The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 11, Issue 1, Article 17, pp. 1-19, 2014

WEAK SOLUTIONS OF NON COERCIVE STOCHASTIC NAVIER-STOKES EQUATIONS IN $\mathbb{R}^2$

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Received 20 February, 2014; accepted 18 November, 2014; published 22 December, 2014.

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ABSTRACT. We prove existence of weak solutions of stochastic Navier-Stokes equations in $\mathbb{R}^2$ which do not satisfy the coercivity condition. The equations are formally derived from the critical point of some variational problem defined on the space of volume preserving diffeomorphisms in $\mathbb{R}^2$. Since the domain of our equation is unbounded, it is more difficult to get tightness of approximating sequences of solutions in comparison with the case of a bounded domain. Our approach is based on uniform a priori estimates on the enstrophy of weak solutions of the stochastic 2D-Navier-Stokes equations with periodic boundary conditions, where the periodicity is growing to infinity combined with a suitable spatial cutoff-technique.

Key words and phrases: Stochastic differential equation; Weak solution.

2000 *Mathematics Subject Classification.* Primary 60H15. Secondary 35R60.

ISSN (electronic): 1449-5910

This research is supported by JSPS and DFG as a member of the International Research Training Group 1529.

The second author would like to give a great thanks to Professor Yoshihiro Shibata who gave him this precious opportunity.

The second author would also like to thank to Professor Okihiro Sawada giving him a very kind advice about our proof during his short stay in Darmstadt.

Finally, the second author would like to give a great thanks to Professor Tadahisa Funaki giving him a continual and good advice.

1. INTRODUCTION

In this paper we study the following type of the stochastic Navier-Stokes equation with respect to $u = (u^1(t,x), u^2(t,x)), t > 0, x \in \mathbb{R}^2$:

$$\frac{\partial u}{\partial t} - \mu \triangle u + (u \cdot \nabla)u - \sqrt{2\mu}\nabla u \cdot \dot{B}(t) + \nabla p = 0, \quad t > 0, x \in \mathbb{R}^2, \tag{1.1}$$

$$\mathrm{div} u = 0, \quad t > 0, x \in \mathbb{R}^2, \tag{1.2}$$

$$u(0) = u_0, \quad x \in \mathbb{R}^2, \tag{1.3}$$

where $p = p(t,x)$ denotes the pressure term, $\mu > 0$ is a constant and $\dot{B}(t) = \frac{d}{dt}(B^1(t), B^2(t))$ the distributional derivative of the two-dimensional Brownian motion $B(t) = (B^1(t), B^2(t))$. Furthermore, u_0 is a deterministic $\mathbf{V}(\mathbb{R}^2)$-valued function on $\mathbb{R}^2$ with compact support. Here $\mathbf{V}(\mathbb{R}^2)$ is the set of functions defined as follows (see Section 2):

$$\mathbf{V}(\mathbb{R}^2) = \mathbf{W}^{1,2}(\mathbb{R}^2; \mathbb{R}^2) \cap \mathbf{H}(\mathbb{R}^2),$$

where

$$\mathbf{H}(\mathbb{R}^2) = \{u \in \mathbf{L}^2(\mathbb{R}^2; \mathbb{R}^2) \,|\, \mathrm{div}\, u = 0\}.$$

Equation (1.1)-(1.3) can be formally derived as the Euler-Lagrange equation satisfied by a critical point of a random energy functional defined on the space of volume preserving diffeomorphisms in $\mathbb{R}^2$ perturbed by Brownian motion (see [9]). In [9], the velocity defined as the time derivative of the associated stationary point satisfies the stochastic Navier-Stokes equation (1.1)-(1.3) and as a corollary, it is shown that the expectation of the solution of (1.1)-(1.3) satisfies the Reynolds equation.

On the other hand, [4] considers an energy functional different from that of [9], and shows that the deterministic Navier-Stokes equation is related to its stationary point. In this paper, we try to study the equation (1.1)-(1.3) not in the case of a two-dimensional torus but on the whole space $\mathbb{R}^2$.

In comparison with the case of stochastic Navier-Stokes equations on a bounded domain, the case of unbounded domains requires more efforts because of the lack of compactness.

In addition, our equation does not satisfy the coercivity condition which usually gives the tightness. Let us explain briefly our strategy taken in this paper to construct the solution of the equation (1.1)-(1.3). In this paper, we partly use the method which is studied in [2], [11] and [14], that is, we construct the solution by taking the limit of the sequence of periodic solutions. First, we consider a family of modified equations with $2l$-period ($l \in \mathbb{N}$) in each variable whose viscosity coefficient is slightly larger than $\mu > 0$, that is, $\frac{2+\delta}{2}\mu$, $\delta > 0$, so that the approximating equations satisfy the coercivity condition. We use a standard Galerkin approximation and construct a solution $u_n^{l,\delta}$. Then, for suitable cutoff functions $\chi_R \uparrow 1_{\mathbb{R}^2}$, it can be shown that the family $u_n^{R,l,\delta} = \chi_R \Pi_n u_n^{l,\delta}$, where Π_n represents the orthogonal projection onto an n-dimensional linear subspace, is uniformly bounded in the space $L^2(\Omega, L^2(0,T; \mathbf{V}(\mathbb{R}^2))$ with respect to R,l,n and δ. Finally, we take a limit of $u_n^{R,l,\delta}$ as $\delta \to 0$, $n \to \infty$ and $R \to \infty$ simultaneously and show that its limit satisfies the equation (1.1)-(1.3) in a weak sense.

So far, there are several known results about weak solutions of stochastic Navier-Stokes equations ([1], [2], [8], [11], [12], [13], [14], [15]). The papers [1] and [13] study the equation with a trace class Wiener process and a spatially homogeneous initial distribution and the existence of the spatially homogeneous weak solution in a weighted Sobolev space is proven. There are also several results about the case of the two-dimensional torus ([3], [5], [10], [15]). Especially, [15] studies the case where the equation does not satisfy the coercivity condition in a two-dimensional torus and [5] discusses the two-dimensional stochastic Euler equation in a bounded domain and periodic case.

Furthermore, [14] shows that there exists a spatially homogeneous weak solution of the equations in $\mathbb{R}^n (n \geq 2)$ with a spatially homogeneous H^1-valued initial distribution independent of the space-time white noise. In [11] and [12], the stochastic Navier-Stokes equations on $\mathbb{R}^n (n \geq 2)$ satisfying the coercivity condition are studied. However, no results are known in the case where the equation does not satisfy the coercivity condition in unbounded domain in $\mathbb{R}^n (n \geq 2)$.

This paper is organized as follows: In Section 2, we introduce notations used in this paper and our main result. Section 3 and Section 4 contain the proof of our main results.

