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ON THE REGULARIZATION OF HAMMERSTEIN'S TYPE OPERATOR EQUATIONS

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ABSTRACT. We have studied Regularization of Hammerstein's Type Operator Equations in general Banach Spaces. In this paper, the results have been employed to establish regularized solutions to Hammerstein's type operator equations in Hilbert spaces by looking at three cases of regularization.

Key words and phrases: Maximal monotone mappings, Hammerstein Operator Equations, Regularization, General Banach Space, Hilbert Space.

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1. INTRODUCTION

The operator equation, central to our study is the operator equation of the Hammerstein's type:

$$u + BAu = w, \tag{1.1}$$

where A and B are operators, u, w elements in a given Banach space X, with u being the unknown element in X.
Since the work of Hammerstein [1], the study of (1.1) has mainly benefited by the compact mappings.
It will become clear in the sequel that breakup of the operator BA from Banach space X into consistuent parts notably:

$$A : X \to X^* \text{ and } B : X^* \to X \text{ with } R(A) \subseteq D(B),$$

where X^* is the dual space, will reveal the interdependence between the theory of non-linear integral equations and that of monotone mappings.
The notion of montone operators were introduced by Zarantenello [12], Minty [11] and Kacurovskii [13]. Monotonicity conditions in the context of variational methods for non-linear operator equation were also used by Vainberg and Kacurovskii [14].
Interest in (1.1) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's functions can, as a rule, be transformed into a form (1.1) [see [2]].
Equation (1.1) is generally called the Hammerstein's type operator equation.
Several existence and uniqueness theorems have been proved for the equation of the Hammerstein's type [2], [7].
Several applications of (1.1) are found in the studies of partial differential equations [15], [16], the theory of optimal control system [17], theory of mechanics, in particular in technical problems [18].
In this paper, we consider X a reflexive Banach space, A, B maximal monotone operators and the equations of study, notably the operator equation of the Hammerstein's type (1.1).
The operators A, B will be perturbed by some small parameter. The approximate solution of the regularized or perturbed Hammerstein operator equation will be extensively studied. The regularized equations will finally be examined in Hilbert spaces so as to unify results.
Finally, this paper provides a more congenial solution space for the approximate solution of the Hammerstein equation in the operator form. The methodology of this paper is to apply regularized techniques to solve operator equations of the Hammerstein's type.

2. PRELIMINARIES

Let X be a real Banach space with dual X^*. The mapping $T : X \to 2^{X^*}$ is said to be monotone if for some elements $u, v \in D(T)$, the inequality:

$$\langle f - g, u - v\rangle \geq 0, \forall [v, g] \in G(T) \tag{2.1}$$

holds for all $f \in Tu$ and $g \in Tv$. The mapping is said to be

- strictly monotone if the equality in (2.1) implies $u = v$,
- monotone if

$$\langle Tu - Tv, u - v\rangle \geq 0, \forall u, v \in D(T), \tag{2.2}$$

- uniformly monotone if for each $u, v \in D(T)$ there exists a strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that $\langle Tu - Tv, u - v\rangle \geq \varphi(||u - v||^2)$,
- strongly monotone if for each $u, v \in D(T)$ there exists $k \in (0, 1)$ such that $\langle Tu - Tv, u - v\rangle \geq k(||u - v||^2)$.

A close study of the definitions shows that every strongly monotone map is uniformly monotone; since $\varphi(t) = kt, k \in (0, 1)$ is strictly monotone function with $\varphi(0) = 0$.
A set $M \subseteq X \times X^*$ is monotone provided that $\langle f - g, u - v\rangle \geq 0$ for any pair $[u, f], [v, g] \in M$. A monotone set M is maximal if it is not a proper subset of a monotone set in $X \times X^*$. The mapping $T : X \to 2^{X^*}$ is said to be maximal monotone if its graph $G(T)$ is a maximal monotone set of $X \times X^*$. Therefore, T is maximal monotone if and only if $\langle f - g, u - v\rangle \geq 0$ implies $u \in D(T)$ and $f \in Tu$. The element $[u, f] \in X \times X^*$ lies in $G(T)$ if and only if $[f, u] \in X^* \times X$ lies in $G(T^{-1})$. Since the monotonicity is invariant under transposition of domain and the range of a map, T is maximal monotone if and only if T^{-1} has this property.
Let X be an arbitrary Banach space with dual space X^*. The mapping J, where $J : X \to 2^{X^*}$ is called a normalized duality mapping on X and is defined by

$$J(u) = \{u^* \in X^* : \langle u^*, u\rangle = ||u||^2 = ||u^*||^2\}. \tag{2.3}$$

From now on X is seen as a Reflexive Banach Space with dual X^*, the mappings $A : X \to 2^{X^*}$ and $B : X^* \to 2^X$ maximal monotone with respective domains $D(A)$ and $D(B)$.
In (1.1) we define a map $C = A^{-1} : X^* \to 2^X$ and for any $\varphi \in Au$, (1.1) may be written as:

$$C\varphi + B\varphi = w, \tag{2.4}$$

where C is maximal monotone [[2], p. 122].

