varepsilon.$$

Choose r such that $\text{Im } S_1$ and $\text{Im } \xi_r$ have the same sign, so we get,

$$\begin{aligned} |\text{Im } S_1| \cdot |\text{Im } \xi_r| &< \frac{n-1}{n}(t_8 + 2nt_5)\varepsilon \\ \implies |\text{Im } S_1| &< \frac{(n-1)(t_8+2nt_5)}{n \sin \frac{2(r-1)\pi}{n}}\varepsilon \le \frac{(n-1)(t_8+2nt_5)}{n \sin \frac{2\pi}{n}}\varepsilon. \end{aligned}$$

For $n \ge 8$ we have $\sin \frac{2\pi}{n} \ge \frac{\sin \frac{2\pi}{8}}{\frac{2\pi}{8}} \cdot \frac{2\pi}{n} = \frac{4\sqrt{2}}{n}$, hence we get,

$$|\text{Im } S_1| < \frac{(n-1)(t_8+2nt_5)}{4\sqrt{2}}\varepsilon. \tag{7.9}$$

Using (7.8),

$$|S_1| < \sqrt{(\text{Re } S_1)^2 + (\text{Im } S_1)^2} < (n-1)\varepsilon\sqrt{t_5^2 + \frac{(t_8+2nt_5)^2}{32}} = t_9\varepsilon.$$

Let us estimate t_9,

$$\begin{aligned} t_9 &= (n-1)\sqrt{t_5^2 + \frac{(t_8+2nt_5)^2}{32}} \\ &< (n-1)\sqrt{(0.11998n^3)^2 + \frac{(0.26304n^6 \ln n + 2n \cdot 0.11998n^3)^2}{32}} \\ &= (n-1)n^6 \ln n \sqrt{\frac{(0.11998n^3)^2}{n^{12}\ln(n)^2} + \frac{\left(0.26304 + 2n \cdot \frac{0.11998n^3}{n^6 \ln n}\right)^2}{32}} \\ &= (n-1)n^6 \ln n \sqrt{\frac{0.11998^2}{n^6 \ln(n)^2} + \frac{\left(0.26304 + 2 \cdot \frac{0.11998}{n^2 \ln n}\right)^2}{32}} \\ &\le (n-1)n^6 \ln n \sqrt{\frac{0.11998^2}{8^6 \ln(8)^2} + \frac{\left(0.26304 + 2 \cdot \frac{0.11998}{8^2 \ln 8}\right)^2}{32}} \\ &< 0.04682(n-1)n^6 \ln n. \end{aligned}$$

∎

In the next lemma we will improve Lemma 4 which provided an estimate for the closeness of the roots of the polynomial to the nth roots of unity. We will closely follow the proof of Lemma 8 in [2].

Lemma 8. *For* $r = 1, 2, ..., n$,

$$|z_r - \xi_r| < t_{10}\varepsilon,$$

where,

$$t_{10} = 0.09718n^6 \ln n.$$

Proof. As per (7.2) in the proof of Lemma 7, for $r = 1, 2, ..., n$, we get,

$$|z_r - \xi_r| = \left|\frac{z_r - \xi_r}{\xi_r}\right| \le \frac{|T_1| + |T_2|}{\left|\xi_r \prod_{j \ne r}(\xi_r - z_j)\right|} \le \frac{\frac{2n}{n-1}|S_1| + |T_2|}{\left|\xi_r \prod_{j \ne r}(\xi_r - z_j)\right|}.$$

Using Lemma 7 and (7.6) and the Corollary and Remark of Lemma 7 to estimate the above terms, we get,

$$\begin{aligned}|z_r - \xi_r| &< \frac{\frac{2n}{n-1}0.04682(n-1)n^6 \ln n\,\varepsilon + 0.12027n^5\varepsilon}{0.97289n}\\ &= n^6 \ln n\,\varepsilon\left[\frac{2\cdot 0.04682 + \frac{0.12027}{n^2\ln n}}{0.97289}\right]\\ &\leq n^6 \ln n\,\varepsilon\left[\frac{2\cdot 0.04682 + \frac{0.12027}{8^2\ln 8}}{0.97289}\right]\\ &< 0.09718n^6\ln n\,\varepsilon = t_{10}\varepsilon.\end{aligned}$$

∎

In the next lemma we will improve Lemma 6. We will follow the same proof as that of Lemma 6, but we use the estimates from Lemma 8.

Lemma 9. *For* $r = 1, 2, ..., n$,

$$\left|\xi_r \prod_{j\neq r}(\xi_j - z_j) - n\right| < t_{11}\varepsilon,$$

where,

$$t_{11} = 0.04945n^8 \ln(n)^2 .$$

Proof. Let $r \in \{1, ..., n\}$. We replace $t_6\sqrt{\varepsilon}$ by $t_{10}\varepsilon$ in (6.3) to get,

$$\begin{aligned}|\Delta_j| = \left|\frac{\xi_j - z_j}{\xi_r - \xi_j}\right| &< \frac{t_{10}\varepsilon}{\left|2\sin\frac{(r-j)\pi}{n}\right|} \qquad (9.1)\\ &\leq \frac{t_{10}\varepsilon}{2\sin\frac{\pi}{n}} < \frac{nt_{10}\varepsilon_n}{6} \leq \frac{n}{6}\cdot 0.09718n^6\ln n\cdot\frac{90}{n^{12}\ln n}\\ &= \frac{0.09718}{6}\cdot\frac{90}{n^5} \leq \frac{0.09718}{6}\cdot\frac{90}{8^5} < 0.00005.\end{aligned}$$

The function $u(x) := \frac{-\ln(1-x)}{x}$ is positive and increasing for $x \in (0, 1)$. Hence we have,

$$u(|\Delta_j|) \leq u(0.00005) < 1.00003.$$

Thus we get,

$$|\ln(1 + \Delta_j)| \leq -\ln(1 - |\Delta_j|) < 1.00003|\Delta_j|. \qquad (9.2)$$

Recall from Lemma 6 that $\gamma = \sum_{j\neq r} |\ln(1 + \Delta_j)|$. Summing up the above over $j = 1, 2, ..., n$ and using (9.1),

$$\gamma < 1.00003 \sum_{j\neq r} |\Delta_j| < \frac{1.00003t_{10}\varepsilon}{2}\sum_{j\neq r}\frac{1}{\left|\sin\frac{(r-j)\pi}{n}\right|}.$$

Using the estimates from Lemma 8 and Lemma 5 we get,

$$\begin{aligned}\gamma &< 1.00003\cdot\frac{0.09718n^6\ln n\,\varepsilon}{2}\cdot 1.01734n\ln n < 0.04944n^7\ln(n)^2\varepsilon \qquad (9.3)\\ &< 0.04944n^7\ln(n)^2\varepsilon_n = 0.04944\cdot\frac{90\ln n}{n^5} \leq 0.04944\cdot\frac{90\ln 8}{8^5} < 0.00029.\end{aligned}$$

In the above, we again used the fact that $\frac{\ln n}{n}$ is strictly decreasing for $n \geq 8$.

The function $g(x) := \frac{\exp(x)-1}{x}$ is positive and increasing for $x > 0$. So we have,

$$g(\gamma) < g(0.00029) = 1.00015$$
$$\implies \exp(\gamma) - 1 < 1.00015\gamma.$$

Substituting back in (6.1), then using our estimate for γ from (9.3) we get,

$$K \leq n[\exp(\gamma) - 1] < 1.00015n\gamma < 1.00015n \cdot 0.04944n^7 \ln(n)^2 \varepsilon < 0.04945n^8 \ln(n)^2 \varepsilon,$$
$$\implies K = \left|\xi_r \prod_{j \neq r}(\xi_j - z_j) - n\right| < 0.04945n^8 \ln(n)^2 \varepsilon = t_{11}\varepsilon.$$

∎

Corollary 9.1. By taking the real and imaginary parts of $\xi_r \prod_{j\neq r}(\xi_j - z_j) - n$ respectively we get,

$$n - t_{11}\varepsilon < \mathrm{Re}\left(\xi_r \prod_{j\neq r}(\xi_j - z_j)\right) < n + t_{11}\varepsilon,$$
$$-t_{11}\varepsilon < \mathrm{Im}\left(\xi_r \prod_{j\neq r}(\xi_j - z_j)\right) < t_{11}\varepsilon.$$

Also,

$$n - t_{11}\varepsilon < \left|\xi_r \prod_{j\neq r}(\xi_j - z_j)\right| < n + t_{11}\varepsilon.$$

Remark 9.1. Some estimations:

$$n - t_{11}\varepsilon \geq n - t_{11}\varepsilon_n > n - 0.04945n^8 \ln(n)^2 \frac{90}{n^{12}\ln n} = \left[1 - \frac{0.04945 \cdot 90 \ln n}{n^5}\right] n$$
$$\geq \left[1 - \frac{0.04945 \cdot 90 \ln 8}{8^5}\right] n > 0.99971n.$$

$$n + t_{11}\varepsilon \leq n + t_{11}\varepsilon_n < n + 0.04945n^8 \ln(n)^2 \frac{90}{n^{12}\ln n} = \left[1 + \frac{0.04945 \cdot 90 \ln n}{n^5}\right] n$$
$$\leq \left[1 + \frac{0.04945 \cdot 90 \ln 8}{8^5}\right] n < 1.00029n.$$

At this point we will divert from the proof outline given in [2]. We can further improve the estimates in Lemma 7, Lemma 8 and Lemma 9 by running the proofs of these lemmas with the new estimates. The next three Lemmas will be respectively identical to the last three, except for the use of better estimates.