2. NOTATIONS AND RESULTS

In this section we introduce several notations appearing later. Set $\boldsymbol{T}_l = (-l, l)^2$, $l \in \mathbb{N}$. We denote by

$$\mathbf{C}^\infty_{per}(l) = \{u \in \mathbf{C}^\infty(\mathbb{R}^2; \mathbb{R}^2) \mid u \text{ is } 2l\text{-periodic in } (x_1, x_2) \in \mathbb{R}^2\},$$

the family of smooth vector fields u having period $2l$ in each variable $(x_1, x_2) \in \mathbb{R}^2$. We also denote by $\mathbf{C}^\infty_{per,\sigma}(l)$ the subspace of divergence free vector fields u satisfying $\int_{\boldsymbol{T}_l} u dx = 0$, that is,

$$\mathbf{C}^\infty_{per,\sigma}(l) = \{u \in \mathbf{C}^\infty_{per}(l) \mid \int_{\boldsymbol{T}_l} u dx = 0,\ \text{div}\, u = 0 \text{ in } \boldsymbol{T}_l\}.$$

We also denote the following function spaces:

$$\mathbf{C}^\infty_0 = \{u \in \mathbf{C}^\infty(\mathbb{R}^2; \mathbb{R}^2) \mid \text{supp}\, u \text{ is compact}\},$$

$$\mathbf{C}^\infty_0(\Omega) = \{u \in \mathbf{C}^\infty_0 \mid \text{supp}\, u \subset \Omega\},$$

$$\mathbf{C}^\infty_{0,\sigma} = \left\{u \in \mathbf{C}^\infty_0 \mid \int_{\mathbb{R}^2} u dx = 0,\ \text{div}\, u = 0 \text{ in } \mathbb{R}^2\right\}.$$

We denote by $\mathbf{H}(l)$ the set of square integrable vector fields u on $\boldsymbol{T}_l$ which are of divergence zero and satisfy $\int_{\boldsymbol{T}_l} u dx = 0$, that is,

$$\mathbf{H}(l) = \left\{u \in \mathbf{L}^2(\boldsymbol{T}_l; \mathbb{R}^2) \mid \int_{\boldsymbol{T}_l} u dx = 0,\ \text{div}\, u = 0 \text{ in } \boldsymbol{T}_l\right\}.$$

Let $\langle u, v\rangle_l = \sum_{i=1}^{2} \int_{\boldsymbol{T}_l} u^i(x) v^i(x) dx$ be its inner product and $|u|_l = \langle u, u\rangle_l^{\frac{1}{2}}$ its norm. In addition, we set

$$\mathbf{V}(l) = \mathbf{W}^{1,2}(\boldsymbol{T}_l; \mathbb{R}^2) \cap \mathbf{H}(l),$$

with its inner product

$$\langle\langle u, v\rangle\rangle_l = \sum_{j=1}^{2} \langle \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j}\rangle_l$$

and associated norm

$$||u||_l = \langle\langle u, u\rangle\rangle_l^{\frac{1}{2}}.$$

Let us set $\mathbb{Z}_0^2 = \mathbb{Z}^2 \backslash \{0\}$ and $\mathbb{T}_l^2 = \mathbb{R}^2/2l\mathbb{Z}^2$. Let

$$\mathbf{H}_{per}(l) = \left\{ u \in \mathbf{L}^2(\mathbb{T}_l^2; \mathbb{R}^2) \mid \int_{\mathbb{T}_l^2} u dx = 0, \ \text{div}\, u = 0 \text{ in } \mathbb{R}^2 \right\},$$

be the Hilbert space with inner product

$$\langle u, v \rangle_{per(l)} = \sum_{k \in \mathbb{Z}_0^2} \hat{u}(k)\hat{v}(k)$$

and associated norm

$$|u|_{per(l)} = \Big(\sum_{k \in \mathbb{Z}_0^2} |\hat{u}(k)|^2 \Big)^{\frac{1}{2}}, \quad u, v \in \mathbf{H}_{per}(l),$$

where $\hat{u}(k)$ represents the $k = (k_1, k_2)$-th Fourier coefficient of the Fourier expansion of u. In addition, we set

$$\mathbf{V}_{per}(l) = \mathbf{W}^{1,2}(\mathbb{T}_l^2; \mathbb{R}^2) \cap \mathbf{H}_{per}(l),$$

with inner product

$$\langle\langle u, v \rangle\rangle_{per(l)} = \sum_{k \in \mathbb{Z}_0^2} (\frac{\pi}{l}|k|)^2 \hat{u}(k)\hat{v}(k)$$

and associated norm

$$||u||_{per(l)} = (\sum_{k \in \mathbb{Z}^2} (\frac{\pi}{l}|k|)^2 |\hat{u}(k)|^2)^{\frac{1}{2}},$$

for $u, v \in \mathbf{V}_{per}(l)$. Note that $|u|_{per(l)} = |u|_l$ if $u \in \mathbf{H}_{per}(l)$ and $||u||_{per(l)} = ||u||_l$ if $u \in \mathbf{V}_{per}(l)$. Similarly, let us set

$$\mathbf{H}(\mathbb{R}^2) = \left\{ u \in \mathbf{L}^2(\mathbb{R}^2; \mathbb{R}^2) \mid \ \text{div}\, u = 0 \text{ in } \mathbb{R}^2 \right\},$$

with its inner product and the norm denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively, and

$$\mathbf{V}(\mathbb{R}^2) = \mathbf{W}^{1,2}(\mathbb{R}^2; \mathbb{R}^2) \cap \mathbf{H}(\mathbb{R}^2),$$

with its inner product and the norm denoted by $\langle\langle \cdot, \cdot \rangle\rangle$ and $||\cdot||$, respectively. For an open set $\Omega \subset \mathbb{R}^2$, let us define

$$\mathbf{H}(\Omega) = \{u \in \mathbf{H}(\mathbb{R}^2) \mid \ \text{supp}\, u \subset \Omega\}, \quad \mathbf{V}(\Omega) = \{u \in \mathbf{V}(\mathbb{R}^2) \mid \ \text{supp}\, u \subset \Omega\}.$$

In addition, define

$$H_\Omega = \{u \mid \int_\Omega |u(x)|^2 dx < \infty\}, \quad V_\Omega = \{u \mid \sum_{j=1}^2 \int_\Omega |\frac{\partial u(x)}{\partial x_j}|^2 dx < \infty\},$$
$$\mathbf{H}_\Omega = \{u \in H_\Omega \mid \ \text{div}\, u = 0\}, \quad \mathbf{V}_\Omega = \{u \in V_\Omega \mid \ \text{div}\, u = 0\}.$$

We denote by H_{loc}, V_{loc} the set of vector-fields whose countable semi norms $||u||_{0,R}$, $||u||_{1,R}$ are finite for all $R \in \mathbb{N}$, respectively, that is,

$$H_{loc} = \{u \in (\mathbf{C}_0^\infty)' \mid ||u||_{0,R} < \infty \text{ for all } R \in \mathbb{N}\},$$
$$V_{loc} = \{u \in (\mathbf{C}_0^\infty)' \mid ||u||_{1,R} < \infty \text{ for all } R \in \mathbb{N}\},$$

where $||u||_{0,R}$, $||u||_{1,R}$ are defined as follows:

$$||u||_{0,R} = \int_{B_R} |u(x)|^2 dx, \quad ||u||_{1,R} = \sum_{i=1}^2 \int_{B_R} |\frac{\partial u(x)}{\partial x_i}|^2 dx,$$

where B_R is the open ball with radius $R \in \mathbb{N}$ centered at the origin. In addition, let us set

$$\mathbf{H}_{\text{loc}} = \{u \in H_{loc} |\ \text{div}\, u(x) = 0, x \in \mathbb{R}^2\},$$
$$\mathbf{V}_{\text{loc}} = \{u \in V_{loc} |\ \text{div}\, u(x) = 0, x \in \mathbb{R}^2\}.$$

