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SUFFICIENT CONDITIONS FOR CERTAIN TYPES OF FUNCTIONS TO BE PARABOLIC STARLIKE

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ABSTRACT. In this paper sufficient conditions are determined for functions of the form $f(z) = \dfrac{z}{1+\sum_{1}^{\infty} b_k z^k}$, $f(z) = \dfrac{z}{(1+z^n)^k}$ and certain other types of functions to be parabolic starlike.

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1. Introduction

Let A denote the class of all analytic functions defined in the open unit disk $\Delta = \{z : |z| < 1\}$ and normalized as $f(0) = 0 = f'(0) - 1$.

A function $f \in A$ is uniformly convex (starlike) if for every arc γ contained in Δ with centre $\xi \in \Delta$, the image arc $f(\gamma)$ is convex (starlike with respect to $f(\xi)$). The class of all uniformly convex functions denoted by UCV was introduced by Goodman [2] in 1991. Ronning [8] and Ma and Minda [3] independently proved that $f \in UCV$ if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \ \ z \in \Delta$$

Further Ronning [8] defined the class S_p of functions $f \in A$ for which

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right|, \ \ z \in \Delta$$

It can be observed easily that $f \in UCV$ if and only if $zf' \in S_p$.

Let $\Omega = \{w : |w - 1| < Rew\}$. It follows that $f \in UCV$ or S_p are equivalent to saying that $1 + \dfrac{zf''(z)}{f'(z)}$ or $\dfrac{zf'(z)}{f(z)}$ lies in Ω respectively.

Note that Ω is a parabolic region, symmetric with respect to the real axis having $(1/2, 0)$ as its vertex. The class S_p and UCV are generalized to the class $S_p(\alpha)$ and $UCV(\alpha)$ as follows:

A function f is in $UCV(\alpha)$ if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha + \left|\frac{zf''(z)}{f'(z)}\right|, \ \ z \in \Delta$$

Similarly $f \in S_p(\alpha)$ if

$$Re\left\{1 + \frac{zf'(z)}{f(z)}\right\} > \alpha + \left|\frac{zf'(z)}{f(z)} - 1\right|, \ \ z \in \Delta$$

If $-1 < \alpha \leq 1$ then the functions in the above classes are univalent. In fact $f \in S_p(\alpha)$ we have $Re\left(\dfrac{zf'(z)}{f(z)}\right) > \dfrac{1+\alpha}{2} > 0$ and hence $f \in S^*$.

In this paper we find sufficient conditions for certain types of functions.

In order to prove our results we need the following.

2. Preliminaries

Lemma 2.1. *If* $-1 < \alpha \leq 1$, $\dfrac{3+\alpha}{4} < a \leq 2 - \alpha + \sqrt{(1-\alpha)(1-3\alpha)}$ *where* a *is any real number such that*

$$R_{a,\alpha} = \begin{cases} a - \frac{1+\alpha}{2} & \frac{3+\alpha}{4} < a \leq \frac{3-\alpha}{2} \\ \sqrt{2(1-\alpha)(a-1)} & a \geq \frac{3-\alpha}{2} \end{cases}$$

then $\left|\dfrac{zf'(z)}{f(z)} - a\right| < R_{a,\alpha}$ *implies* $f \in S_p(\alpha)$. *In particular we have* $f \in S_p(\alpha)$ *if*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{1-\alpha}{2}$$

The lemma with $\alpha = 0$ *is in* [6].

Proof. The result follows from the fact that $R_{a,\alpha}$ given in the lemma is the minimum of all distances from $a > \dfrac{1+\alpha}{2}$, and any point on the parabola $|w - 1| + \alpha < Re\, w$. ∎

Lemma 2.2. [1] *For* $|z| \leq r < 1$, $|z_k| = R > r$ *we have*

$$\left|\frac{z}{z - z_k} + \frac{r^2}{R^2 - r^2}\right| \leq \frac{Rr}{R^2 - r^2}$$

3. MAIN RESULTS

Theorem 3.1. *Let* a *and* α *be as defined in Lemma 2.1. Let*
$f(z) = \dfrac{z}{\left(1 + \sum_1^\infty b_k z^k\right)}$.
If the coefficients of $f(z)$ *satisfy*

$$\sum_{k=1}^{\infty} \{R_{a,\alpha} + |(1-a) - k|\}|b_k| \leq R_{a,\alpha} - |1-a|$$

then $f \in S_p(\alpha)$.