Lemma 10 (Improvement of Lemma 7)**.** *We have,*

$$|S_1| < t_{13}\varepsilon,$$

where,

$$t_{13} = (n-1)\varepsilon\sqrt{t_{12}^2 + \frac{(t_{12} + 2nt_5)^2}{32}} < 0.02727(n-1)n^5,$$

with,

$$t_{12} = 0.12387n^5.$$

Proof. We will mimic the proof of Lemma 7. Using Lemma 7 to improve (7.4),

$$|\mathrm{Im}\, T_1| < \frac{2n|S_1|}{n-1} < \frac{2nt_9\varepsilon}{n-1} = \frac{2n \cdot 0.04682(n-1)n^6 \ln n\, \varepsilon}{n-1} = 0.09364n^7 \ln n\, \varepsilon.$$

As in Lemma 7, we intend to use Equation (7.3) to estimate $\mathrm{Re}\, T_1$. We will use the above inequality to estimate $\mathrm{Im}\, T_1$. We will use the estimate for T_2 from Lemma 7, namely (7.6). And we will use the Corollary and Remark of Lemma 9 as well. So for $r \in \{1, 2, ..., n\}$ we get,

$$\begin{aligned}
\mathrm{Re}\, T_1 &\geq -\left[\frac{|T_2| \cdot |\lambda_r| + \mathrm{Im}\, T_1 \cdot \mathrm{Im}\, \lambda_r}{\mathrm{Re}\, \lambda_r}\right] > -\left[\frac{|T_2|(n + t_{11}\varepsilon_n) + \mathrm{Im}\, T_1 \cdot t_{11}\varepsilon_n}{n - t_{11}\varepsilon_n}\right] \\
&> -\left[\frac{0.12027n^5\varepsilon \cdot 1.00029n + 0.09364n^7 \ln n\, \varepsilon \cdot 0.04945n^8 \ln(n)^2\, \varepsilon_n}{0.99971n}\right] \\
&= -\left[\frac{0.12027 \cdot 1.00029n^6 + 0.09364 \cdot 0.04945n^{15} \ln(n)^3 \cdot \frac{90}{n^{12}\ln n}}{0.99971n}\right]\varepsilon \\
&= -\left[\frac{0.12027 \cdot 1.00029n^5 + 0.09364 \cdot 0.04945 \cdot 90 \ln(n)^2\, n^2}{0.99971}\right]\varepsilon \\
&= -\left[\frac{0.12027 \cdot 1.00029 + 0.09364 \cdot 0.04945 \cdot \frac{90\ln(n)^2}{n^3}}{0.99971}\right] n^5\varepsilon \\
&\geq -\left[\frac{0.12027 \cdot 1.00029 + 0.09364 \cdot 0.04945 \cdot \frac{90\ln(8)^2}{8^3}}{0.99971}\right] n^5\varepsilon \\
&> -0.12387n^5\varepsilon = -t_{12}\varepsilon.
\end{aligned}$$

We can now follow the proof of Lemma 7 from (7.7) to (7.9) in the same way except for replacing t_8 with t_{12}. We then get,

$$|\mathrm{Im}\, S_1| < \frac{(n-1)(t_{12} + 2nt_5)}{4\sqrt{2}}\varepsilon.$$

Using (7.8),

$$|S_1| < \sqrt{(\mathrm{Re}\, S_1)^2 + (\mathrm{Im}\, S_1)^2} < (n-1)\varepsilon\sqrt{t_5^2 + \frac{(t_{12} + 2nt_5)^2}{32}} = t_{13}\varepsilon.$$

Let us estimate t_{13},

$$\begin{aligned}
t_{13} &= (n-1)\sqrt{t_5^2 + \frac{(t_{12} + 2nt_5)^2}{32}} \\
&< (n-1)\sqrt{(0.11998n^3)^2 + \frac{(0.12387n^5 + 2n \cdot 0.11998n^3)^2}{32}} \\
&= (n-1)n^5\sqrt{\frac{(0.11998n^3)^2}{n^{10}} + \frac{\left(0.12387 + 2n \cdot \frac{0.11998n^3}{n^5}\right)^2}{32}} \\
&= (n-1)n^5\sqrt{\frac{0.11998^2}{n^4} + \frac{\left(0.12387 + 2 \cdot \frac{0.11998}{n}\right)^2}{32}} \\
&\leq (n-1)n^5\sqrt{\frac{0.11998^2}{8^4} + \frac{\left(0.12387 + 2 \cdot \frac{0.11998}{8}\right)^2}{32}} \\
&< 0.02727(n-1)n^5.
\end{aligned}$$

∎

Lemma 11 (Improvement of Lemma 8). *For* $r = 1, 2, ..., n$,

$$|z_r - \xi_r| < t_{14}\varepsilon,$$

where,

$$t_{14} = 0.0696n^5.$$

Proof. We will mimic the proof of Lemma 8. Due to (7.2), for $r = 1, 2, ..., n$ we get,

$$|z_r - \xi_r| = \left|\frac{z_r - \xi_r}{\xi_r}\right| \leq \frac{|T_1| + |T_2|}{\left|\xi_r \prod_{j\neq r}(\xi_r - z_j)\right|} \leq \frac{\frac{2n}{n-1}|S_1| + |T_2|}{\left|\xi_r \prod_{j\neq r}(\xi_r - z_j)\right|}.$$

Using Lemma 10, (7.6) and the Corollary and Remark of Lemma 9 to estimate the various terms above we get,

$$\begin{aligned}
|z_r - \xi_r| &< \frac{\frac{2n}{n-1}0.02727(n-1)n^5\varepsilon + 0.12027n^5\varepsilon}{0.99971n} \\
&= \left[\frac{2 \cdot 0.02727 + \frac{0.12027}{n}}{0.99971}\right] n^5\varepsilon \\
&\leq \left[\frac{2 \cdot 0.02727 + \frac{0.12027}{8}}{0.99971}\right] n^5\varepsilon \\
&< 0.0696n^5\varepsilon = t_{14}\varepsilon.
\end{aligned}$$

∎

Lemma 12 (Improvement of Lemma 9). *For* $r = 1, 2, ..., n$,

$$\left|\xi_r \prod_{j\neq r}(\xi_j - z_j) - n\right| < t_{15}\varepsilon,$$

where,

$$t_{15} = 0.03542n^7 \ln n.$$

Proof. Let $r \in \{1, 2, ..., n\}$. We can replace $t_6\sqrt{\varepsilon}$ by $t_{14}\varepsilon$ in (6.3) to get,

$$|\Delta_j| = \left|\frac{\xi_j - z_j}{\xi_r - \xi_j}\right| < \frac{t_{14}\varepsilon}{\left|2\sin\frac{(r-j)\pi}{n}\right|}. \tag{12.1}$$

By (9.2),

$$|\ln(1 + \Delta_j)| < 1.00003|\Delta_j|.$$

Recall from Lemma 6 that $\gamma = \sum_{j\neq r} |\ln(1 + \Delta_j)|$. Summing up the above over $j = 1, 2, ..., n$ and using (12.1) we get,

$$\gamma < 1.00003 \sum_{j\neq r} |\Delta_j| < \frac{1.00003t_{14}\varepsilon}{2} \sum_{j\neq r} \frac{1}{\left|\sin\frac{(r-j)\pi}{n}\right|}.$$

Using estimates from Lemma 11 and Lemma 5, we obtain,

$$\gamma < 1.00003 \cdot \frac{0.0696n^5\varepsilon}{2} \cdot 1.01734n\ln n < 0.03541n^6 \ln n\,\varepsilon.$$