Furthermore, we denote by $\mathbf{V}'$ the topological dual space of $\mathbf{V}(\mathbb{R}^2)$ and by $(u, \phi)_{-1}$ the pair of $u \in \mathbf{V}'$ and $\phi \in \mathbf{V}(\mathbb{R}^2)$. We denote by $\mathbf{V}_{\text{loc}}{}'$ the space $\mathbf{V}'$ with topology given by countable semi-norms

$$|u|_{\mathbf{V},R} := \sup\{|(u,\phi)_{-1}|\ ;\ ||\phi|| \leq 1,\ \phi \in \mathbf{C}^{\infty}_{0,\sigma}, \text{supp}\ u \subset B_R\},$$

where B_R is the open ball with radius $R \in \mathbb{N}$ at centered origin. Note that the divergence appearing in each class is understood in the distributional sense. Let $Au = -\mu\mathbb{P}\triangle u$ be the Stokes operator with domain

$$D(A) = \mathbf{V}(\mathbb{R}^2) \cap \mathbf{W}^{2,2}(\mathbb{R}^2;\mathbb{R}^2),$$

where $\mathbb{P}$ represents the Leray projection. It is well known that A is a non negative self adjoint linear operator. Furthermore, let B be defined by

$$\langle B(u,v), w\rangle = \int_{\mathbb{R}^2} (u(x)\cdot\nabla)v(x)\cdot w(x)dx, \quad u,v,w \in \mathbf{C}^{\infty}_{0,\sigma},$$

and $G : \mathbf{V}_{\text{loc}} \to \mathrm{L}_{\text{H.S}}(\mathbb{R}^2;\mathbf{H}_{\text{loc}})$ be defined by

$$Gu = -\sqrt{2\mu}\nabla u,$$

where $\mathrm{L}_{\text{H.S}}(\mathbb{R}^2;\mathbf{H}_{\text{loc}})$ denotes the space of Hilbert-Schmidt operators from $\mathbb{R}^2$ to $\mathbf{H}_{\text{loc}}$. Let us denote by $\mathbf{U} = \mathbf{W}^{k_0,2}(\mathbb{R}^2;\mathbb{R}^2) \cap \mathbf{H}(\mathbb{R}^2)$, $k_0 > 2$ the Sobolev space equipped with its norm $||u||_{k_0} = |(1-\triangle)^{\frac{k_0}{2}}u|^{\frac{1}{2}}$ and $\mathbf{U}'$ its dual space with norm $||\cdot||_{\mathbf{U}'}$. We denote by $\mathbf{U}'_{loc}$ the space $\mathbf{U}'$ with topology given by countable semi-norms:

$$||g||_{\mathbf{U}',R} := \sup_{\phi\in C^{\infty}_{0,\sigma}, \text{supp}\ \phi\subset B_R, ||\phi||_{k_0}\leq 1} |g(\phi)|,$$

for each $R \in \mathbb{N}$. By Sobolev's embedding theorem, we see that

$$\mathbf{W}^{k_0-1,2}(D;\mathbb{R}^2) \subset C_b(D;\mathbb{R}^2) \subset L^{\infty}(D;\mathbb{R}^2),$$

for any bounded domain D in $\mathbb{R}^2$. This implies that B can be uniquely extended to a $\mathbf{U}'$-valued bilinear operator on $\mathbf{H}_{\text{loc}} \times \mathbf{H}_{\text{loc}}$. Indeed,

$$\begin{aligned} |\int_{B_R} u_i \frac{\partial u_j}{\partial x_i}\phi_j dx| &= |\int_{B_R} u_i \frac{\partial \phi_j}{\partial x_i} u_j dx| \\ &\leq |u_i|_{L^2(B_R)} |\frac{\partial \phi_j}{\partial x_i}|_{L^\infty(B_R)} |u_j|_{L^2(B_R)} \\ &\leq |u_i|_{L^2(B_R)} ||\phi_j||_{k_0} |u_j|_{L^2(B_R)}, \quad u \in C^{\infty}_0, \quad \phi \in \mathbf{U}, \quad j = 1,2, \end{aligned}$$

holds. This implies that B is a $\mathbf{U}'$-valued bilinear operator on $\mathbf{H}_{\text{loc}} \times \mathbf{H}_{\text{loc}}$. In our equation, the noise is finite-dimensional and thus its covariance is trivially of finite trace, so the square root is Hilbert-Schmidt. The abstract stochastic evolution equation associated with (1.1)-(1.3) is defined as follows:

$$\left\{ \begin{array}{l} du(t) + [Au(t) + B(u(t),u(t))]dt + Gu(t)\cdot dB(t) = 0, \quad t > 0, \\ u(0) = u_0. \end{array} \right. \tag{2.1}$$

Definition 2.1. *We say $\{u(t), B(t)\}_{t\geq 0}$ is a weak solution of (2.1) if*

(1) $u(t)$ *is an adapted process on a probability space* $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$.
(2) $u \in L^2(0,T;\mathbf{V}_{loc}) \cap L^{\infty}(0,T;\mathbf{H}_{loc})$*, a.s.*

(3) $\{B(t), \mathcal{F}_t\}_{t\geq 0}$ *is a two-dimensional Brownian motion.*
(4) *For a.e.* $t \in [0,T]$ *and every* $\phi \in \mathbf{C}^\infty_{0,\sigma}$, P*-a.s., the following equality*

$$\langle u(t), \phi\rangle - \langle u_0, \phi\rangle + \int_0^t \langle u(s), A\phi\rangle ds$$
$$= \int_0^t \langle B(u(s), \phi), u(s)\rangle ds - \int_0^t (G\phi)^* u(s) \cdot dB(s),$$

holds.

Remark 2.1. *The term containing* ∇p *drops out in the weak form of the solution since* $\int \nabla p \cdot \phi dx = -\int p \, div \, \phi dx = 0$ *holds.*

Remark 2.2. *We can regard the second condition of the Definition 2.1 as*

$$u \in \bigcap_{R\in\mathbb{N}} L^2(0,T;\mathbf{V}_{B_R}) \cap L^\infty(0,T;\mathbf{H}_{B_R}), a.s.$$

Now we can formulate our main result in this paper.

Theorem 2.1. *Let* $u_0 \in \mathbf{V}(\mathbb{R}^2)$ *has compact support. Then, there exists a weak solution of (2.1).*

3. PROOF OF THEOREM 2.1

We will separate the proof into four steps.

step 1. We denote by $A_{l,\delta}$ the Stokes operator with viscosity $\frac{2+\delta}{2}\mu$, that is,

$$A_{l,\delta} u = -\frac{2+\delta}{2}\mu \mathbb{P}\triangle u,$$

with domain

$$D(A_{l,\delta}) = \mathbf{V}_{per}(l) \cap \mathbf{W}^{2,2}(\mathbb{T}^2_l; \mathbb{R}^2).$$

Note that $A_{l,\delta}$ is a strictly positive definite self-adjoint operator and has a compact resolvent. Let $0 < \lambda_1^{(l,\delta)} \leq \lambda_2^{(l,\delta)} \leq \ldots$ be the eigenvalues of $A_{l,\delta}$ and $e_1^{(l)}, e_2^{(l)}, \ldots$ the associated normalized eigenfunctions. Let us prepare a complete filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}; \{\mathcal{F}_t\}_{t\geq 0})$ on which a two-dimensional $\mathcal{F}_t$-Brownian motion $B = \{B(t)\}_{t\geq 0}$ is defined. Then, let us consider the following finite-dimensional stochastic differential equation:

(3.1)
$$\begin{cases} du_n^{l,\delta}(t) + [A_{l,\delta} u_n^{l,\delta}(t) + \Pi_n B(u_n^{l,\delta}(t), u_n^{l,\delta}(t))]dt + \Pi_n G u^{l,\delta}(t) \cdot dB(t) = 0, \quad t > 0, \\ u^{l,\delta}(0) = \Pi_n u_0^{(l)}, \end{cases}$$

where $u_0^{(l)}$ is the Fourier expansion of u_0 in $\mathbf{H}_{per}(l)$, that is, $u_0^{(l)} = \sum_k \hat{u_0}(k) e_k^{(l)}$, where $\hat{u_0}(k)$ denotes the k-th Fourier coefficient and Π_n is the orthogonal projection onto the linear subspace spanned by $(e_k^{(l)})_{|k|\leq n}$. By standard arguments (see also Lemma 4.1), we see that there exists a unique solution $u_n^{l,\delta}$ for each l, δ and n.