Proof. Since $f(z) = \dfrac{z}{\left(1 + \sum_1^\infty b_k z^k\right)}$.
We have

$$\left|\frac{zf'}{f} - a\right| = \left|1 - a - \frac{\sum_1^\infty k b_k z^k}{1 + \sum_1^\infty b_k z^k}\right|$$

So we can conclude that $f \in S_p(\alpha)$ if

$$\left|(1-a) - \frac{\sum_1^\infty k b_k z^k}{1 + \sum_1^\infty b_k z^k}\right| \leq R_{a,\alpha}$$

that is, if

$$\left|(1-a) + \sum_{k=1}^{\infty} (1-a-k) b_k z^k\right| \leq R_{a,\alpha} \left|1 + \sum_1^\infty b_k z^k\right|$$

The above inequality holds whenever

$$|1-a| + \sum_{k=1}^{\infty} |(1-a) - k||b_k| r^k \leq R_{a,\alpha}\left(1 - \sum_1^\infty |b_k| r^k\right)$$

That is

$$\sum_{k=1}^{\infty} \{|(1-a) - k| + R_{a,\alpha}\}|b_k| r^k \leq R_{a,\alpha} - |1-a|$$

Since $r < 1$ and $r^k < 1$ we have

$$\sum_{k=1}^{\infty} \{R_{a,\alpha} + |(1-a) - k|\}|b_k| \leq R_{a,\alpha} - |1-a|$$

which proves the theorem. ∎

In particular if $\alpha = 0$ and $a = 1$, then $R_{a,\alpha} = 1/2$ we have

Corollary 3.2. *Let* $f(z) = \dfrac{z}{1 + \sum_{1}^{\infty} b_k z^k}$.

Whenever $\sum_{1}^{\infty}(2k+1)|b_k| < 1$ *then* $f \in S_p$.

Remark 3.1. If $f(z) = \dfrac{z}{1 + b_1 z}$
then $f \in S_p$ whenever $3|b_1| \leq 1$
$\Rightarrow |b_1| \leq 1/3$ which is true from Goodman [2].

Corollary 3.3. *Let* $f(z) = \dfrac{z}{1 + \sum_{1}^{\infty} b_k z^k}$. *If the coefficients of* $f(z)$ *satisfy the condition given in Theorem 3.1, then*

$$f(z) = \int_0^z \left(1 + \sum_{1}^{\infty} b_k t^k\right)^{-1} dt$$

is in $UCV(\alpha)$. *In particular if* $\sum_{1}^{\infty}(2k+1)|b_k| \leq 1$ *then* f *is in* UCV.

Theorem 3.4. *Let* n, k *be fixed positive integers and let* $p(z) = \dfrac{z}{(1+z^n)^k}$. *Let* α, a *be as in Lemma 2.1. Then* $p(z) \in S_p(\alpha)$ *for*

$$|z| < \left(\frac{R_{a,\alpha} - |1-a|}{|1-a-kn| + R_{a,\alpha}}\right)^{\frac{1}{n}}$$

Proof. Since $p(z) = \dfrac{z}{(1+z^n)^k}$.

$$\left|\frac{zp'}{p} - a\right| = \left|1 - a - \frac{knz^n}{1+z^n}\right|$$

if

$$\left|1 - a - \frac{knz^n}{1+z^n}\right| < R_{a,\alpha}$$

then $p(z) \in S_p(\alpha)$.
That is,

$$|(1-a) + (1-a-kn)z^n| < R_{a,\alpha}|1+z^n|$$

This holds whenever,

$$|1-a| + |1-a-kn||z|^n < R_{a,\alpha} - R_{a,\alpha}|z|^n$$

which simplifies to

$$|z| < \left(\frac{R_{a,\alpha} - |1-a|}{|1-a-kn| + R_{a,\alpha}}\right)^{\frac{1}{n}}$$

which proves Theorem 3.4. ∎

By taking $\alpha = 0$ and $a = 1$ in the above theorem and $R_{a,\alpha} = 1/2$ we have

Corollary 3.5. $p(z) = \dfrac{z}{(1+z^n)^k}$ *is in* S_p
whenever $|z| < \left(\dfrac{1}{2kn+1}\right)^{1/n}$.