Theorem 2.1. [3]. *Suppose that X is reflexive, that A and B are maximal monotone operators on X and that*

$$D(A) \cap intD(B) \neq \emptyset \tag{2.5}$$

then $A + B$ is maximal.

Combining (2.4) with the two theorems [[2] p. 106 - 107], we have the following:

Lemma 2.2. *If $A : X \to 2^{X^*}$ and $B : X^* \to 2^X$ maximal monotone mappings, the map $A^{-1} + B$ coercive and the condition (2.5) holds, then (1.1) or (2.4) with any $w \in X$ has at least one solution.*

The formulation of a regularized equation of Hammerstein's type requires the following definitions and notations.

Definition 2.1. Let the space $\mathcal{Z}$ be defined by

$$\mathcal{Z} = X \times X^* = \{\varsigma = [u, \varphi] : u \in X, \varphi \in X^*\}. \tag{2.6}$$

With the natural linear operation, +, defined by $\alpha\varsigma_1 + \beta\varsigma_2 = [\alpha u_1 + \beta u_2, \alpha\varphi_1 + \beta\varphi_2]$ for real numbers α and β and $\varsigma_1 = [u_1, \varphi_1], \varsigma_2 = [u_2, \varphi_2]$ are elements of $\mathcal{Z}$.
For any $\varsigma \in \mathcal{Z}$, let $||\varsigma||_{\mathcal{Z}} = \{||u||^2 + ||\varphi||^2\}^{\frac{1}{2}}$ then the space $\mathcal{Z}$ is a Banach space with $\mathcal{Z}^* = X^* \times X$ as its dual. The duality pairing of the spaces $\mathcal{Z}$ and $\mathcal{Z}^*$ is defined by the product

$$\langle \eta^*, \varsigma\rangle = \langle \psi, u\rangle + \langle \varphi, v\rangle \tag{2.7}$$

of the elements $\varsigma = [u, \varphi] \in \mathcal{Z}$ and $\eta^* = [\psi, v] \in \mathcal{Z}^*$.

Lemma 2.3. *Let $\{\varsigma_n\}$ be a sequence in $\mathcal{Z}$, where $\varsigma_n = [u_n, \varphi_n]$ and let $\varsigma_0 = [u_0, \varphi_0]$. As $n \to \infty$, the following relations are equiavelent:*

$$\varsigma_n \rightharpoonup \varsigma_0, ||\varsigma_n||_{\mathcal{Z}} \to ||\varsigma_0||_{\mathcal{Z}} \tag{2.8}$$

and

(2.9) $$u_n \rightharpoonup u_0, \varphi_n \rightharpoonup \varphi_0, ||u_n|| \to ||u_0||, ||\varphi_n|| \to ||\varphi_0||.$$

Proof. The implication $(2.9) \Rightarrow (2.8)$ is obvious.
$[u_n, \phi_n] \rightharpoonup [u_0, \phi_0] = \varsigma_n \rightharpoonup \varsigma_0$ and $||\varsigma_n||^2_{\mathcal{Z}} = (||u_n||^2 + ||\phi_n||^2_*) = ||\varsigma_0||^2_{\mathcal{Z}}$
On the other hand, we assume (2.8) is valid and let $\eta^* = [\psi, v]$ be an arbitrary element of $\mathcal{Z}^*$. Then the relation $\langle \eta^*, \varsigma_n \rangle$ implies

(2.10) $$\langle \psi, u_n \rangle + \langle \phi_n, v \rangle \to \langle \psi, u_0 \rangle + \langle \phi_0, v \rangle$$

If we put $v = 0$ in (2.10) then

$$\langle \psi, u_n \rangle \to \langle \psi, u_0 \rangle \qquad \forall \psi \in \mathcal{X}^*$$

that is, $u_n \rightharpoonup u_0$. Also if $\psi = 0$ then $phi_n \rightharpoonup \phi_0$. By the weak convergence these imply the inequalities