Substituting back in (6.1) we finally get,

$$K \leq n[\exp(\gamma) - 1] < 1.00015n \cdot \gamma < 1.00015n \cdot 0.03541n^6 \ln n\,\varepsilon < 0.03542n^7 \ln n\,\varepsilon$$

$$\implies K = \left|\xi_r \prod_{j\neq r}(\xi_j - z_j) - n\right| < t_{15}\varepsilon.$$

∎

Corollary 12.1. By taking the real and imaginary parts of $\xi_r \prod_{j\neq r}(\xi_j - z_j) - n$ respectively we get,

$$n - t_{15}\varepsilon < \text{Re}\,\left(\xi_r \prod_{j\neq r}(\xi_j - z_j)\right) < n + t_{15}\varepsilon,$$

$$-t_{15}\varepsilon < \text{Im}\,\left(\xi_r \prod_{j\neq r}(\xi_j - z_j)\right) < t_{15}\varepsilon.$$

Also,

$$n - t_{15}\varepsilon < \left|\xi_r \prod_{j\neq r}(\xi_j - z_j)\right| < n + t_{15}\varepsilon.$$

Remark 12.1. Some estimations:

$$n - t_{15}\varepsilon \geq n - t_{15}\varepsilon_n > n - 0.03542n^7 \ln n \cdot \frac{90}{n^{12}\ln n} = n\left[1 - \frac{0.03542\cdot 90}{n^6}\right]$$
$$\geq n\left[1 - \frac{0.03542\cdot 90}{8^6}\right] > 0.99998n.$$

$$n + t_{15}\varepsilon \leq n + t_{15}\varepsilon_n < n + 0.03542n^7 \ln n \cdot \frac{90}{n^{12}\ln n} = n\left[1 + \frac{0.03542\cdot 90}{n^6}\right]$$
$$\leq n\left[1 + \frac{0.03542\cdot 90}{8^6}\right] < 1.00002n.$$

Lemmas 2, 3, 10, 11 and 12 will be used in the proof of the main theorem.

3. PROOF OF MAIN THEOREM

We will follow the outline of the proof given in [2], but will will make key improvements in some of the estimations.

Theorem A. *Proof.* For $r \in \{1, 2, ..., n\}$ define:

$$T_1' := 1 - (1-\varepsilon)^n + \sum_{j=1}^{3}(-1)^j \frac{nS_j}{n-j}(\xi_r^{n-j} - (1-\varepsilon)^{n-j}),$$

$$T_2' := \sum_{j=4}^{n-1}(-1)^j \frac{nS_j}{n-j}(\xi_r^{n-j} - (1-\varepsilon)^{n-j}).$$

Then by (7.1) we have,

$$\frac{\xi_r - z_r}{\xi_r} = \frac{T_1' + T_2'}{\xi_r \prod_{j\neq r}(\xi_r - z_j)}.$$

We can mimic the proof of Lemma 7 from (7.2) to (7.3) to obtain the following inequality,

$$\text{(A.1)} \qquad \text{Re}\, T_1' \geq -\left[\frac{|T_2'|\cdot|\xi_r \prod_{j\neq r}(\xi_r - z_j)| + \text{Im}\, T_1' \cdot \text{Im}\,(\xi_r \prod_{j\neq r}(\xi_r - z_j))}{\text{Re}\,(\xi_r \prod_{j\neq r}(\xi_r - z_j))}\right].$$

Note that,

$$\begin{aligned}|\mathrm{Im}\ T_1'| &= \left|\sum_{j=1}^{3}(-1)^j\frac{n}{n-j}\cdot \mathrm{Im}\ \left[S_j(\xi_r^{n-j}-(1-\varepsilon)^{n-r})\right]\right| \\ &\le \sum_{j=1}^{3}|(-1)^j|\cdot\frac{n}{n-j}\cdot|S_j|\cdot|\xi_r^{n-j}-(1-\varepsilon)^{n-r}| \\ &\le \frac{n}{n-1}|S_1|\cdot 2+\frac{n}{n-2}|S_2|\cdot 2+\frac{n}{n-3}|S_3|\cdot 2.\end{aligned}$$

Using Lemma 10 to estimate $|S_1|$ and (4.4) to estimate $|S_2|$ and $|S_3|$,

$$\begin{aligned}|\mathrm{Im}\ T_1'| &\le \frac{2n}{n-1}|S_1|+\frac{2n}{n-2}|S_2|+\frac{2n}{n-3}|S_3| \\ &< \frac{2n}{n-1}t_{13}\varepsilon+\frac{2n}{n-2}\binom{n-1}{2}t_4^2\varepsilon+\frac{2n}{n-3}\binom{n-1}{3}t_4^3\varepsilon\sqrt{\varepsilon} \\ &< \frac{2n}{n-1}0.02727(n-1)n^5\varepsilon+n(n-1)(0.34634n\sqrt{n})^2\varepsilon \\ &+ \frac{n(n-1)(n-2)}{3}(0.34634n\sqrt{n})^3\varepsilon\sqrt{\varepsilon_n} \\ &< 2\cdot 0.02727n^6\varepsilon+n^2\cdot 0.34634^2n^3\varepsilon \\ &+ \frac{n^3}{3}\cdot 0.34634^3n^4\sqrt{n}\varepsilon\cdot\frac{\sqrt{90}}{n^6\sqrt{\ln n}} \\ &= 2\cdot 0.02727n^6\varepsilon+0.34634^2n^5\varepsilon+0.34634^3\cdot\frac{n\sqrt{n}\sqrt{90}}{3\sqrt{\ln n}}\varepsilon \\ &= \left[2\cdot 0.02727+\frac{0.34634^2}{n}+\frac{0.34634^3\sqrt{90}}{3n^4\sqrt{n}\sqrt{\ln n}}\right]n^6\varepsilon \\ &= \left[2\cdot 0.02727+\frac{0.34634^2}{8}+\frac{0.34634^3\sqrt{90}}{3\cdot 8^4\sqrt{8}\sqrt{\ln 8}}\right]n^6\varepsilon \\ &< 0.06955n^6\varepsilon.\end{aligned}$$

Now, let us estimate T_2',

$$\begin{aligned}|T_2'| &\le \sum_{j=4}^{n-1}|(-1)^j|\frac{n|S_j|}{n-j}|\xi_r^{n-j}-(1-\varepsilon)^{n-j}| \\ &< \sum_{j=4}^{n-1}\frac{n|S_j|}{n-j}\cdot 2 = 2\sum_{j=3}^{n-1}\frac{n}{n-j}|S_j| \\ &< 2\sum_{j=4}^{n-1}\frac{n}{n-j}\binom{n-1}{j}t_4^j\sqrt{\varepsilon}^j = 2\sum_{j=4}^{n-1}\binom{n}{j}t_4^j\sqrt{\varepsilon}^j \\ &= 2\left[(1+t_4\sqrt{\varepsilon})^n-1-nt_4\sqrt{\varepsilon}-\binom{n}{2}(t_4\sqrt{\varepsilon})^2-\binom{n}{3}(t_4\sqrt{\varepsilon})^3-(t_4\sqrt{\varepsilon})^n\right] \\ &< 2\left[(1+t_4\sqrt{\varepsilon})^n-1-nt_4\sqrt{\varepsilon}-\binom{n}{2}(t_4\sqrt{\varepsilon})^2-\binom{n}{3}(t_4\sqrt{\varepsilon})^3\right]\end{aligned}$$

Consider the function:

$$f(x) := \frac{(1+x)^n - 1 - nx - \binom{n}{2}x^2 - \binom{n}{3}x^3}{n^4x^4}.$$

Note that f is positive and increasing for $x > 0$. Since $t_4\sqrt{\varepsilon} \le t_4\sqrt{\varepsilon_n} < \frac{1}{n}$,

$$\begin{aligned}
f(t_4\sqrt{\varepsilon}) < f\left(\frac{1}{n}\right) &= \left(1+\frac{1}{n}\right)^n - 1 - \frac{n}{n} - \frac{n(n-1)}{2n^2} - \frac{n(n-1)(n-2)}{6n^3} \\
&< e - 1 - \frac{n}{n} - \frac{n(n-1)}{2n^2} - \frac{n(n-1)(n-2)}{6n^3} \\
&= e - 1 - 1 - \left(\frac{1}{2} - \frac{1}{2n}\right) - \left(\frac{1}{6} - \frac{3}{6n} + \frac{2}{6n^2}\right) \\
&= e - 1 - 1 - \frac{1}{2} - \frac{1}{6} + \frac{1}{n} - \frac{1}{3n^2} \\
&< e - 2 - \frac{1}{2} - \frac{1}{6} + \frac{1}{8} + 0 < 0.17662.
\end{aligned}$$