Let $0 \leq \chi_R \leq 1$, $R > 0$ be a $C_0^\infty(\mathbb{R})$-function which is equal to 1 in B_R, 0 outside B_{2R} and satisfies $|\chi_R'(x)| \leq c$ for some uniform constant $c > 0$. Let $u_n^{l,\delta,R} = \chi_R u_n^{l,\delta}$. Now let us obtain an a priori estimate of $\mathbf{E}^{\mathbf{P}}\{||u_n^{l,\delta,R}(t)||^2\}$. Let $\{K_m\}_{m\geq 1}$ be an increasing sequence of compact sets in $\mathbb{R}^2$ such that

$$K_1 \subset K_2 \subset \cdots \uparrow \mathbb{R}^2,$$

and assume that $K_m \subset K^i_{m+1}$, where K^i_{m+1} denotes the interior of K_{m+1}. For each K_m, choose two bounded open sets Ω_{K_m}, Ω'_{K_m} such that $K_m \subset \Omega_{K_m} \subset \bar{\Omega}_{K_m} \subset \Omega'_{K_m} \subset K^i_{m+1}$ holds. For each $R \in \mathbb{N}$ and compact set K, let us choose $l = l(R,K) \in \mathbb{N}$ such that $(-l,l)^2 \supset \Omega_K \cup B_{4R} \cup$ supp u_0 holds. Then, for such $l(R,K)$,

$$\begin{aligned} ||u_n^{l,\delta,2R}(t)||^2 =& ||\chi_{2R} u_n^{l,\delta}(t)||^2 \\ =& |\chi_{2R} u_n^{l,\delta}(t)|^2 + \sum_{j=1}^{2} |\frac{\partial}{\partial x_j}\chi_{2R} u_n^{l,\delta}(t)|^2 \\ \le& C\left(|u_n^{l,\delta}(t)|_l^2 + ||u_n^{l,\delta}(t)||_l^2\right), \end{aligned} \tag{3.2}$$

holds for some constant $C > 0$, since χ_R and $\frac{\partial \chi_R}{\partial x_j}$, $R > 0$ are bounded functions. On the other hand, since $u_n^{l,\delta}$ is a solution of (3.1), we obtain the following uniform estimate:

$$\mathbf{E}^{\mathbf{P}}\{||u_n^{l,\delta}(t)||_l^2\} \le ||u_0^{(l)}||_l^2, \tag{3.3}$$

(see Lemma 4.2). By Parseval's formula,

$$||u_0^{(l)}||_l^2 = ||u_0||_l^2. \tag{3.4}$$

As a result, for any $R \in \mathbb{N}$ and compact set $K \subset \mathbb{R}^2$, we have

$$\mathbf{E}^{\mathbf{P}}\left\{||u_n^{l(R,K),\delta,2R}(t)||^2\right\} \le ||u_0||^2,$$

which means that

$$\sup_{n,R\in\mathbb{N},\delta>0,K\subset\mathbb{R}^2 \text{ compact}} \mathbf{E}^{\mathbf{P}}\left\{||u_n^{l(R,K),\delta,2R}(t)||^2\right\} < \infty. \tag{3.5}$$

Furthermore, note that the following estimate holds (See (4.10) in Lemma 4.3):

$$\sup_{n\ge 1,\delta>0} \mathbf{E}^{\mathbf{P}}\{\sup_{t\in[0,T]} |u_n^{l,\delta}(t)|^2_{\mathbf{H}(l)}\} \le C||u_0^{(l)}||_l^2. \tag{3.6}$$

Thus, using $|\chi_{2R} u_n^{l,\delta}(t)|^2 \le |u_n^{l,\delta}(t)|^2_{\mathbf{H}(l)}$ and (3.4), we see

$$\sup_{n,R\in\mathbb{N},\delta>0,K\subset\mathbb{R}^2 \text{ compact}} \mathbf{E}^{\mathbf{P}}\left\{\sup_{t\in[0,T]} |u_n^{l(R,K),\delta,2R}(t)|^2\right\} < \infty. \tag{3.7}$$

step 2. The following lemma is essential for tightness for a family of probability laws related to our problems. Set

$$\mathbb{W}_T = C([0,T];\mathbf{U}'_{loc}) \cap L^2(0,T;\mathbf{H}_{loc}) \cap L^2_w(0,T;\mathbf{V}(\mathbb{R}^2)) \cap C([0,T];\mathbf{H}_\sigma),$$

where $L^2_w(0,T;\mathbf{V}(\mathbb{R}^2))$ is the space $L^2(0,T;\mathbf{V}(\mathbb{R}^2))$ equipped with its weak topology and $\mathbf{H}_\sigma$ represents the $\mathbf{H}_{loc}$ endowed with its weak topology. Let τ be the corresponding supremum norm on $\mathbb{W}_T$ and $\mathcal{B}$ be its topological σ-field.

Furthermore, for the canonical process $X(t) = X(t,w) = w(t)$, $w \in \mathbb{W}_T$, we set $\mathcal{B}_t := \sigma(X(s),\ s \le t)$ and by standard argument we can assume that $(\mathcal{B}_t)_{t\ge 0}$ satisfies the usual condition, that is, it is right-continuous and contains all $\mathbf{P}$-null set. Then, the following Lemma holds.

Lemma 3.1. *(See Lemma 2.7 in* [12]*). A set $K \subset \mathbb{W}_T$ is τ-relatively compact if*

1. $\displaystyle\sup_{u\in K} \int_0^T ||u(t)||^2 dt < \infty,$
2. $\displaystyle\sup_{u\in K} \sup_{t,s\in[0,T],|t-s|<\delta} ||u(t)-u(s)||_{\mathbf{U}'} \to 0, \quad \delta \to 0,$

$$3. \quad \sup_{u\in K}\sup_{t\in[0,T]}|u(t)|^2<\infty,$$

hold.

Later we will use the above Lemma. Let us set $\delta=\delta_k\equiv\frac{1}{k}$ and $n=k$, $k\in\mathbb{N}$. For each compact set $K\subset\mathbb{R}^2$, let us choose $R=R_k$, $l=l_k$ and $\delta=\delta_k$ such that $(-l_k,l_k)\supset B_{4R_k}\cup K\cup \text{supp } u_0$, $k=1,2,\cdots$ and (3.5) and (3.7) hold for any $R=R_k$, $l=l_k$ and $\delta=\delta_k$. Let us set P^k the probability law of $u_k^{\delta_k}$ on $\mathbb{W}_T$. We denote by $\mathcal{D}$ the family of functions Ψ defined on $\mathbf{H}(\mathbb{R}^2)$ whose form are of

$$\Psi(u)=\psi(\langle u,\phi_1\rangle,\cdots,\langle u,\phi_n\rangle),$$

for some $n\in\mathbb{N}$, where $\psi\in C_0^2(\mathbb{R}^n)$ and for all $\phi_i\in\mathbf{C}^\infty_{\sigma,0}$, $i=1,\cdots,n$. Let us define a linear operator $\mathcal{L}_k$ on $\mathcal{D}$, $k=1,2,\cdots$, as

$$\begin{aligned}\mathcal{L}_k\Psi(u)=&\frac{1}{2}\sum_{i,j=1}^n\frac{\partial^2\psi}{\partial\alpha_i\partial\alpha_j}(\langle u,\phi_1\rangle,\cdots,\langle u,\phi_n\rangle)\left\{(-\Pi_kGu)^*\phi_i)\cdot((-\Pi_kGu)^*\phi_j)^*\right\}\\&+\sum_{i=1}^n\frac{\partial\psi}{\partial\alpha_i}(\langle u,\phi_1\rangle,\cdots,\langle u,\phi_n\rangle)\left\{\frac{2+\delta_k}{2}\mu\langle u,\triangle\phi_i\rangle+\langle\Pi_k(u\cdot\nabla)\Pi_k\phi_i,u\rangle\right\},\end{aligned}$$

for $\Psi\in\mathcal{D}$. In these settings, we formulate the martingale problem associated to our equations.