(2.11) $$||u_0|| \leq \liminf_{n\to\infty} ||u_n|| and ||\phi_0||_* \leq \liminf_{n\to\infty} ||\phi_n||_*$$

since

$$||\varsigma_n||_{\mathcal{Z}} \to ||\varsigma_0|| = (||u_0||^2 + ||\phi_0||^2_*)^{\frac{1}{2}},$$

one can consider that $||u_{n_k}|| \to a$ and $||\phi_{n_k}|| \to b$ with

$$a^2 + b^2 = ||u_0||^2 + ||\phi_0||^2_*,$$

where $\{u_{n_k}\}$ and $\{\phi_{n_k}\}$ are subsequences of $\{u_n\}$ and $\{\phi_n\}$, respectively.
Then (2.11) implies $||u_0|| = a$ and $||\phi_0||_* = b$.
Thus, $||u_n|| \to ||u_0||$ and $||\phi_n||_* \to ||\phi_0||_*$. ∎

Lemma 2.4. *If $J : X \to X^*$ and $J^* : X^* \to X$ are the normalized duality mapping on X and X^*, respectively, then the operator $J_{\mathcal{Z}} : \mathcal{Z} \mapsto \mathcal{Z}^*$, defined by*

(2.12) $$J_{\mathcal{Z}}\varsigma = [Ju, J^*\varphi] \quad \forall \varsigma = [u, \varphi] \in \mathcal{Z},$$

is a normalized duality mapping on $\mathcal{Z}$; and conversely, every normalized duality mapping on $\mathcal{Z}$ has the form (2.10).

Proof. We verify for $J_{\mathcal{Z}}$ all the conditions of a normalized duality mapping as spelt out in (2.3). For this reason consider an arbitrary element $\varsigma = [u, \phi] \in \mathcal{Z}$ and using the properties of the operators J and J^*, we obtain
$\langle J_{\mathcal{Z}}\varsigma, \varsigma \rangle = \langle Ju, u \rangle + \langle \phi, J^*\phi \rangle = ||u||^2 + ||\phi||^2_* = ||\varsigma||^2_{\mathcal{Z}}$;
$||J_{\mathcal{Z}}\varsigma||_{\mathcal{Z}^*} = (||Ju||^2_* + ||J^*\phi||^2)^{\frac{1}{2}} = (||u||^2 + ||\phi||^2_*)^{\frac{1}{2}} = ||\varsigma||_{\mathcal{Z}}$
that is $J_{\mathcal{Z}} : \mathcal{Z} \mapsto \mathcal{Z}^*$ is dual mapping on $\mathcal{Z}$.
Let the operator $\bar{J}_{\mathcal{Z}} : \mathcal{Z} \mapsto \mathcal{Z}^*$ be such that

(2.13) $$\langle \bar{J}_{\mathcal{Z}}\varsigma, \varsigma \rangle = ||\varsigma||^2_{\mathcal{Z}}, \qquad ||\bar{J}_{\mathcal{Z}}\varsigma||^*_{\mathcal{Z}} = ||\varsigma||_{\mathcal{Z}}$$

Assuming that $\bar{J}_{\mathcal{Z}}\varsigma = \eta^* = [\psi, v]$, $\psi \in \mathcal{X}^*$, $v \in \mathcal{X}$, we write the inequality

$$\langle \bar{J}_{\mathcal{Z}}\varsigma, \varsigma \rangle = \langle \eta^*, \varsigma \rangle = \langle \psi, u \rangle + \langle \phi, v \rangle,$$

which by virtue of (2.13) gives

$$||u||^2 + ||\phi||_*^2 = \langle \psi, u\rangle + \langle \phi, v\rangle = ||\psi||_*^2 + ||v||^2 \tag{2.14}$$

Next we show that $\psi = Ju$, $v = J^*\phi$.
It is necessary to establish that $\langle \psi, u\rangle = ||\psi||_*||u||$, and $\langle \phi, v\rangle = ||\phi||_*||v||$ From (2.14) we assume on the contrary
$||u||^2+||\phi||_*^2 < ||\psi||_*||u||+||v||||\phi||_* \leq 2^{-1}(||\psi||_*^2+||u||^2)+2^{-1}(||v||^2+||\phi||_*^2) = ||u||^2+||\phi||_*^2$
which is false. Hence (2.13) may be written as:
$||u||^2 + ||\phi||_*^2 < ||\psi||_*||u|| + ||v||||\phi||_* \leq 2^{-1}(||\psi||_*^2 + ||u||^2) + 2^{-1}(||v||^2 + ||\phi||_*^2)$
Then

$$0 = (||u|| - ||\psi||_*)^2 + (||\phi||_* - ||v||)^2,$$

That is, $\psi = Ju$ and $v = J^*\phi$.
∎

Lemma 2.5. *The space $\mathcal{Z}$ is strictly convex if and only X and X^* are strictly convex.*