Thus we have,

$$(1 + t_4\sqrt{\varepsilon})^n - 1 - nt_4\sqrt{\varepsilon} - \binom{n}{2}(t_4\sqrt{\varepsilon})^2 - \binom{n}{3}(t_4\sqrt{\varepsilon})^3 < 0.17662n^4(t_4\sqrt{\varepsilon})^4.$$

Hence we get,

$$|T_2'| < 2 \cdot 0.17662n^4t_4^4\varepsilon^2 < 2 \cdot 0.17662n^4 \cdot 0.34634^4n^6\varepsilon^2 < 0.00509n^{10}\varepsilon^2.$$

Substituting back in (A.1), then using the Corollary and Remark of Lemma 12 to estimate $\operatorname{Re} T_1'$,

$$\begin{aligned}
\operatorname{Re} T_1' &\ge -\left[\frac{|T_2'| \cdot |\xi_r \prod_{j\ne r}(\xi_r - z_j)| + \operatorname{Im} T_1' \cdot \operatorname{Im} \xi_r \prod_{j\ne r}(\xi_r - z_j))}{\operatorname{Re}\,(\xi_r \prod_{j\ne r}(\xi_r - z_j))}\right] \\
&> -\left[\frac{|T_2'| \cdot (n + t_{15}\varepsilon) + \operatorname{Im} T_1' \cdot t_{15}\varepsilon}{n - t_{15}\varepsilon}\right] \\
&> -\left[\frac{0.00509n^{10}\varepsilon^2 \cdot 1.00002n + 0.06955n^6\varepsilon \cdot 0.03542n^7 \ln n\, \varepsilon}{0.99998n}\right] \\
&\ge -\left[\frac{0.00509 \cdot 1.00002n^{10} + 0.06955 \cdot 0.03542n^{12}\ln n}{0.99998}\right]\varepsilon^2 \\
&= -\left[\frac{\frac{0.00509\cdot 1.00002}{n^2 \ln n} + 0.06955 \cdot 0.03542}{0.99998}\right] n^{12}\ln n\, \varepsilon^2 \\
&\ge -\left[\frac{\frac{0.00509\cdot 1.00002}{8^2 \ln 8} + 0.06955 \cdot 0.03542}{0.99998}\right] n^{12}\ln n\, \varepsilon^2 \\
&> -n \cdot 0.00251n^{11}\ln n\, \varepsilon^2.
\end{aligned}$$

Let $t_{16} := 0.00251n^{11}\ln n$. Then we can write,

$$\operatorname{Re} T_1' > -nt_{16}\varepsilon^2 \tag{A.2}$$

On the other hand we have,

$$\begin{aligned}\operatorname{Re} T_1' &= \operatorname{Re}\left[1-(1-\varepsilon)^n+\sum_{j=1}^{3}(-1)^j\frac{nS_j}{n-j}\left(\xi_r^{n-j}-(1-\varepsilon)^{n-j}\right)\right]\\ &= \operatorname{Re}\left[1-(1-\varepsilon)^n+\sum_{j=1}^{3}(-1)^{j-1}\frac{nS_j}{n-j}\left((1-\varepsilon)^{n-j}-\overline{\xi_r}^j\right)\right]\\ &= 1-(1-\varepsilon)^n\\ &+\sum_{j=1}^{3}(-1)^{j-1}\left[\frac{n(1-\varepsilon)^{n-j}}{n-j}\operatorname{Re} S_j-\frac{n\operatorname{Re} S_j}{n-j}\operatorname{Re}\xi_r^j-\frac{n\operatorname{Im} S_j}{n-j}\operatorname{Im}\xi_r^j\right].\end{aligned}$$

Define:

For $j = 1, 2, 3$, $a_j := \dfrac{\operatorname{Re} S_j}{n-j}$, $b_j := \dfrac{\operatorname{Im} S_j}{n-j}$. For $r = 1, 2, ..., n$: $\theta_r := \dfrac{2(r-1)\pi}{n}$.

Thus, substituting in above:

$$\begin{aligned}\operatorname{Re} T_1' &= 1-(1-\varepsilon)^n+\sum_{j=1}^{3}(-1)^{j-1}\left[n(1-\varepsilon)^{n-j}a_j-na_j\cos j\theta_r-nb_j\sin j\theta_r\right]\\ &< n\varepsilon+\sum_{j=1}^{3}(-1)^{j-1}\left[n(1-\varepsilon)^{n-j}a_j-na_j\cos j\theta_r-nb_j\sin j\theta_r\right].\end{aligned}$$

We have, due to (A.2),

$$\begin{aligned}0 &< nt_{16}\varepsilon^2+\operatorname{Re} T_1'\\ &< n\left(t_{16}\varepsilon^2+\varepsilon+\sum_{j=1}^{3}(-1)^{j-1}\left[(1-\varepsilon)^{n-j}a_j-a_j\cos j\theta_r-b_j\sin j\theta_r\right]\right).\end{aligned}$$

Dividing by n,

$$0 < t_{16}\varepsilon^2+\varepsilon+\sum_{j=1}^{3}(-1)^{j-1}\left[(1-\varepsilon)^{n-j}a_j-a_j\cos j\theta_r-b_j\sin j\theta_r\right]. \tag{A.3}$$

From this point onwards the objective is to eliminate the a_j and b_j through various substitutions and estimates. First, we can immediately eliminate b_1, b_2 and b_3 by observing that $\sin x$ is an odd function. Indeed, replacing θ_r in (A.3) by θ_{n+1-r}, i.e. by $2\pi-\theta_r$ we get,

$$0 < t_{16}\varepsilon^2+\varepsilon+\sum_{j=1}^{3}(-1)^{j-1}\left[(1-\varepsilon)^{n-j}a_j-a_j\cos j\theta_r+b_j\sin j\theta_r\right]. \tag{A.4}$$

Adding the two inequalities (A.3) and (A.4) eliminates the sine terms,

$$0 < t_{16}\varepsilon^2+\varepsilon+\sum_{j=1}^{3}(-1)^{j-1}a_j\left[(1-\varepsilon)^{n-j}-\cos j\theta_r\right].$$

Note that $1-\varepsilon < 1$ and $|\cos x| < 1$, so we have,

$$0 < t_{16}\varepsilon^2+\varepsilon+a_3\left[(1-\varepsilon)^{n-3}-\cos 3\theta_r\right]+\sum_{j=1}^{2}(-1)^{j-1}a_j\left[(1-\varepsilon)^{n-j}-\cos j\theta_r\right]$$

$$< t_{16}\varepsilon^2 + \varepsilon + |a_3| \cdot [1+1] + \sum_{j=1}^{2} (-1)^{j-1} a_j \left[(1-\varepsilon)^{n-j} - \cos j\theta_r\right]$$

$$= t_{16}\varepsilon^2 + \varepsilon + 2|a_3| + \sum_{j=1}^{2} (-1)^{j-1} a_j \left[((1-\varepsilon)^{n-j} - 1) - (\cos j\theta_r - 1)\right]$$

$$= t_{16}\varepsilon^2 + \varepsilon + 2|a_3| + \sum_{j=1}^{2} (-1)^{j} a_j \left[(1-(1-\varepsilon)^{n-j}) - (1 - \cos j\theta_r)\right]$$

$$= t_{16}\varepsilon^2 + \varepsilon + 2|a_3| - \left[\sum_{j=1}^{2} (-1)^{j} a_j (1 - \cos j\theta_r)\right] + \sum_{j=1}^{2} (-1)^{j} a_j (1-(1-\varepsilon)^{n-j})$$

$$\leq t_{16}\varepsilon^2 + \varepsilon + 2|a_3| - \left[\sum_{j=1}^{2} (-1)^{j} a_j (1 - \cos j\theta_r)\right] + V. \tag{A.5}$$

where $V := |a_1(1-(1-\varepsilon)^{n-1}) - a_2(1-(1-\varepsilon)^{n-2})|$. Let us estimate it,

$$\begin{aligned} V &\leq |a_1|(1-(1-\varepsilon)^{n-1}) + |a_2|(1-(1-\varepsilon)^{n-2}) \\ &< |a_1|(n-1)\varepsilon + |a_2|(n-2)\varepsilon = (|\mathrm{Re}\, S_1| + |\mathrm{Re}\, S_2|)\,\varepsilon \\ &\leq (|\mathrm{Re}\, S_1| + |S_2|)\,\varepsilon. \end{aligned}$$