Definition 3.1. *We say that a probability measure P defined on $(\mathbb{W}_T,\mathcal{B})$ is a solution of $(\mathcal{L}_k,\mathcal{D})$-martingale problem starting at $u\in\mathbf{H}(\mathbb{R}^2)$ if*

(1) $P(x(0)=u)=1$,

(2) $\Psi(x)(t)-\Psi(x)(0)-\int_0^t\mathcal{L}_k\Psi(x)(s)ds,\quad t\in[0,T]$,

is a $\mathcal{B}_t$-local martingale under P.

Note that P^k is a solution of $(\mathcal{L}_k,\mathcal{D})$-martingale problem starting at Π_ku_0. We shall prove the following lemmas:

Lemma 3.2. *The family of probability measures $(P^k)_{k=1,2,\ldots}$ is relatively compact in $\mathbb{W}_T$.*

Suppose that Lemma 3.2 is proven, we denote by $\bar{P}$ its limit. We define a linear operator $\mathcal{L}$ on $\mathcal{D}$ as

$$\begin{aligned}\mathcal{L}\Psi(u)=&\frac{1}{2}\sum_{i,j=1}^n\frac{\partial^2\psi}{\partial\alpha_i\partial\alpha_j}(\langle u,\phi_1\rangle,\cdots,\langle u,\phi_n\rangle)\left\{(-Gu)^*\phi_i)\cdot((-Gu)^*\phi_j)^*\right\}\\&+\sum_{i=1}^n\frac{\partial\psi}{\partial\alpha_i}(\langle u,\phi_1\rangle,\cdots,\langle u,\phi_n\rangle)\left\{\mu\langle u,\triangle\phi_i\rangle+\langle(u\cdot\nabla)\phi_i,u\rangle\right\}.\end{aligned}$$

We shall prove following lemma.

Lemma 3.3. *The probability measure $\bar{P}$ is a solution of $(\mathcal{L},\mathcal{D})$-martingale problem starting at u_0.*

The following Lemma is used for tightness criterion of the set of probability law of $(u_k^{\delta_k})$.

Lemma 3.4. *Let (X_n) be a sequence of continuous $\mathbf{V}'$-valued random variables on $(\Omega,\mathcal{F},P)$ satisfying the following conditions:*

1. $\sup_{n\geq1}\mathbf{E}^P\{\int_0^T||X_n(t)||^2dt\}<\infty.$
2. *For any $\epsilon,\epsilon'>0$, there exists $\delta>0$ such that for any stopping times $(T_n)_{n\geq1}$,*

$0 \le T_n \le T$, $\sup_{n\ge 1}\sup_{t\in[0,\delta)} P(||X_n(T_n+t)-X_n(T_n)||_{\mathbf{U}'} > \epsilon') < \epsilon$ *holds.*

3. $\sup_{n\ge1} \mathbf{E}^P\{\sup_{t\in[0,T]} |X_n(t)|^2\} < \infty.$

Let P_n be the law of X_n on $\mathbb{W}_T$. Then, $(P_n)_{n\ge1}$ is tight in $\mathbb{W}_T$.

Proof. The assertion is proven by similar methods of Lemma 3.12 in [15] hence we omit the proof. ∎

We shall prove Lemma 3.2:

Proof. By (3.5) and (3.7), we see that $(u_k^{l_k,\delta_k,R_k})$ satisfies the conditions 1 and 3 of Lemma 3.4. Let us set

$$\begin{aligned} u_n^{l,\delta}(t) =&\Pi_n u_0^{(l)} - \int_0^t A_\delta u_n^{l,\delta}(s)ds \\ &- \int_0^t \Pi_n B(u_n^{l,\delta}(s), u_n^{l,\delta}(s))ds - \int_0^t \Pi_n G u_n^{l,\delta}(s)\, dB_s \\ =&J_0^{l,n} + J_1^{n,l,\delta}(t) + J_2^{n,l,\delta}(t) + J_3^{n,l,\delta}(t). \end{aligned}$$

Let $(T_k)_{k\ge1}$, $0\le T_k \le T$ be a sequence of stopping times. Then,

$$\sup_{k\ge1} \mathbf{E}^P\left\{||\chi_{R_k} J_1^{k,\delta_k,l_k}(t+T_k) - \chi_{R_k} J_1^{k,\delta_k,l_k}(T_k)||_{\mathbf{U}'}\right\} < C_1 t^{\frac{1}{2}}, \tag{3.8}$$

$$\sup_{k\ge1} \mathbf{E}^P\left\{||\chi_{R_k} J_2^{k,\delta_k,l_k}(t+T_k) - \chi_{R_k} J_2^{k,\delta_k,l_k}(T_k)||_{\mathbf{U}'}\right\} < C_2 t, \tag{3.9}$$

$$\sup_{k\ge1} \mathbf{E}^P\left\{||\chi_{R_k} J_3^{k,\delta_k,l_k}(t+T_k) - \chi_{R_k} J_3^{k,\delta_k,l_k}(T_k)||_{\mathbf{U}'}\right\} < C_3 t^{\frac{1}{2}}, \tag{3.10}$$

hold for some constant C_i, $i=1,2,3$ independent of k. Indeed, by noting that $(-l_k,l_k) \supset B_{4R_k} \cup \text{supp}\, u_0$, $k=1,2,\cdots$ holds, we obtain

$$\begin{aligned} &||\chi_{R_k} J_1^{k,\delta_k,l_k}(t+T_k) - \chi_{R_k} J_1^{k,\delta_k,l_k}(T_k)||_{\mathbf{U}'} \\ \le&\frac{2+\delta_k}{2}\mu c_1 \int_{T_k}^{t+T_k} ||u_k^{l_k,\delta_k,2R_k}(s)||ds \le \frac{2+\delta_k}{2}\mu c_1 (\int_0^T ||u_k^{l_k,\delta_k,2R_k}(s)||^2 ds)^{\frac{1}{2}} t^{\frac{1}{2}}. \end{aligned}$$

Thus, by (3.5), we obtain (3.8). As for J_2^{k,δ_k,l_k}, we have

$$\begin{aligned} &||\chi_{R_k} J_2^{k,\delta_k,l_k}(t+T_k) - \chi_{R_k} J_2^{k,\delta_k,l_k}(T_k)||_{\mathbf{U}'} \\ \le& \int_{T_k}^{t+T_k} ||\chi_{R_k}(u_k^{l_k,\delta_k}(s)\cdot\nabla)u_k^{l_k,\delta_k}(s)||_{\mathbf{U}'} ds \le c_2 \int_{T_k}^{t+T_k} |u_k^{l_k,\delta_k,2R_k}(s)|^2_{\mathbf{H}(l_k)} ds \\ \le& c_2 \left(\sup_{t\in[0,T]} |u_k^{l_k,\delta_k,2R_k}(t)|^2_{\mathbf{H}(l_k)}\right) t = c_2 \left(\sup_{t\in[0,T]} |u_k^{l_k,\delta_k,2R_k}(t)|^2\right) t. \end{aligned}$$