Proof. The space $\mathcal{Z}$ is strictly convex if and only if $J_{\mathcal{Z}}$ is strcitly monotone operator. On the other hand, $J_{\mathcal{Z}}$ is strictly monotone if and only if J and J^* are strictly monotone since
$J_{\mathcal{Z}}\varsigma = [Ju, J^*\phi]$ for all $\varsigma = [u, \phi]$, (see [4], p45). ∎

The equation (1.1) is written as the system:

$$Au - \varphi = 0, u + B\varphi = w, \varphi \in Au\ ,$$

which is equivalent to the operator equation:

$$T\varsigma = h, \tag{2.15}$$

where $T : \mathcal{Z} \mapsto 2^{\mathcal{Z}^*}$ such that

$$T\varsigma = [Au - \varphi, u + B\varphi] \in \mathcal{Z}^*, \varsigma = [u, \varphi] \in \mathcal{Z}, h = [0_{X^*}, \omega] \in \mathcal{Z}^*\ .$$

Lemma 2.6. *The operator T is monotone.*

Proof. Let $\varsigma_1 = [u_1, \varphi_1], \varsigma_2 = [u_2, \varphi_2] \in \mathcal{Z}$, the equality

$$\langle T\varsigma_1 - T\varsigma_2, \varsigma_1 - \varsigma_2\rangle = \langle Au_1 - Au_2, u_1 - u_2\rangle + \langle \varphi_1 - \varphi_2, B\varphi_1 - B\varphi_2\rangle \tag{2.16}$$

holds. Additionally, the equality is valid if $u_i \in D(A), \varphi_i \in D(B)$ for all $i = 1, 2$. At that, the inclusion $\varphi_i \in Au_i$ should hold, that is, $u_i \in A^{-1}\varphi_i, i = 1, 2$
Therefore (2.12) may be written as
$\langle T\varsigma_1 - T\varsigma_2, \varsigma_1 - \varsigma_2\rangle = \langle \varphi_1 - \varphi_2, A^{-1}\varphi_1 - A^{-1}\varphi_2\rangle + \langle \varphi_1 - \varphi_2, B\varphi_1 - B\varphi_2\rangle$, where $\varphi_i \in D(B), \varphi_i \in D(A^{-1}), i = 1, 2.$
With this condition it follows that if (2.5) is satisfied then the operator T is maximal monotone on its domain $D(T)$.
∎

Let the operators A, B be maximal monotone. Let $D(A) = X$ and condition (2.5) be satisfied. Let N be a non-empty closed solution set of (1.1). Then we have the following result:

Lemma 2.7. *The set N is convex.*

Proof. Let M be a solution of (2.4) then M is convex and closed in $\mathcal{Z}$ (see [2] p105). In view of (2.15) , and since $M = \{[u, \phi] : u \in N, \phi \in Au\}$, then $M = N \times A(N)$. Therefore N is convex. ∎

Next we consider a regularized Hammerstein type operator equation in general Banach space

$$u + (B + \alpha J^*)(A + \alpha J)u = w^\delta, \tag{2.17}$$

where $\alpha > 0, \delta > 0$. w^δ is $\delta-$approximation of w, where $||w - w^\delta|| \leq \delta$

Lemma 2.8. *The equation (2.13) is uniquely solvable for every element $w^\delta \in X$.*

Proof. Let $B^\alpha = B + \alpha J^*$, $A^\alpha = A + \alpha J^*$.
We introduce an operator
$T^\alpha = [A^\alpha u - \phi, u + B^\phi] = T\varsigma + \alpha J_{\mathcal{Z}}\varsigma, \varsigma = [u, \phi]$, $J_{\mathcal{Z}}\varsigma = [Ju, J^*\phi]$
The solvability of (2.15) is equivalent to the solvability of equation

$$T^\alpha \varsigma^\delta_\alpha = h^\delta, \qquad h^\delta = [0^*_{\mathcal{X}}, \omega^\delta] \tag{2.18}$$

where $T^\alpha = T + \alpha J_{\mathcal{Z}}$ is a maximal monotone. The conclusion of the Lemma follows from (see [5], [6] and [7]
∎