Using Remark 3.1 to estimate $|\mathrm{Re}\, S_1|$, then using (4.4) to estimate $|S_2|$,

$$V < \left((n-1)t_5\varepsilon + \binom{n-1}{2} t_4^2 \varepsilon\right)\varepsilon = (n-1)\left(t_5 + \frac{n-2}{2} t_4^2\right)\varepsilon^2.$$

Let $t_{17} = (n-1)\left[t_5 + \frac{n-2}{2} t_4^2\right]$. Then $V < t_{17}\varepsilon^2$. Let us estimate t_{17},

$$\begin{aligned} t_{17} &= (n-1)\left[t_5 + \frac{n-2}{2} t_4^2\right] \\ &< n\left[0.11998 n^3 + \frac{n}{2}(0.34634 n\sqrt{n})^2\right] \\ &= n \cdot n^4 \left[\frac{0.11998}{n} + \frac{0.34634^2}{2}\right] \\ &\leq \left[\frac{0.11998}{8} + \frac{0.34634^2}{2}\right] n^5 < 0.07498 n^5. \end{aligned}$$

Substituting back in our initial inequality (A.5) we get,

$$0 < t_{16}\varepsilon^2 + \varepsilon + 2|a_3| - \left[\sum_{j=1}^{2} (-1)^{j} a_j (1 - \cos j\theta_r)\right] + t_{17}\varepsilon^2$$

$$= \varepsilon - \left[\sum_{j=1}^{2} (-1)^{j} a_j (1 - \cos j\theta_r)\right] + 2|a_3| + t_{16}\varepsilon^2 + t_{17}\varepsilon^2. \tag{A.6}$$

Let us estimate $|a_3|$,

$$|a_3| = \left|\frac{\mathrm{Re}\, S_3}{n-3}\right| = \frac{1}{n-3}\left|\mathrm{Re} \sum_{1 \leq i < j < k \leq n-1} w_i w_j w_k\right|$$

$$= \frac{1}{n-3}\left|\sum_{i<j<k}(\mathrm{Re}\, w_i \cdot \mathrm{Re}\, w_j \cdot \mathrm{Re}\, w_k - \mathrm{Re}\, w_i \cdot \mathrm{Im}\, w_j \cdot \mathrm{Im}\, w_k \right.$$
$$\left. -\mathrm{Re}\, w_j \cdot \mathrm{Im}\, w_k \cdot \mathrm{Im}\, w_i - \mathrm{Re}\, w_k \cdot \mathrm{Im}\, w_i \cdot \mathrm{Im}\, w_j)\right|$$
$$\leq \frac{1}{n-3}\sum_{1\leq i<j<k\leq n-1}(|\mathrm{Re}\, w_i|\cdot|\mathrm{Re}\, w_j|\cdot|\mathrm{Re}\, w_k| + |\mathrm{Re}\, w_i|\cdot|\mathrm{Im}\, w_j|\cdot|\mathrm{Im}\, w_k|$$
$$+|\mathrm{Re}\, w_j|\cdot|\mathrm{Im}\, w_k|\cdot|\mathrm{Im}\, w_i| + |\mathrm{Re}\, w_k|\cdot|\mathrm{Im}\, w_i|\cdot|\mathrm{Im}\, w_j|)$$
$$\leq \frac{1}{n-3}\sum_{i<j<k}(|\mathrm{Re}\, w_i|\cdot|\mathrm{Re}\, w_j|\cdot|\mathrm{Re}\, w_k| + |\mathrm{Re}\, w_i|\cdot|w_j|\cdot|w_k|$$
$$+|\mathrm{Re}\, w_j|\cdot|w_k|\cdot|w_i| + |\mathrm{Re}\, w_k|\cdot|w_i|\cdot|w_j|)\,.$$

We will use Remark 3.1 and Lemma 2 to estimate the above terms,

$$|a_3| < \frac{1}{n-3}\sum_{i<j<k}\left[t_5^3\varepsilon^3 + 3t_5t_4^2\varepsilon^2\right] = \frac{1}{n-3}\binom{n-1}{3}\left[t_5^3\varepsilon^3 + 3t_5t_4^2\varepsilon^2\right]$$
$$\leq \frac{(n-1)(n-2)}{6}\left[t_5^3\varepsilon_n + 3t_5t_4^2\right]\varepsilon^2$$
$$< \frac{n^2}{6}\left[(0.11998n^3)^3\cdot\frac{90}{n^{12}\ln n} + 3\cdot 0.11998n^3(0.34634n\sqrt{n})^2\right]\varepsilon^2$$
$$= \frac{n^2}{6}\left[0.11998^3n^9\cdot\frac{90}{n^{12}\ln n} + 3\cdot 0.11998\cdot 0.34634^2n^6\right]\varepsilon^2$$
$$= \frac{n^8}{6}\left[0.11998^3n^3\cdot\frac{90}{n^{12}\ln n} + 3\cdot 0.11998\cdot 0.34634^2\right]\varepsilon^2$$
$$\leq \frac{n^8}{6}\left[0.11998^3\cdot\frac{90}{8^9\ln 8} + 3\cdot 0.11998\cdot 0.34634^2\right]\varepsilon^2 < 0.0072n^8\varepsilon^2.$$

Thus, $2|a_3| < 2\cdot 0.0072n^8\varepsilon^2 = 0.0144n^8\varepsilon^2$. Let $t_{18} := 0.0144n^8$.

Let us collect all the ε^2 terms. Define, $t_{19} := t_{16} + t_{17} + t_{18}$. Then we have,

$$t_{19} = t_{16} + t_{17} + t_{18} < 0.00251n^{11}\ln n + 0.07498n^5 + 0.0144n^8$$
$$= \left[0.00251 + \frac{0.07498}{n^6\ln n} + \frac{0.0144}{n^3\ln n}\right]n^{11}\ln n$$
$$\leq \left[0.00251 + \frac{0.07498}{8^6\ln 8} + \frac{0.0144}{8^3\ln 8}\right]n^{11}\ln n < 0.00253n^{11}\ln n\,.$$

Thus, in (A.6) we can write,

$$0 < \varepsilon + a_1(1-\cos\theta_r) - a_2(1-\cos 2\theta_r) + t_{19}\varepsilon^2. \tag{A.7}$$

Define:

$$m_2 := n \bmod 2, \quad m_4 := n \bmod 4$$

In (A.7), put $\theta_r = \frac{2\pi}{n}\cdot\frac{n-m_2}{2}$ and then $\theta_r = \frac{2\pi}{n}\cdot\frac{n-m_4}{4}$. Using basic trigonometric simplifications we respectively get,

$$0 < \varepsilon + a_1\left[1 + \cos\frac{m_2\pi}{n}\right] - a_2\left[1 - \cos\frac{2m_2\pi}{n}\right] + t_{19}\varepsilon^2, \tag{A.8}$$

$$0 < \varepsilon + a_1 \left[1 - \sin\frac{m_4\pi}{2n}\right] - a_2\left[1 + \cos\frac{m_4\pi}{n}\right] + t_{19}\varepsilon^2. \tag{A.9}$$

We will now find a suitable inequality to eliminate a_2. We will do so by using our original contrapositive assumption, ($\diamondsuit$). So we will now take a minor detour from the proof and then return to equations (A.8) and (A.9) once we have an inequality to eliminate a_2.