Thus, by (3.7), we obtain (3.9). Concerning J_3^{k,δ_k,l_k}, we see that

$$\begin{aligned} &\mathbf{E}||\chi_{R_k} J_3^{k,\delta_k,l_k}(t+T_k) - \chi_{R_k} J_3^{k,\delta_k,l_k}(T_k)||_{\mathbf{U}'} \\ \le& c_3 \mathbf{E}\left\{ \left(\int_{T_k}^{t+T_k} ||\nabla u_k^{l_k,\delta_k,2R_k}(s)||^2_{L_{\mathrm{H.S}}(\mathbb{R}^2;\mathbf{V}'(l_k))} ds \right)^{\frac{1}{2}} \right\} \end{aligned}$$

$$\leq c_3 \mathbf{E}\left\{\left(\int_{T_k}^{t+T_k} |u_k^{l_k,\delta_k,2R_k}(s)|^2_{\mathbf{H}(l_k)} ds\right)^{\frac{1}{2}}\right\}$$
$$\leq c_3 \left(\mathbf{E}\left\{\sup_{t\in[0,T]} |u_k^{l_k,\delta_k,2R_k}(s)|^2_{\mathbf{H}(l_k)}\right\}\right)^{\frac{1}{2}} t^{\frac{1}{2}} \leq c_3 \left(\mathbf{E}\left\{\sup_{t\in[0,T]} |u_k^{l_k,\delta_k,2R_k}(s)|^2\right\}\right)^{\frac{1}{2}} t^{\frac{1}{2}}.$$

We obtain (3.10) from (3.7). From the estimates (3.8) - (3.10) and Chebyshev's inequality, we obtain that $(u_k^{\delta_k})_{k\geq 1}$ satisfies the condition 2 of Lemma 3.4. Thus, the conclusion follows from Lemma 3.4. ∎

We shall prove Lemma 3.3.

Proof. The assertion is shown similarly to Theorem 2.10 in [12] or Lemma 3.5 in [15]. By Lemma 3.2, there exists a convergent subsequence, which is denoted by (k) again, such that $\lim_{k\to} P^k = \bar{P}$ weakly in $\mathbb{W}_T$. From this, it follows that the condition 1 of Definition 3.1 holds for $\bar{P}$. As for the condition 2, since P^k is a solution of $(\mathcal{L}_k, \mathcal{D})$-martingale problem starting at $\Pi_k u_0$,

$$\mathbf{E}^{P^k}\left\{\left(\Psi(x)(t) - \Psi(x)(s) - \int_s^t \mathcal{L}_k\Psi(x)(u)du\right)\Theta(x)\right\} = 0, \tag{3.11}$$

holds for any $0 \leq s < t \leq T$ and any $\mathcal{B}_s$-measurable bounded continuous function Θ. Let $x_k \to x$ in $\mathbb{W}_T$. Then, we have

$$\sup_{k\geq 1}\left\{\left(\sup_{t\in[0,T]}\left(|x_k(t)|^2 + |x(t)|^2\right)\right) + \int_0^T \left(|x_k(t)|^2 + |x(t)|^2\right) dt\right\} < \infty. \tag{3.12}$$

In addition,

$$\lim_{k\to\infty}\left(||x_k - x||_{L^2(0,T;\mathbf{H}_{\mathbf{B_R}})} + \sup_{t\in[0,T]} |\langle x_k(t) - x(t), \phi\rangle|\right) = 0, \tag{3.13}$$

for any $\phi \in \mathbf{C}_{\sigma,0}$ and $R > 0$. Set

$$G_k(t,x) \equiv \Psi(x)(t) - \Psi(x)(0) - \int_0^t \mathcal{L}_k\Psi(x)(u)du$$
$$= f_1(t,x) + f_2^k(t,x) + f_3^k(t,x) + f_4^k(t,x),$$

where

$$f_1(t,x) = \psi(\langle x(t),\phi\rangle_1^n) - \psi(\langle x(s),\phi\rangle_1^n),$$
$$f_2^k(t,x) = -\sum_{i,j=1}^n \frac{1}{2}\int_s^t \frac{\partial^2\psi}{\partial\alpha_i\partial\alpha_j}(\langle x(u),\phi\rangle_1^n)\Big((-\Pi_k Gx(u))^*\phi_i \cdot ((-\Pi_k Gx(u))^*\phi_j)^*\Big)du,$$
$$f_3^k(t,x) = -\frac{2+\delta_k}{2}\sum_{i=1}^n \int_s^t \frac{\partial\psi}{\partial\alpha_i}(\langle x(u),\phi\rangle_1^n)\mu\langle x(u),\triangle\phi_i\rangle du,$$
$$f_4^k(t,x) = -\sum_{i=1}^n \int_s^t \frac{\partial\psi}{\partial\alpha_i}(\langle x(u),\phi\rangle_1^n)\langle\Pi_k(x(u)\cdot\nabla)\Pi_k\phi_i, x(u)\rangle du,$$

and $\langle x(u),\phi\rangle_1^n = (\langle x(u),\phi_1\rangle,\cdots,\langle x(u),\phi_n\rangle)$. Then $\{G_k(t,x_k)\}_{k\geq 1}$ is equicontinuous in t. Indeed, ϕ is a smooth function with a compact support and by (3.13), $f_1(t,x_k)$ is equicontinuous. From

$$(-Gx_k(u))^*\phi_i \cdot ((-Gx_k(u))^*\phi_j)^* \leq C_\phi |x_k(u)|^2,$$

$$|\langle \Pi_k (x_k(u) \cdot \nabla)\phi_i, x_k(u)\rangle| \leq C_\phi |x_k(u)|^2,$$

for some $C_\phi > 0$, it follows that $|f_2^k(t, x_k) - f_2^k(s, x_k)|$, $|f_3^k(t, x_k) - f_3^k(s, x_k)|$ and $|f_4^k(t, x_k) - f_4^k(s, x_k)|$ are bounded from above by

$$C_{\phi,\psi} \sup_{k\geq 1} \left(\sup_{u\in[0,T]} |x_k(u)|^2 \right) (t - s),$$

for some constant $C_{\phi,\psi} > 0$. The equicontinuity of $\{G_k(x_k, t)\}_{k\geq 1}$ is shown by (3.12). Set

$$\begin{aligned} G(t, x) &\equiv \Psi(x)(t) - \Psi(x)(0) - \int_0^t \mathcal{L}\Psi(x)(u)du, \\ &= f_1(t, x) + f_2(t, x) + f_3(t, x) + f_4(t, x), \end{aligned}$$

where

$$\begin{aligned} f_2(t, x) &= -\sum_{i,j=1}^n \frac{1}{2} \int_s^t \frac{\partial^2 \psi}{\partial \alpha_i \partial \alpha_j} (\langle x(u), \phi\rangle_1^n) \Big((-Gx(u))^* \phi_i \cdot ((-Gx(u))^* \phi_j)^* \Big) du, \\ f_3(t, x) &= -\sum_{i=1}^n \int_s^t \frac{\partial \psi}{\partial \alpha_i} (\langle x(u), \phi\rangle_1^n) \mu \langle x(u), \triangle \phi_i\rangle du, \\ f_4(t, x) &= -\sum_{i=1}^n \int_s^t \frac{\partial \psi}{\partial \alpha_i} (\langle x(u), \phi\rangle_1^n) \langle (x(u) \cdot \nabla)\phi_i, x(u)\rangle du. \end{aligned}$$