Theorem 2.9. *Let a solution set N of equation (1.1) be non empty and closed, $A : X \to X^*$ be a maximal monotone locally bounded mapping with $D(A) = X$. Assume that $B : X^* \mapsto 2^X$ is also maximal monotone mapping. Let w^δ be a $\delta-$ approxiamtion of w, such that $||w - w^\delta|| \leq \delta$. If $\frac{\delta}{\alpha} \to 0$, and also $\alpha \to 0$, then the sequence $\{u^\delta_\alpha\}$ of solutions of the regularized equation (2.13) converges strongly in X to the solution $u^* \in N$ which is defined as*

$$||u^*||^2 + ||Au^*||^2 = min\{||u||^2 + ||Au||^2 : u \in N\}.$$

Proof. Consider the operator equation (2.15) and (2.18) with solutions $\varsigma, \varsigma^\delta_\alpha$ respectively. Subtracting (2.15) from (2.18), we have

$$T^\alpha \varsigma^\delta_\alpha - T\varsigma = h^\delta - h$$

or equivalently

$$T\varsigma^\delta_\alpha - T\varsigma + \alpha I_{\mathcal{Z}}\varsigma^\delta_\alpha = h^\delta - h$$

Multiply the above through with $\varsigma^\delta_\alpha - \varsigma$, we have

$$\langle T\varsigma^\delta_\alpha - T\varsigma, \varsigma^\delta_\alpha - \varsigma\rangle + \alpha\langle I_{\mathcal{Z}}\varsigma^\delta_\alpha, \varsigma^\delta_\alpha - \varsigma\rangle \leq \langle h^\delta - h, \varsigma^\delta_\alpha - \varsigma\rangle \tag{2.19}$$

By the monotonicty of T, we have

$$\alpha\langle I_{\mathcal{Z}}\varsigma^\delta_\alpha, \varsigma^\delta_\alpha - \varsigma\rangle \leq \langle h^\delta - h, \varsigma^\delta_\alpha - \varsigma\rangle \leq ||h^\delta - h||||\varsigma^\delta_\alpha - \varsigma|| \tag{2.20}$$

That is,

$$\alpha\langle I_{\mathcal{Z}}\varsigma^\delta_\alpha, \varsigma^\delta_\alpha\rangle \leq \delta||\varsigma^\delta_\alpha - \varsigma||$$

or $$\alpha(||\varsigma^\delta_\alpha||^2 - ||\varsigma^\delta_\alpha||||\varsigma||) \leq \delta||\varsigma^\delta_\alpha - \varsigma||$$
which is a quadratic in $||\varsigma^\delta_\alpha||$, giving an estimate $0 \leq ||\varsigma^\delta_\alpha|| \leq w$ where

$$w = ||\varsigma|| + \frac{\delta}{\alpha} + \sqrt{(||\varsigma|| + \frac{\delta}{\alpha})^2 + \frac{4\delta||\varsigma||}{\alpha}} \tag{2.21}$$

Thus the sequence $\{\varsigma^\delta_\alpha\}$ is bounded. Hence there is a subset $\{\varsigma^\delta_\beta\}$ (see [8] p 53) which converges as $\beta \to 0$ to some element $\bar{\varsigma} \in \mathcal{Z}$

Next we show that $T\bar{\varsigma} = h$.
Take an arbitrary element $\varsigma \in \mathcal{Z}$. Since T is monotone, then
$0 \leq \langle T\varsigma - T^{\alpha}\varsigma_{\alpha}^{\delta}, \varsigma - \varsigma_{\alpha}^{\delta}\rangle = \langle T\varsigma - h, \varsigma - \varsigma_{\alpha}^{\delta}\rangle + \beta\langle I_{\mathcal{Z}}\varsigma_{\alpha}^{\delta}, \varsigma - \varsigma_{\alpha}^{\delta}\rangle + \langle h - h^{\delta}, \varsigma - \varsigma_{\alpha}^{\delta}\rangle$
$\leq \langle T\varsigma - h, \varsigma - \varsigma_{\alpha}^{\delta}\rangle + \beta\langle I_{\mathcal{Z}}\varsigma_{\alpha}^{\delta}, \varsigma - \varsigma_{\alpha}^{\delta}\rangle + \delta||\varsigma - \varsigma_{\alpha}^{\delta}||$
As $\beta, \delta \to 0$, we have
$\langle T\varsigma - h, \varsigma - \bar{\varsigma}\rangle \geq 0$
which means

$$T\bar{\varsigma} = h$$

implying clearly that $\bar{\varsigma}$ is a solution of (2.15)
Finally, from (2.19) we have the expression $\langle I_{\mathcal{Z}}\varsigma_{\beta}^{\delta}, \varsigma_{\beta}^{\delta} - \varsigma\rangle \leq \frac{\delta}{\beta}||\varsigma_{\beta}^{\delta} - \varsigma||$
As $\beta, \frac{\delta}{\beta} \to 0, \varsigma_{\beta}^{\delta} \to \bar{\varsigma}$. This implies $\langle I_{\mathcal{Z}}\bar{\varsigma}, \bar{\varsigma} - \varsigma\rangle \leq 0$. Thus $||\bar{\varsigma}|| \leq ||\varsigma||$
Therefore, since ς is arbitrary we have:

$$||\bar{\varsigma}|| = min_{\varsigma \in N}||\varsigma||.$$

∎

3. MAIN RESULTS

We now discuss our results in Hilbert spaces by looking at various types of regularization of the parameters of the Hammerstein's type operator equation.