Corollary 3.6. $F(z) = \displaystyle\int_0^z \frac{dt}{(1+t^n)^k}$ *is in* UCV *if*

$$|z| < \left(\frac{1}{2kn+1}\right)^{1/n}.$$

Theorem 3.7. *Let* $Q(z)$ *be a polynomial of degree* n*. Let* $R = \min\{|\xi| : Q(\xi) = 0, \xi \neq 0\}$*. Let* a, α *be as in Lemma 2.1. Then the function* $F(z) = z(Q(z))^{\beta/n}$ *belongs to* $S_p(\alpha)$ *for*

$$|z| = r < \frac{R(R_{a,\alpha} - |1-a|)}{(|\beta| + R_{a,\alpha} - |1-a|)}$$

Proof. $F(z) = z[Q(z)]^{\beta/n}$ implies

$$\frac{zp'}{p} = 1 + \beta/n \sum_{k=1}^{n} \frac{z}{z - z_k}$$

$$\frac{zp'}{p} - a = 1 - a + \beta/n \sum_{k=1}^{n} \left(\frac{z}{z-z_k} + \frac{r^2}{R^2 - r^2} - \frac{r^2}{R^2 - r^2}\right)$$

If

$$\left|(1-a) + \beta/n \sum_{k=1}^{n} \left(\frac{z}{z-z_k} + \frac{r^2}{R^2 - r^2} - \frac{r^2}{R^2 - r^2}\right)\right| < R_{a,\alpha}$$

then $f \in S_p(\alpha)$ by Lemma 2.1.
Using Lemma 2.2 we see that $f \in S_p(\alpha)$ if

$$|1-a| + \frac{|\beta| Rr}{R^2 - r^2} + \frac{|\beta| r^2}{R^2 - r^2} < R_{a,\alpha}$$

This is satisfied if $r < \dfrac{R(R_{a,\alpha} - |1-a|)}{(|\beta| + R_{a,\alpha} - |1-a|)}$.
We get the required result. ∎

Corollary 3.8. *Let* $Q(z)$ *be a polynomial of degree* n*. Let* R *be as in Theorem 3.7. Then the function* $F(z) = z[Q(z)]^{\beta/n}$ *belongs to* S_p *for* $|z| < \dfrac{R}{1+2|\beta|}$.
This follows by taking $\alpha = 0$, $a = 1$ *in Theorem 3.7.*

Remark 3.2.

(1) Let $\beta = n$, $Q(z) = 1 - \dfrac{z^n}{(2n+1)^n}$ then $F(z) = z - \dfrac{z^{n+1}}{(2n+1)^n}$ is in S_p.
(2) The function $g(z) = z + z^2 + z^3 \in S_p$ for $|z| < 1/5$.
Since $g(z) = z(1 + z + z^2) = zQ(z)$.
Taking $\beta = 2$, $n = 2$ we get
$R = \min\{|\xi| : 1 + z + z^2 = 0\}$. Therefore $R = 1$.
Hence $g \in S_p$ for $|z| < 1/5$.
(3) The function $\dfrac{z}{(1-z^n)^{2/n}} \in S_p$.

Corollary 3.9. *Let $Q(z)$, R be as in Theorem 3.7. Then the function*

$$G(z) = \int_0^z [Q(t)]^{\beta/n} dt \in UCV(\alpha)$$

for $|z| = r < \dfrac{R(R_{a,\alpha} - |1-a|)}{(|\beta| + R_{a,\alpha} - |1-a|)}$.

Corollary 3.10.

$$g(z) = \int_0^z [Q(t)]^{\beta/n} dt \in UCV$$

for $|z| = r$ *where* $r = \dfrac{R}{1+2|\beta|}$.

Corollary 3.11.

$$\int_0^z \frac{dz}{(1-z^n)^{2/n}} \in UCV$$

This function maps the unit disk onto a regular polygon of n *sides.*

Before we proceed to our next result, we need the following lemma.

Lemma 3.12. [4] *If the function* g *is regular in* Δ *and* $|g(z)| < 1$ *in* Δ*, then for all* $\xi \in \Delta$ *and* $z \in \Delta$ *the following inequalities hold*

$$\left| \frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)} \right| \leq \left| \frac{\xi - z}{1 - \bar{z}\xi} \right|$$

and

$$|g'(z)| \leq \frac{|1 - g(z)|^2}{1 - |z|^2}$$

the equalities hold only in the case

$$g(z) = \epsilon \frac{z+u}{1+\bar{u}z} \text{ where } |\epsilon| = 1 \text{ and } |u| < 1$$

We also need the following.

Remark 3.3. [7]

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a||z|}$$

for all $g \in \Delta$ for z defined in Lemma 3.12.