We will show that

$$\lim_{k\to\infty} G_k(t, x_k) = G(t, x), \tag{3.14}$$

for each t. Since ψ has a compact support and by (3.13), we have $\lim_{k\to\infty} |f_1(x_k, t) - f_1(x, t)| = 0$. Concerning f_2^k, $|f_2^k(t, x_k) - f_2(t, x)|$ can be rewritten as

$$\begin{aligned} &\Big| \sum_{i,j=1}^n \frac{1}{2} \int_0^t \frac{\partial^2 \psi}{\partial \alpha_i \partial \alpha_j} (\langle x_k(u), \phi\rangle_1^n) \\ &(-\Pi_k G x_k(u))^* \phi_i \cdot \Big(((-\Pi_k G x_k(u))^* \phi_j)^* - ((-Gx(u))^* \phi_j)^* \Big) du \\ &+ \frac{1}{2} \int_0^t \left(\frac{\partial^2 \psi}{\partial \alpha_i \partial \alpha_j} (\langle x_k(u), \phi\rangle_1^n) - \frac{\partial^2 \psi}{\partial \alpha_i \partial \alpha_j} (\langle x(u), \phi\rangle_1^n) \right) \\ &\Big((-Gx(u))^* \phi_i \cdot ((-Gx(u))^* \phi_j)^* \Big) du \\ &+ \frac{1}{2} \int_0^t \frac{\partial^2 \psi}{\partial \alpha_i \partial \alpha_j} (\langle x_k(u), \phi\rangle_1^n) \Big((-\Pi_k G x_k(u))^* \phi_i - (-Gx(u))^* \phi_i \Big) ((-Gx(u))^* \phi_j)^* du \Big|, \end{aligned}$$

which is bounded from above by

$$\begin{aligned} &\sum_{i,j=1}^n \Big| \frac{1}{2} \int_0^t \frac{\partial^2 \psi}{\partial \alpha_i \partial \alpha_j} (\langle x_k(u), \phi\rangle_1^n) \\ &(-\Pi_k G x_k(u))^* \phi_i \cdot \Big(((-\Pi_k G x_k(u))^* \phi_j)^* - \big((-Gx(u))^* \phi_j\big)^* \Big) du \Big| \\ &+ \sum_{i,j=1}^n \Big| \frac{1}{2} \int_0^t \left(\frac{\partial^2 \psi}{\partial \alpha_i \partial \alpha_j} (\langle x_k(u), \phi\rangle_1^n) - \frac{\partial^2 \psi}{\partial \alpha_i \partial \alpha_j} (\langle x(u), \phi\rangle_1^n) \right) \end{aligned}$$

$$\Big((-Gx(u))^*\phi_i\cdot((-Gx(u))^*\phi_j)^*\Big)du\Big| + \sum_{i,j=1}^{n}\Big|\frac{1}{2}\int_0^t\frac{\partial^2\psi}{\partial\alpha_i\partial\alpha_j}(\langle x_k(u),\phi\rangle_1^n)$$
$$\Big((-\Pi_kGx_k(u))^*\phi_i-(-Gx(u))^*\phi_i\Big)((-Gx(u))^*\phi_j)^*du\Big|$$
$$= I + II + III. \tag{3.15}$$

Then,

$$\begin{aligned}I \le & C_{\phi,\psi}(\sum_{i,j=1}^{n}\int_0^t\Big|\Big(((-\Pi_kGx_k(u))^*\phi_j)^*-((-Gx_k(u))^*\phi_j)^*\Big)(-\Pi_kGx_k(u))^*\phi_i\Big|du\\ &+\sum_{i,j=1}^{n}\int_0^t\Big|(-\Pi_kGx_k(u))^*\phi_i\cdot\Big(((-Gx_k(u))^*\phi_j)^*-((-Gx(u))^*\phi_j)^*\Big)\Big|du.\end{aligned}$$

The right hand side is bounded from above by

$$C_{\phi,\psi}\sup_{l\ge1}||x_l||_{L^2(0,T;\mathbf{H}(\mathbb{R}^2))}\Big(||x_k-x||_{L^2(0,T;\mathbf{H}_{\mathbf{B_R}})}+\sum_{j=1}^{2}|\partial_j\phi-\partial_j\Pi_k\phi|^2\Big),$$

for a large enough $R>0$, where $C_{\phi,\psi}>0$ is some constant. Similarly, we have

$$III \le C_{\phi,\psi}||x||_{L^2(0,T;\mathbf{H}(\mathbb{R}^2))}\Big(||x_k-x||_{L^2(0,T;\mathbf{H}_{\mathbf{B_R}})}+\sum_{j=1}^{2}|\partial_j\phi-\partial_j\Pi_k\phi|^2\Big),$$

for a large $R>0$. On the other hand,

$$II \le C_{\phi,\psi}\sum_{i,j=1}^{n}\int_0^T\left|\frac{\partial^2\psi}{\partial\alpha_i\partial\alpha_j}(\langle x_k(u),\phi\rangle_1^n)-\frac{\partial^2\psi}{\partial\alpha_i\partial\alpha_j}(\langle x(u),\phi\rangle_1^n)\right|du\Big(\sup_{u\in[0,T]}|x(u)|^2\Big),$$

for some constant $C_{\phi,\psi}>0$. Here $\frac{\partial^2\psi}{\partial x_i\partial x_j}$, $i,j=1,\cdots,n$ are bounded continuous. By (3.12) and (3.13), it follows that I, II and III converge to 0 as $k\to\infty$, hence, $\lim_{k\to\infty}|f_2^k(t,x_k)-f_2(t,x)|=0$. We will check f_3^k. Indeed,

$$\begin{aligned}&|f_3^k(t,x_k)-f_3(t,x)|\\ =&\Big|-\delta_k\mu\sum_{i=1}^{n}\int_0^t\frac{\partial\psi}{\partial\alpha_i}(\langle x_k(u),\phi\rangle_1^n)\langle x_k(u),\triangle\phi_i\rangle du\\ &+\mu\sum_{i=1}^{n}\int_0^t\frac{\partial\psi}{\partial\alpha_i}(\langle x(u),\phi\rangle_1^n)\langle x(u),\triangle\phi_i\rangle-\frac{\partial\psi}{\partial\alpha_i}(\langle x_k(u),\phi\rangle_1^n)\langle x_k(u),\triangle\phi_i\rangle du\Big|.\end{aligned}$$

The right hand side is bounded from above by

$$\begin{aligned}&\delta_kC_{\psi,\phi}\sup_{k\ge1}\left(||x_k||_{L^2(0,T;\mathbf{H}(\mathbb{R}^2))}\right)\\ &+\mu\sum_{i=1}^{n}\Big|\int_0^t\frac{\partial\psi}{\partial\alpha_i}(\langle x(u),\phi\rangle_1^n)\langle x(u)-x_k(u),\triangle\phi_i\rangle\\ &+\left(\frac{\partial\psi}{\partial\alpha_i}(\langle x(u),\phi\rangle_1^n)-\frac{\partial\psi}{\partial\alpha_i}(\langle x_k(u),\phi\rangle_1^n)\right)\langle x_k(u),\triangle\phi_i\rangle du\Big|.\end{aligned}$$

For a large $R>0$, this is bounded from above by

$$\delta_kC_{\psi,\phi}\sup_{k\ge1}||x_k||_{L^2(0,T;\mathbf{H}(\mathbb{R}^2))}+C_{\psi,\phi}||x-x_k||_{L^2(0,T;\mathbf{H}_{B_R})}$$

(3.16)

$$+ T^{\frac{1}{2}} C_\phi \sum_{i=1}^{n} \left(\int_0^t \left| \frac{\partial \psi}{\partial \alpha_i}(\langle x(u), \phi \rangle_1^n) - \frac{\partial \psi}{\partial \alpha_i}(\langle x_k(u), \phi \rangle_1^n) \right|^2 du \right)^{\frac{1}{2}} \sup_{k \geq 1} ||x_k||_{L^2(0,T;\mathbf{H}(\mathbb{R}^2))}.$$