Case I

We consider a regularized Hammerstein type operator equation in H:

$$u + (B + \alpha I)(A + \alpha I)u = w, \tag{3.1}$$

where $\alpha > 0$. The equivalent operator form of this regularized equation is :

$$T^{\alpha}\varsigma_{\alpha} = h, \tag{3.2}$$

where $T^{\alpha} = (T + \alpha I)$.

Theorem 3.1. *Let a solution set N of (3.1) be a non-empty and closed, $A : H \to H$ be a maximal monotone and locally bounded mapping with $D(A) = H$. Assume that $B : H \to 2^H$ is a maximal monotone mapping too and the condition of the maximal monotone is satisfied. If $\alpha \to 0$, then the solution ς_{α} of the regularized equation (3.1) satisfies (2.11) and also*

$$||\varsigma_{\alpha}|| = min_{\varsigma \in \mathcal{Z}}||\varsigma||,$$

where $\varsigma \in \mathcal{Z}$ is in the solution set of the Hammerstein's type operator equation (2.11).

Proof. Let $\varsigma \in \mathcal{Z}$ be arbitrary. Since T is monotone,
$0 \leq \langle T\varsigma - T\varsigma_{\alpha}, \varsigma - \varsigma_{\alpha}\rangle = \langle T\varsigma - h + \alpha\varsigma_{\alpha}, \varsigma - \varsigma_{\alpha}\rangle = \langle T\varsigma - h, \varsigma - \varsigma_{\alpha}\rangle + \alpha\langle\varsigma_{\alpha}, \varsigma - \varsigma_{\alpha}\rangle$. As $\alpha \to 0$ we have

$$0 \leq \langle T\varsigma - h, \varsigma - \varsigma_{\alpha}\rangle$$

implying that $T\varsigma_{\alpha} = h$, that is, ς_{α} is a solution of (3.2).
Subtracting (2.11) from (3.2) and multiplying through by $\varsigma_{\alpha} - \varsigma$, we have

$$0 \leq \langle T^{\alpha}\varsigma_{\alpha} - T\varsigma, \varsigma_{\alpha} - \varsigma\rangle$$
$$= \langle T\varsigma_{\alpha} - h, \varsigma_{\alpha} - \varsigma\rangle + \alpha\langle\varsigma_{\alpha}, \varsigma_{\alpha} - \varsigma\rangle.$$

By the monotonicity of T we have the required results

$$||\varsigma_\alpha|| = min_{\varsigma \in \mathcal{Z}}||\varsigma||.$$

∎

Case II

Next we consider a regularized Hammerstein's type operator equation in H

$$u + (B + \alpha I)(A + \alpha I)u = w^\delta, \tag{3.3}$$

where $\alpha > 0, \delta > 0$. The equivalent operator form of this equation is:

$$T^\alpha \varsigma_\alpha^\delta = h^\delta, \tag{3.4}$$

where $T^\alpha = T + \alpha I$ and w^δ is $\delta-$approximation of w, where $||w - w^\delta|| \geq \delta$ and $h^\delta = [0_H, w^\delta]$.

Lemma 3.2. *The equation (3.3) is uniquely solvable for every element $w^\delta \in X$.*

Proof. Refer to the proof of Lemma (2.8). ∎

Theorem 3.3. *Let a solution set N of equation (1.1) be non empty and closed, $A : X \to X^*$ be a maximal monotone locally bounded mapping with $D(A) = X$. Assume that $B : X^* \mapsto 2^X$ is also maximal monotone mapping. Let w^δ be a $\delta-$ approxiamtion of w, such that $||w - w^\delta|| \leq \delta$. If $\frac{\delta}{\alpha} \to 0$, and also $\alpha \to 0$, then the sequence $\{u_\alpha^\delta\}$ of solutions of the regularized equation (3.4) converges strongly in X to the solution $u^* \in N$ which is defined as*

$$||u^*||^2 + ||Au^*||^2 = min\{||u||^2 + ||Au||^2 : u \in N\}.$$

Proof. Refer to the proof of Theorem (2.9). ∎

Case III

The parameters and conditions of regularization are:
$B^h, A^h, w^\delta, \alpha > 0, ||A^h u - Au||_H \leq hg(||u||), ||B^h u - Bu||_H \leq hg(||u||), h > 0, \delta > 0$.
Let the operators A, B, A^h, B^h be maximal monotone mappings on $H \mapsto 2^H$. The regularized form of (2.11) is given by