By ($\diamondsuit$) we have,

$$\begin{aligned} 1 - c_n\varepsilon &< |w_j - (1-\varepsilon)| \\ \implies 1 - 2c_n\varepsilon + c_n^2\varepsilon^2 &< |w_j|^2 - 2(1-\varepsilon)\mathrm{Re}\, w_j + (1-\varepsilon)^2. \end{aligned}$$

Thus we can write,

$$0 < (\mathrm{Re}\, w_j)^2 + (\mathrm{Im}\, w_j)^2 - 2(1-\varepsilon)\mathrm{Re}\, w_j + -2(1-c_n)\varepsilon + (1-c_n^2)\varepsilon^2.$$

By Remark 3.1 we get,

$$\begin{aligned} 0 &< t_5^2\varepsilon^2 + (\mathrm{Im}\, w_j)^2 - 2\mathrm{Re}\, w_j + 2t_5\varepsilon^2 - 2(1-c_n)\varepsilon + (1-c_n^2)\varepsilon^2 \\ &= (\mathrm{Im}\, w_j)^2 - 2(\mathrm{Re}\, w_j) - 2(1-c_n)\varepsilon + (t_5^2 + 2t_5 + 1 - c_n^2)\varepsilon^2 \\ &< (\mathrm{Im}\, w_j)^2 - 2(\mathrm{Re}\, w_j) - 2(1-c_n)\varepsilon + (t_5+1)^2\varepsilon^2. \end{aligned}$$

Define $J := \sum_{j=1}^{n-1}(\mathrm{Im}\, w_j)^2$. Summing up the above inequality over $j = 1, ..., n-1$,

$$\begin{aligned} 0 &< \sum_{j=1}^{n-1}(\mathrm{Im}\, w_j)^2 - 2\mathrm{Re}\, S_1 - 2(n-1)(1-c_n)\varepsilon + (n-1)(t_5+1)^2\varepsilon^2 \\ &= J - 2(n-1)a_1 - 2(n-1)(1-c_n)\varepsilon + (n-1)(t_5+1)^2\varepsilon^2. \end{aligned} \tag{A.10}$$

We will now use Remark 3.1 and Lemma 10 to estimate J, which will in turn help us to estimate a_2,

$$\begin{aligned} (n-2)a_2 = \mathrm{Re}\, S_2 &= \mathrm{Re} \sum_{1\le j<k\le n-1} w_j w_k \\ &= \sum_{1\le j<k\le n-1} [\mathrm{Re}\, w_j \cdot \mathrm{Re}\, w_k - \mathrm{Im}\, w_j \cdot \mathrm{Im}\, w_k] \\ &\ge -\sum_{j<k} |\mathrm{Re}\, w_j| \cdot |\mathrm{Re}\, w_k| - \sum_{j<k} \mathrm{Im}\, w_j \cdot \mathrm{Im}\, w_k \\ &> -\binom{n-1}{2} t_5^2\varepsilon^2 - \frac{1}{2}\left(\sum_{j=1}^{n-1}\mathrm{Im}\, w_j\right)^2 + \frac{1}{2}\sum_{j=1}^{n-1}(\mathrm{Im}\, w_j)^2 \\ &= -\binom{n-1}{2} t_5^2\varepsilon^2 - \frac{1}{2}(\mathrm{Im}\, S_1)^2 + \frac{1}{2}\sum_{j=1}^{n-1}(\mathrm{Im}\, w_j)^2 \\ &\ge -\binom{n-1}{2} t_5^2\varepsilon^2 - \frac{1}{2}|S_1|^2 + \frac{1}{2}\sum_{j=1}^{n-1}(\mathrm{Im}\, w_j)^2 \\ &\ge -\binom{n-1}{2} t_5^2\varepsilon^2 - \frac{1}{2}t_{13}^2\varepsilon^2 + \frac{1}{2}J. \end{aligned}$$

Thus we can write,

$$J \le 2(n-2)a_2 + \left[(n-1)(n-2)t_5^2 + t_{13}^2\right]\varepsilon^2.$$

Let $t_{20} := (n-1)(n-2)t_5^2 + t_{13}^2$. We will estimate it using Lemmas 3 and 10,

$$\begin{aligned} t_{20} &= (n-1)(n-2)t_5^2 + t_{13}^2 < n^2 t_5^2 + t_{13}^2 \\ &< n^2(0.11998n^3)^2 + (0.02727(n-1)n^5)^2 \\ &< 0.11998^2 n^8 + 0.02727^2 n^{12} \\ &= n^{12}\left[\frac{0.11998^2}{n^4} + 0.02727^2\right] \\ &\le n^{12}\left[\frac{0.11998^2}{8^4} + 0.02727^2\right] < 0.00075n^{12}. \end{aligned}$$

Thus we have,

$$J \le 2(n-2)a_2 + t_{20}\varepsilon^2 < 2(n-2)a_2 + t_{20}\varepsilon^2.$$

Substituting back in (A.10),

$$\begin{aligned} 0 &< 2(n-2)a_2 + t_{20}\varepsilon^2 - 2(n-1)a_1 - 2(n-1)(1-c_n)\varepsilon + (n-1)(t_5+1)^2\varepsilon^2 \\ &= 2(n-2)a_2 - 2(n-1)a_1 - 2(n-1)(1-c_n)\varepsilon + \left[t_{20} + (n-1)(t_5+1)^2\right]\varepsilon^2. \end{aligned}$$

Let $t_{21} := \frac{t_{20}+(n-1)(t_5+1)^2}{2}$. By above and the estimate for t_5 from Lemma 3,

$$\begin{aligned} t_{21} &= \frac{t_{20} + (n-1)(t_5+1)^2}{2} < \frac{t_{20} + n(t_5+1)^2}{2} \\ &< \frac{0.00075n^{12} + n\left(0.11998n^3+1\right)^2}{2} \\ &= n^{12}\left[\frac{0.00075 + \frac{n\cdot n^6}{n^{12}}\left(0.11998 + \frac{1}{n^3}\right)^2}{2}\right] \\ &= n^{12}\left[\frac{0.00075 + \frac{1}{n^5}\left(0.11998 + \frac{1}{n^3}\right)^2}{2}\right] \\ &\le n^{12}\left[\frac{0.00075 + \frac{1}{8^5}\left(0.11998 + \frac{1}{8^3}\right)^2}{2}\right] < 0.00038n^{12}. \end{aligned}$$

Thus we can write,

$$0 < (n-2)a_2 - (n-1)a_1 - (n-1)(1-c_n)\varepsilon + t_{21}\varepsilon^2 \tag{A.11}$$

Now we will return to the equations (A.8) and (A.9). To eliminate a_2, multiply (A.11) by $\frac{1}{n-2}\left[1 - \cos\frac{2m_2\pi}{n})\right] > 0$ and add to (A.8). We then get,

$$\begin{aligned} &\varepsilon\left[1 - \frac{(n-1)(1-c_n)}{n-2}\left(1 - \cos\frac{2m_2\pi}{n}\right)\right] \\ &+a_1\left[1 + \cos\frac{m_2\pi}{n} - \frac{n-1}{n-2}\left(1 - \cos\frac{2m_2\pi}{n}\right)\right] \\ &+\varepsilon^2\left[t_{19} + \frac{t_{21}}{n-2}\left(1 - \cos\frac{2m_2\pi}{n}\right)\right] > 0. \end{aligned} \tag{A.12}$$

Similarly, multiply (A.11) by $\frac{1}{n-2}\left[1 + \cos\frac{m_4\pi}{n})\right] > 0$ and add to (A.9). We then get,

$$\varepsilon\left[1 - \frac{(n-1)(1-c_n)}{n-2}\left(1 + \cos\frac{m_4\pi}{n}\right)\right]$$

$$-a_1\left[-1+\sin\frac{m_4\pi}{2n}+\frac{n-1}{n-2}\left(1+\cos\frac{m_4\pi}{n}\right)\right]$$
$$+\varepsilon^2\left[t_{19}+\frac{t_{21}}{n-2}\left(1+\cos\frac{m_4\pi}{n}\right)\right]>0. \tag{A.13}$$

Define:

$$\alpha := 1+\cos\frac{m_2\pi}{n}-\frac{n-1}{n-2}\left[1-\cos\frac{2m_2\pi}{n}\right],$$
$$\beta := -1+\sin\frac{m_4\pi}{2n}+\frac{n-1}{n-2}\left[1+\cos\frac{m_4\pi}{n}\right],$$
$$t_{22} := t_{19}+\frac{t_{21}}{n-2}\left(1-\cos\frac{2m_2\pi}{n}\right),$$
$$t_{23} := t_{19}+\frac{t_{21}}{n-2}\left(1+\cos\frac{m_4\pi}{n}\right).$$

Then we can rewrite the above two inequalities (A.12) and (A.13) as follows:

$$\varepsilon\left[1-\frac{(n-1)(1-c_n)}{n-2}\left(1-\cos\frac{2m_2\pi}{n}\right)\right]+\alpha a_1+t_{22}\varepsilon^2>0. \tag{A.14}$$

$$\varepsilon\left[1-\frac{(n-1)(1-c_n)}{n-2}\left(1+\cos\frac{m_4\pi}{n}\right)\right]-\beta a_1+t_{23}\varepsilon^2>0. \tag{A.15}$$

Let us estimate $\alpha, \beta, t_{22}, t_{23}$:

$$\alpha = 1+\cos\frac{m_2\pi}{n}-\frac{n-1}{n-2}\left[1-\cos\frac{2m_2\pi}{n}\right]\le 1+1-\frac{n-1}{n-2}\cdot 0 = 2.$$