Theorem 3.13. *Let* β *be a complex number and the function* $g \in A$, $g(z) = z + a_2 z^2 + \ldots$*. If* $\left| \dfrac{g''(z)}{g'(z)} \right| < 1$ *for all* $z \in \Delta$ *and* β *satisfies the condition*

$$|\beta| \leq \frac{R_{a,\alpha} + |1-a|}{\max_{|z|\leq 1} \left(\frac{|z|^2 + 2|a_2||z|}{1+2|a_2||z|} \right)}$$

where $R_{a,\alpha}$ *and* a *are as in Lemma 2.1 then the function*

$$G_\beta(z) = \int_0^z (g'(u))^\beta du \text{ is in } UCV(\alpha)$$

Proof. $G_\beta(z)$ is regular in Δ.
Consider $p(z) = \dfrac{1}{|\beta|}\dfrac{G''_\beta(z)}{G'_\beta(z)}$
where β satisfies the inequality given in the Theorem 3.13.
The function $p(z)$ is regular in Δ and we have $p(z) = \dfrac{\beta}{|\beta|}\dfrac{g''(z)}{g'(z)}$ and so $|p(z)| < 1$ for all $z \in \Delta$ and $p(0) = 2|a_2|$.
Therefore

$$\frac{1}{|\beta|}\left|\frac{G''_\beta(z)}{G'_\beta(z)}\right| \le \frac{|z| + 2|a_2|}{1 + 2|a_2||z|}$$

For all $z \in \Delta$. Therefore we get

$$\begin{aligned}\left|\frac{zG''_\beta(z)}{G'_\beta(z)}\right| &\le |\beta||z|\left(\frac{|z| + 2|a_2|}{1 + 2|a_2||z|}\right)\\ &\le |\beta|\max\left(\frac{|z|^2 + 2|a_2||z|}{1 + 2|a_2||z|}\right)\end{aligned}$$

and so

$$\left|\frac{zG''_\beta(z)}{G'_\beta(z)}\right| \le R_{a,\alpha} + |1 - a|$$

This proves the theorem. ∎

Theorem 3.14. *Let r be complex number and the function $g \in A$, $g(z) = z + a_2 z^2 + \dots$. If $\left|\dfrac{g''(z^n)}{g'(z^n}\right| \le \dfrac{1}{n}$ for all $z \in \Delta$ and n a positive integer, $|r| < R_{a,\alpha} + |1 - a|$ where $R_{a,\alpha}$ are as in Lemma 2.1 then the function*

$$G_{r,n}(z) = \int_0^z [g'(u^n)]^r du \text{ is in } UCV(\alpha)$$

Proof. Let us consider the function

$$f(z) = \int_0^z (g'(u^n))^r du$$

The function $p(z) = \dfrac{1}{|r|}\dfrac{f''(z)}{f'(z)}$ where the constant $|r|$ satisfies the inequality given in Theorem 3.14 is regular in Δ.
It follows that

$$p(z) = \frac{r}{|r|}\left[\frac{nz^{n-1}g''(z^n)}{g'(z^n)}\right]$$

and so we have $|p(z)| < 1$ for all $z \in \Delta$.
Also $p(0) = 0$ and applying Schwarz-Lemma we havve

$$\frac{1}{|r|}\left|\frac{f''(z)}{f'(z)}\right| \le |z|$$

and hence we obtain

$$\left|\frac{zf''(z)}{f'(z)}\right| \le |r||z|^2$$

This proves the required result. ∎

REFERENCES

[1] A. GANGADHARAN, V. RAVICHANDRAN and T. N. SHANMUGAM, Radii of convexity and strong starlikeness for some classes of analytic functions, *Journal of Math. Anal. and Appl.*, **211** (1997), pp. 301–313.

[2] A. W. GOODMAN, On uniformly convex functions, *Ann. Polon. Math.*, **56** (1991), pp. 87–92.

[3] W. MA and D. MINDA, Uniformly convex functions, *Ann. Polon. Math.*, **57** (1992), pp. 165–175.

[4] Z. NEHARI, *Conformal Mappings*, McGraw Hill Comp., New York (1992), Dover Pub. Inc., 1995.

[5] V. PESCAR, Some integral operators and their univalent, *J. Analysis*, **5** (1997), pp. 157–162.

[6] V. RAVICHANDRAN and T. N. SHANMUGAM, Radius problems for analytic functions, *Chinese Journal of Mathematics*, **23**(4) (1995), pp. 343–351.

[7] M. O. READE, H. SILVERMAN and P. G. TODOROV, Classes of rational functions, *Contemporary Mathematics*, **38** (1985), pp. 99–103.

[8] F. RONNING, Uniformly starlike functions, *Ann. Uni. Sklodowski*, **47** (1997), pp. 123–134.