$$u + (B^h + \alpha I_H)(A^h + \alpha I_H)u = w^\delta. \tag{3.5}$$

Then equation (3.5) is represented by

$$T_h^{\alpha\delta} \varsigma_h^{\alpha\delta} = h_h^\delta, \tag{3.6}$$

where
$T_h^{\alpha\delta}\varsigma_h^{\alpha\delta} = [(A^h + \alpha I_H)u_h^{\alpha\delta} - \phi_h^{\alpha\delta}, u_h^{\alpha\delta} + (B^h + \alpha I_H)\phi_h^{\alpha\delta}],\ h_h^\delta = [0, w^\delta], \varsigma_h^{\alpha\delta} = [u_h^{\alpha\delta}, \phi_h^{\alpha\delta}]$.
Furthermore,

$$T_h^{\alpha\delta} = T_h^\delta + \alpha I_{\mathcal{Z}}, \mathcal{Z} = H \times H, \tag{3.7}$$

where $T_h^{\alpha\delta}\varsigma_h^{\alpha\delta} = [A^h u_h^{\alpha\delta} - \phi_h^{\alpha\delta}, u_h^{\alpha\delta} + B^h\phi_h^{\alpha\delta}]$, $I_{\mathcal{Z}} = [I_H u_h^{\alpha\delta}, I_H \phi_h^{\alpha\delta}]$.
Let $\varsigma_{h_1}^{\alpha\delta}, \varsigma_{h_2}^{\alpha\delta} \in \mathcal{Z}$. Then clearly

$$0 \leq \langle T_h^{\alpha\delta}\varsigma_{h_1}^{\alpha\delta} - T_h^{\alpha\delta}\varsigma_{h_2}^{\alpha\delta}, \varsigma_{h_1}^{\alpha\delta} - \varsigma_{h_2}^{\alpha\delta}\rangle$$

that is $T_h^{\alpha\delta}$ is monotone. By Lemma 3.2, (3.6) is uniquely solvable for $T_h^{\alpha\delta}$ which is maximal monotone.
We verify the requirements inherent in Theorem 3.3. We subtract (2.11) from (3.6) and multiply through by $\varsigma_h^{\alpha\delta} - \varsigma$ to obtain

$$\langle T_h^{\alpha\delta}\varsigma_h^{\alpha\delta} - T\varsigma, \varsigma_h^{\alpha\delta} - \varsigma\rangle = \langle h_h^\delta - h, \varsigma_h^{\alpha\delta} - \varsigma\rangle$$

or

$$\langle T_h^\delta \varsigma_h^{\alpha\delta} - T\varsigma, \varsigma_h^{\alpha\delta} - \varsigma\rangle + \alpha\langle I_{\mathcal{Z}}\varsigma_h^{\alpha\delta}, \varsigma_h^{\alpha\delta} - \varsigma\rangle = \langle h_h^\delta - h, \varsigma_h^{\alpha\delta} - \varsigma\rangle$$

implying that

(3.8) $$\alpha\langle \varsigma_h^{\alpha\delta}, \varsigma_h^{\alpha\delta} - \varsigma\rangle = \langle h_h^{\delta} - h, \varsigma_h^{\alpha\delta} - \varsigma\rangle - \langle T_h^{\delta}\varsigma_h^{\alpha\delta} - T\varsigma, \varsigma_h^{\alpha\delta} - \varsigma\rangle.$$

However,

$$-\langle T_h^{\delta}\varsigma_h^{\alpha\delta} - T\varsigma, \varsigma_h^{\alpha\delta} - \varsigma\rangle \le hg(||u||)||u_h^{\alpha\delta} - u|| + h(g(||\phi||)||\phi_h^{\alpha\delta} - \phi||).$$

Also

$$\langle h_h^{\delta} - h, \varsigma_h^{\alpha\delta} - \varsigma\rangle \le \delta||\varsigma_h^{\alpha\delta} - \varsigma||.$$