For the lower bound on α we have,

$$\alpha = 1+\cos\frac{m_2\pi}{n}-\frac{n-1}{n-2}\left[1-\cos\frac{2m_2\pi}{n}\right]\ge 1+\left(1-\frac{m_2^2\pi^2}{2n^2}\right)-\frac{n-1}{n-2}$$
$$=\frac{n-3}{n-2}-\frac{m_2^2\pi^2}{2n^2}\ge\frac{8-3}{8-2}-\frac{1\cdot\pi^2}{2\cdot 8^2}>0.$$

Hence we have $0<\alpha\le 2$. Now let us estimate β,

$$\beta = -1+\sin\frac{m_4\pi}{2n}+\frac{n-1}{n-2}\left[1+\cos\frac{m_4\pi}{n}\right]$$
$$\le -1+\frac{m_4\pi}{2n}+\frac{2(n-1)}{n-2}=\frac{n}{n-2}+\frac{m_4\pi}{2n}\le\frac{8}{6}+\frac{3\pi}{2\cdot 8}<1.92239.$$

For the lower bound on β we have,

$$\beta = -1+\sin\frac{m_4\pi}{2n}+\frac{n-1}{n-2}\left[1+\cos\frac{m_4\pi}{n}\right]\ge -1+\frac{n-1}{n-2}>0.$$

Hence we have $0<\beta<1.92239$. Now let us estimate t_{22},

$$t_{22} = t_{19}+\frac{t_{21}}{n-2}\left(1-\cos\frac{2m_2\pi}{n}\right)$$
$$\le t_{19}+\frac{t_{21}}{n-2}\cdot\frac{2m_2^2\pi^2}{n^2}\le t_{19}+\frac{4}{3}\cdot\frac{t_{21}}{n}\cdot\frac{2m_2^2\pi^2}{n^2}.$$

The last step follows from the fact that for $n \geq 8$, $\frac{n}{n-2} \leq \frac{4}{3}$. Further estimating t_{22},

$$\begin{aligned} t_{22} &\leq 0.00253 n^{11} \ln n + \frac{4}{3} \cdot \frac{0.00038 n^{12}}{n} \cdot \frac{2m_2^2 \pi^2}{n^2} \\ &\leq 0.00253 n^{11} \ln n + \frac{8\pi^2}{3} \cdot 0.00038 n^9 \\ &= n^{11} \ln n \left[0.00253 + \frac{8\pi^2}{3n^2 \ln n} \cdot 0.00038 \right] \\ &\leq n^{11} \ln n \left[0.00253 + \frac{8\pi^2}{3 \cdot 8^2 \ln 8} \cdot 0.00038 \right] < 0.00261 n^{11} \ln n . \end{aligned}$$

Lastly, let us estimate t_{23},

$$\begin{aligned} t_{23} &= t_{19} + \frac{t_{21}}{n-2} \left(1 + \cos \frac{m_4 \pi}{n} \right) \leq t_{19} + \frac{2t_{21}}{n-2} \leq t_{19} + \frac{4}{3} \cdot \frac{2t_{21}}{n} \\ &\leq 0.00253 n^{11} \ln n + \frac{4}{3n} \cdot 2 \cdot 0.00038 n^{12} = n^{11} \ln n \left[0.00253 + \frac{8}{3 \ln n} \cdot 0.00038 \right] \\ &\leq n^{11} \ln n \left[0.00253 + \frac{8}{3 \ln 8} \cdot 0.00038 \right] < 0.00302 n^{11} \ln n . \end{aligned}$$

Now we can multiply (A.14) by $\frac{\beta}{\varepsilon}$ and (A.15) by $\frac{\alpha}{\varepsilon}$ and add the two equations to eliminate a_1. We can thus write,

$$Ac_n - B + (t_{22}\beta + t_{23}\alpha)\varepsilon > 0.$$

where, upon futher regrouping and simplification of terms we have,

$$\begin{aligned} A &:= \frac{n-1}{n-2} \left(1 - \cos \frac{2m_2\pi}{n} \right) \beta + \frac{n-1}{n-2} \left(1 + \cos \frac{m_4 \pi}{n} \right) \alpha \\ &= \frac{n-1}{n-2} \left[\left(1 + \cos \frac{m_4\pi}{n} \right) \left(1 + \cos \frac{m_2 \pi}{n} \right) - \left(1 - \cos \frac{2m_2\pi}{n} \right) \left(1 - \sin \frac{m_4 \pi}{2n} \right) \right], \\ B &:= \beta \left[\frac{n-1}{n-2} \left(1 - \cos \frac{2m_2\pi}{n} \right) - 1 \right] + \alpha \left[\frac{n-1}{n-2} \left(1 + \cos \frac{m_4\pi}{n} \right) - 1 \right] \\ &= \frac{1}{n-2} \left[\cos \frac{m_2\pi}{n} + \sin \frac{m_4\pi}{2n} \right] + \frac{n-1}{n-2} \left[\cos \frac{m_2\pi}{n} \cos \frac{m_4\pi}{n} - \cos \frac{2m_2\pi}{n} \sin \frac{m_4\pi}{2n} \right]. \end{aligned}$$

Since $0 \leq \frac{2m_2\pi}{n} < \frac{\pi}{2}$ and $0 \leq \frac{m_4\pi}{n} < \frac{\pi}{2}$, we note that,

$$A > \frac{n-1}{n-2} \left[(1+0)(1+0) - (1-0)(1-0) \right] = 0$$

So now we can write,

$$c_n > \frac{1}{A} \left[B - (t_{22}\beta + t_{23}\alpha)\varepsilon \right] \tag{A.16}$$

Let us estimate the following:

$$\begin{aligned} \beta t_{22} + \alpha t_{23} &< 1.92239 t_{22} + 2t_{23} \\ &= 1.92239 \cdot 0.00261 n^{11} \ln n + 2 \cdot 0.00302 n^{11} \ln n \\ &< 0.01106 n^{11} \ln n . \end{aligned}$$

Thus we have,

$$(\beta t_{22} + \alpha t_{23})\varepsilon \leq (\beta t_{22} + \alpha t_{23})\varepsilon_n < 0.01106 n^{11} \ln n \cdot \frac{90}{n^{12} \ln n}$$

$$= \frac{0.01106 \cdot 90}{n} = \frac{0.9954}{n} < \frac{1}{n} < \frac{1}{n-2}.$$

Also, note that,

$$A \leq \frac{n-1}{n-2}\left[(1+1)(1+1)-0\right] = \frac{4(n-1)}{n-2}.$$

Substituting in (A.16), we obtain,

$$c_n > \frac{1}{A}\left(B - (t_{22}\beta + t_{23}\alpha)\varepsilon\right) > \frac{n-2}{4(n-1)}\left(B - \frac{1}{n-2}\right).$$

From this point onwards the proof is almost identical to the one given by Chijiwa in [2]. We present it here for the sake of completeness.

The proof is now divided into four cases. First suppose that $n \equiv 0 \pmod 4$, i.e. $m_2 = m_4 = 0$. Then,

$$\begin{aligned} B &= \frac{1}{n-2}\left[\cos\frac{0\pi}{n} + \sin\frac{0\pi}{2n}\right] + \frac{n-1}{n-2}\left[\cos\frac{0\pi}{n}\cos\frac{0\pi}{n} - \cos\frac{0\pi}{n}\sin\frac{0\pi}{2n}\right] \\ &= \frac{1}{n-2}\left[1+0\right] + \frac{n-1}{n-2}\left[1-0\right] = \frac{1}{n-2} + \frac{n-1}{n-2} = \frac{n}{n-2}. \end{aligned}$$

Thus for c_n we obtain,

$$\begin{aligned} c_n &> \frac{n-2}{4(n-1)}\left(B - \frac{1}{n-2}\right) = \frac{n-2}{4(n-1)}\left(\frac{n}{n-2} - \frac{1}{n-2}\right) \\ &= \frac{n-2}{4(n-1)}\left(\frac{n-1}{n-2}\right) = \frac{1}{4}, \end{aligned}$$

which contradicts $c_n = \frac{1}{4}$.