Therefore from (3.8) we have:

$$\langle \varsigma_h^{\alpha\delta}, \varsigma_h^{\alpha\delta} - \varsigma\rangle \le \frac{\delta}{\alpha}||\varsigma_h^{\alpha\delta} - \varsigma||_{\mathcal{Z}} + \frac{h}{\alpha}[g(||u||)||u_h^{\alpha\delta} - u|| + g(||\phi||)||\phi_h^{\alpha\delta} - \phi||],$$

or equivalently

$$||\varsigma_h^{\alpha\delta}||^2 - ||\varsigma_h^{\alpha\delta}||||\varsigma|| \le \frac{\delta}{\alpha}||\varsigma_h^{\alpha\delta}||_{\mathcal{Z}} + ||\varsigma||_{\mathcal{Z}} + \frac{h}{\alpha}[g(||u||)||u_h^{\alpha\delta} - u|| + g(||\phi||)||\phi_h^{\alpha\delta} - \phi||],$$

which is quadratic equation in $||\varsigma_h^{\alpha\delta}||$. An argument similar to the proof of Theorem (3.3) shows that $||\varsigma_h^{\alpha\delta}||$ is bounded and therefore there exists a subset $\{\varsigma_h^{\beta\delta}\}$ which converges as $\beta \to 0$ to some element $\bar{\varsigma} \in \mathcal{Z}$.

Next we show that$\bar{\varsigma}$ is a solution of (2.11). Since T is monotone, we have

$$0 \le \langle T\varsigma - T\varsigma_h^{\beta\delta}, \varsigma - \varsigma_h^{\beta\delta}\rangle = \langle T\varsigma - T_h^{\delta}\varsigma_h^{\beta\delta}, \varsigma - \varsigma_h^{\beta\delta}\rangle + \langle T_h^{\delta}\varsigma_h^{\beta\delta} - T\varsigma_h^{\beta\delta}, \varsigma - \varsigma_h^{\beta\delta}\rangle$$

$$= \langle T\varsigma - h, \varsigma - \varsigma_h^{\beta\delta}\rangle + \langle h - h^{\delta}, \varsigma - \varsigma_h^{\beta\delta}\rangle + \langle h^{\delta} - T_h^{\delta}\varsigma_h^{\beta\delta}, \varsigma - \varsigma_h^{\beta\delta}\rangle + \langle T_h^{\delta}\varsigma_h^{\beta\delta} - T\varsigma_h^{\beta\delta}, \varsigma - \varsigma_h^{\beta\delta}\rangle$$
$$\le \langle T\varsigma - h, \varsigma - \varsigma_h^{\beta\delta}\rangle + \delta||\varsigma - \varsigma_h^{\beta\delta}|| + \beta\langle I_{\mathcal{Z}}\varsigma_h^{\beta\delta}, \varsigma - \varsigma_h^{\beta\delta}\rangle + \langle T_h^{\delta}\varsigma_h^{\beta\delta} - T\varsigma_h^{\beta\delta}, \varsigma - \varsigma_h^{\beta\delta}\rangle.$$

However,

$$\langle T_h^{\delta}\varsigma_h^{\beta\delta} - T\varsigma_h^{\beta\delta}, \varsigma - \varsigma_h^{\beta\delta}\rangle \le h[g(||u_h^{\beta\delta}||)||u - u_h^{\beta\delta}|| + g(||\phi_h^{\beta\delta}||)||\phi - \phi_h^{\beta\delta}||].$$

Substituting we have

$$0 \le \langle T\varsigma - h, \varsigma - \varsigma_h^{\beta\delta}\rangle + \delta||\varsigma - \varsigma_h^{\beta\delta}|| + \beta\langle I_{\mathcal{Z}}\varsigma_h^{\beta\delta}, \varsigma - \varsigma_h^{\beta\delta}\rangle + h[g(||u_h^{\beta\delta}||)||u - u_h^{\beta\delta}|| + g(||\phi_h^{\beta\delta}||)||\phi - \phi_h^{\beta\delta}||]$$
.

As $\beta, \delta, h \to 0$, we have

$$0 \le \langle T\varsigma - h, \varsigma - \bar{\varsigma}\rangle\ .$$

Hence by the monotonicity of T, we have

$$T\bar{\varsigma} = h.$$

Finally from (3.8),

$$\langle \varsigma_h^{\alpha\delta}, \varsigma_h^{\alpha\delta} - \varsigma\rangle \le \frac{\delta}{\alpha}||\varsigma_h^{\alpha\delta} - \varsigma||_{\mathcal{Z}} + \frac{h}{\alpha}[g(||u||)||u_h^{\alpha\delta} - u|| + g(||\phi||)||\phi_h^{\alpha\delta} - \phi||]$$
.

As $\frac{\delta}{\alpha}, \frac{h}{\alpha} \to 0$, we have

$$\langle \varsigma_h^{\alpha\delta}, \varsigma_h^{\alpha\delta} - \varsigma\rangle \le 0$$

implying that

$$||\varsigma_h^{\alpha\delta}|| \le ||\varsigma||.$$

Hence,

$$||\varsigma_h^{\alpha\delta}|| \le min_{\varsigma \in N}||\varsigma||.$$

Therefore Theorem 3.3 is satisfied.

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