Now suppose $n \equiv 1 \pmod 4$ i.e $m_2 = m_4 = 1$. Note that $n \geq 9$. In the following calculations we will use the fact that $\cos x > 1 - \frac{x^2}{2}$, $\sin x < x$ for $x > 0$, and (1.2),

$$\begin{aligned} B &= \frac{1}{n-2}\left[\cos\frac{\pi}{n} + \sin\frac{\pi}{2n}\right] + \frac{n-1}{n-2}\left[\frac{1+\cos\frac{2\pi}{n}}{2} - \cos\frac{2\pi}{n}\sin\frac{\pi}{2n}\right] \\ &> \frac{1}{n-2}\left[1 - \frac{\pi^2}{2n^2} + \frac{3}{2n}\right] + \frac{n-1}{n-2}\left[\frac{1}{2} + \cos\frac{2\pi}{n}\left(\frac{1}{2} - \sin\frac{\pi}{2n}\right)\right] \\ &> \frac{1}{n-2}\left[1 - \frac{\pi^2}{2n^2} + \frac{3}{2n}\right] + \frac{n-1}{n-2}\left[\frac{1}{2} + \left(1 - \frac{2\pi^2}{n^2}\right)\left(\frac{1}{2} - \frac{\pi}{2n}\right)\right] \\ &= \frac{1}{n-2}\left[1 - \frac{\pi^2}{2n^2} + \frac{3}{2n}\right] + \frac{n-1}{n-2}\left[\frac{1}{2} + \frac{1}{2} - \frac{\pi^2}{n^2} - \frac{\pi}{2n} + \frac{\pi^3}{n^3}\right] \\ &= \frac{1}{n-2}\left[1 - \frac{\pi^2}{2n^2} + \frac{3}{2n}\right] + \frac{n-1}{n-2}\left[1 - \frac{\pi^2}{n^2} - \frac{\pi}{2n} + \frac{\pi^3}{n^3}\right] \\ &= \frac{1}{n-2}\left[1 - \frac{\pi^2}{2n^2} + \frac{3}{2n} + (n-1) - \frac{(n-1)\pi^2}{n^2} - \frac{(n-1)\pi}{2n} + \frac{(n-1)\pi^3}{n^3}\right] \\ &= \frac{1}{n-2}\left[n - \frac{\pi^2}{2n^2} + \frac{3}{2n} - \frac{\pi^2}{n} + \frac{\pi^2}{n^2} - \frac{\pi}{2} + \frac{\pi}{2n} + \frac{\pi^3}{n^2} - \frac{\pi^3}{n^3}\right] \\ &= \frac{1}{n-2}\left[n - \left(\frac{\pi}{2} + \frac{2\pi^2 - 3 - \pi}{2n} - \frac{\pi^2 + 2\pi^3}{2n^2} + \frac{\pi^3}{n^3}\right)\right]. \end{aligned}$$

Let $f(n) := \frac{\pi}{2} + \frac{2\pi^2-3-\pi}{2n} - \frac{\pi^2+2\pi^3}{2n^2} + \frac{\pi^3}{n^3}$. We claim that $f(n) < 2$ for all $n \geq 9$, $n \equiv 1 \pmod 4$. By direct verification, $f(9) < 2$ and $f(13) < 2$. For $n \geq 17$ we have,

$$f(n) < \frac{\pi}{2} + \frac{2\pi^2 - 3 - \pi}{2n} + \frac{\pi^3}{n^3} \leq \frac{\pi}{2} + \frac{2\pi^2 - 3 - \pi}{2 \cdot 17} + \frac{\pi^3}{17^3} < 2.$$

Hence we get,

$$B > \frac{1}{n-2}\left[n-2\right] = 1.$$

Thus for c_n we obtain,

$$c_n > \frac{n-2}{4(n-1)}\left[B - \frac{1}{n-2}\right] > \frac{n-2}{4(n-1)}\left[\frac{n-3}{n-2}\right] > \frac{n-3}{4(n-1)},$$

which contradicts $c_n = \frac{n-3}{4(n-1)}$.

Now suppose $n \equiv 2 \pmod 4$ i.e $m_2 = 0$, $m_4 = 2$. Note that $n \geq 10$.

$$\begin{aligned} B &= \frac{1}{n-2}\left[\cos\frac{0\pi}{n} + \sin\frac{2\pi}{2n}\right] + \frac{n-1}{n-2}\left[\cos\frac{0\pi}{n}\cos\frac{2\pi}{n} - \cos\frac{0\pi}{n}\sin\frac{2\pi}{2n}\right] \\ &= \frac{1}{n-2}\left[1 + \sin\frac{\pi}{n}\right] + \frac{n-1}{n-2}\left[\cos\frac{2\pi}{n} - \sin\frac{\pi}{n}\right] \\ &= \frac{1}{n-2}\left[1 + (n-1)\cos\frac{2\pi}{n} - (n-2)\sin\frac{\pi}{n}\right] \\ &> \frac{1}{n-2}\left[1 + (n-1)\left(1 - \frac{2\pi^2}{n^2}\right) - (n-2)\frac{\pi}{n}\right] \\ &= \frac{1}{n-2}\left[n - \frac{2(n-1)\pi^2}{n^2} - (n-2)\frac{\pi}{n}\right]. \end{aligned}$$

Let $f(n) := \frac{2(n-1)\pi^2}{n^2} + (n-2)\frac{\pi}{n}$. We claim that $f(n) \leq 5$. By direct verification, $f(10) < 5$. For $n \geq 14$ we have,

$$f(n) = \frac{2(n-1)\pi^2}{n^2} + (n-2)\frac{\pi}{n} < \frac{2\pi^2}{n} + \pi \leq \frac{2\pi^2}{14} + \pi < 5$$

Hence we get,

$$B \geq \frac{1}{n-2}\left[n-5\right] = \frac{n-5}{n-2}.$$

Thus for c_n we obtain,

$$c_n > \frac{n-2}{4(n-1)}\left(B - \frac{1}{n-2}\right) = \frac{n-2}{4(n-1)}\left(\frac{n-6}{n-2}\right) = \frac{n-6}{4(n-1)},$$

which contradicts $c_n = \frac{n-6}{4(n-1)}$.

Finally suppose $n \equiv 3 \pmod 4$ i.e $m_2 = 3$, $m_4 = 2$. Note that $n \geq 11$. Then we have,

$$\begin{aligned} B &= \frac{1}{n-2}\left[\cos\frac{\pi}{n} + \sin\frac{3\pi}{2n}\right] + \frac{n-1}{n-2}\left[\cos\frac{\pi}{n}\cos\frac{3\pi}{n} - \cos\frac{2\pi}{n}\sin\frac{3\pi}{2n}\right] \\ &> \frac{1}{n-2}\left[\cos\frac{\pi}{n} + \sin\frac{3\pi}{2n}\right] + \frac{n-1}{n-2}\left[\cos\frac{\pi}{n}\cos\frac{3\pi}{n} - \sin\frac{3\pi}{2n}\right] \\ &= \frac{1}{n-2}\left[\cos\frac{\pi}{n} + (n-1)\cos\frac{\pi}{n}\cos\frac{3\pi}{n} - (n-2)\sin\frac{3\pi}{2n}\right] \\ &> \frac{1}{n-2}\left[1 - \frac{\pi^2}{2n^2} + (n-1)\left(1 - \frac{\pi^2}{2n^2}\right)\left(1 - \frac{9\pi^2}{2n^2}\right) - (n-2)\frac{3\pi}{2n}\right] \end{aligned}$$

$$= \frac{1}{n-2}\left[n - \frac{\pi^2}{2n} - \frac{9(n-1)\pi^2}{2n^2}\left(1 - \frac{\pi^2}{2n^2}\right) - (n-2)\frac{3\pi}{2n}\right]$$
$$= \frac{1}{n-2}\left[n - \frac{\pi^2}{2n} - \frac{3(n-2)\pi}{2n} - \frac{9(n-1)\pi^2}{2n^2} + \frac{9(n-1)\pi^4}{4n^4}\right].$$

Let $f(n) := \frac{\pi^2}{2n} + \frac{3(n-2)\pi}{2n} + \frac{9(n-1)\pi^2}{2n^2} - \frac{9(n-1)\pi^4}{4n^4}$. We claim that $f(n) \leq 8$. By direct verification, $f(11) < 8$ and $f(15) < 8$. For $n \geq 19$, we have,

$$f(n) < \frac{\pi^2}{2n} + \frac{3\pi}{2} + \frac{9\pi^2}{2n} - 0 \leq \frac{\pi^2}{2 \cdot 19} + \frac{3\pi}{2} + \frac{9\pi^2}{2 \cdot 19} < 8.$$

Hence we get,

$$B \geq \frac{1}{n-2}[n-8] = \frac{n-8}{n-2}.$$

Thus for c_n we obtain,

$$c_n > \frac{n-2}{4(n-1)}\left(B - \frac{1}{n-2}\right) = \frac{n-2}{4(n-1)}\left(\frac{n-9}{n-2}\right) = \frac{n-9}{4(n-1)},$$

which contradicts $c_n = \frac{n-9}{4(n-1)}$.

Hence we have completed the main proof.

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