Here $\frac{\partial \psi}{\partial x_i}$ is bounded continuous and $\delta_k \to 0$. From (3.12) and (3.13), we obtain that $|f_3^k(t, x_k) - f_3(t, x)| \to 0$ as $k \to \infty$. As for f_4^k,

$$\Big| \sum_{i=1}^{n} \int_0^t \frac{\partial \psi}{\partial \alpha_i}(\langle x_k(u), \phi \rangle_1^n) \langle \Pi_k (x_k(u) \cdot \nabla) \phi_i, x_k(u) \rangle - \frac{\partial \psi}{\partial \alpha_i}(\langle x(u), \phi \rangle_1^n) \langle (x(u) \cdot \nabla) \phi_i, x(u) \rangle du \Big|$$

is equal to

$$\begin{aligned} \Big| \sum_{i=1}^{n} \int_0^t & \frac{\partial \psi}{\partial \alpha_i}(\langle x_k(u), \phi \rangle_1^n) \Big(\langle \Pi_k (x_k(u) \cdot \nabla) \Pi_k \phi_i, x_k(u) \rangle - \langle \Pi_k (x_k(u) \cdot \nabla) \phi_i, x_k(u) \rangle \Big) \\ &+ \frac{\partial \psi}{\partial \alpha_i}(\langle x_k(u), \phi \rangle_1^n) \left(\langle \Pi_k (x_k(u) \cdot \nabla) \phi_i, x_k(u) \rangle - \langle (x(u) \cdot \nabla) \phi_i, x(u) \rangle \right) \\ &+ \left(\frac{\partial \psi}{\partial \alpha_i}(\langle x_k(u), \phi \rangle_1^n) - \frac{\partial \psi}{\partial \alpha_i}(\langle x(u), \phi \rangle_1^n) \right) \langle (x(u) \cdot \nabla) \phi_i, x(u) \rangle du \Big|. \end{aligned}$$

And this is bounded from above by

$$\begin{aligned} & \sum_{i=1}^{n} \left| \int_0^t \frac{\partial \psi}{\partial \alpha_i}(\langle x_k(u), \phi \rangle_1^n) \langle \Pi_k (x_k(u) \cdot \nabla)(\Pi_k \phi_i - \phi_i), x_k(u) \rangle du \right| \\ &+ \sum_{i=1}^{n} \left| \int_0^t \frac{\partial \psi}{\partial \alpha_i}(\langle x_k(u), \phi \rangle_1^n) \langle \Pi_k ((x_k(u) - x(u)) \cdot \nabla) \phi_i, x_k(u) \rangle du \right| \\ &+ \sum_{i=1}^{n} \left| \int_0^t \frac{\partial \psi}{\partial \alpha_i}(\langle x_k(u), \phi \rangle_1^n) \langle ((\Pi_k x(u) - x(u)) \cdot \nabla) \phi_i, x_k(u) \rangle du \right| \\ &+ \sum_{i=1}^{n} \left| \int_0^t \frac{\partial \psi}{\partial \alpha_i}(\langle x_k(u), \phi \rangle_1^n) \langle ((x(u) \cdot \nabla) \phi_i, x_k(u) - x(u) \rangle du \right| \\ &+ C_\phi \sum_{i=1}^{n} \int_0^t \left| \frac{\partial \psi}{\partial \alpha_i}(\langle x_k(u), \phi \rangle_1^n) - \frac{\partial \psi}{\partial \alpha_i}(\langle x(u), \phi \rangle_1^n) \right| du \left(\sup_{u \in [0,T]} |x(u)|^2 \right) \\ =& I + II + III + IV + V. \end{aligned}$$

Note that $\lim_{k \to \infty} \Pi_k \phi(u) = \phi(u)$ for all u. Since ϕ is a C^∞-vector fields with compact support and from (3.12), it is easy to see $\lim_{k \to \infty} I = 0$. By proceeding similarly to f_3^k, we have $\lim_{k \to \infty} V = 0$ by (3.12) and (3.13). It is easy to check that $\lim_{k \to \infty} IV = 0$. In addition, we have

$$II \leq C_{\phi,\psi} \sup_{k \geq 1} \left(\sup_{u \in [0,T]} |x_k(u)|^2 \right) ||x_k - x||_{L^2(0,T;\mathbf{H}_{B_R})},$$

Thus, $\lim_{k \to \infty} II = 0$ follows from (3.12) and (3.13). Similarly, we have

$$III \leq C_{\phi,\psi} \left(\int_0^T |\Pi_k x(u) - x(u)|^2_{\mathbf{H}_{B_R}} du \right)^{\frac{1}{2}} \sup_{l \geq 1} \sup_{u \in [0,T]} |x_l(u)|^2.$$

We know that $\lim_{k\to\infty} |\Pi_k x(u) - x(u)|_{\mathbf{H}(B_R)} = 0$, a.e-$u$. From this and (3.12), we have $\lim_{k\to\infty} II = 0$. Thus $\lim_{k\to\infty} |f_4^k(t, x_k) - f_4(t, x)| = 0$. As a result, we obtain (3.14). From equicontinuity, the convergence in (3.14) also holds uniformly in t:

$$\lim_{k\to\infty} \sup_{t\in[0,T]} |G_k(t, x_k) - G(t, x)| = 0. \tag{3.17}$$

Let $K \subset \mathbb{W}_T$ be a compact set. Then, we obtain

$$\lim_{k\to\infty} \sup_{t\in[0,T],x\in K} |G_k(t, x) - G(t, x)| = 0. \tag{3.18}$$

By Prohorov's theorem for the relative compactness of $(P^k)_{k\geq 1}$ in $\mathbb{W}_T$, for each $\eta > 0$,

$$\lim_{k\to\infty} P^k \left(\sup_{t\in[0,T]} |G_k(t) - G(t)| > \eta \right) = 0. \tag{3.19}$$

For $M > 0$, let us set $\tau_M = \inf\{t > 0; |G(t)| > M\}$. Set $\eta = 1$. Let us define $\tau^k = \inf\{t > 0; |G_k(t) - G(t)| > \eta\}$ and $\tau_{M,k} = \min\{\tau_M, \tau^k\}$. Then $\lim_{k\to\infty} P^k(\tau^k < T) = 0$ follows from (3.19). In addition, we see the following uniform boundedness:

$$\sup_{k\geq 1} \sup_{t\in[0,T]} |G_k(t \wedge \tau_{M,k})| \leq M + \eta. \tag{3.20}$$

Let τ_M be left-continuous, $\bar{P}$-a.s., that is, $\bar{P}(\tau_M = \tau_{M-}) = 1$. The function $x \mapsto G(t \wedge \tau_M(x), x)$ is continuous on the set of $\tau_M(x) = \tau_{M-}(x)$. Indeed, this follows from (3.17). From (3.18), (3.20) and $\tau_{M,k} \to \tau_M$, we obtain

$$\begin{aligned} 0 &= \lim_{k\to\infty} \mathbf{E}^{P^k} \left\{ \left(G_k(t \wedge \tau_{M,k}) - G_k(s \wedge \tau_{M,k})\right) \Theta \right\} \\ &= \mathbf{E}^{\bar{P}} \left\{ \left(G(t \wedge \tau_M) - G(s \wedge \tau_M)\right) \Theta \right\}. \end{aligned} \tag{3.21}$$

This shows that

$$\Psi(x)(t) - \Psi(x)(0) - \int_0^t \mathcal{L}\Psi(x)(u)du,$$

is a local martingale under $\bar{P}$. The proof is complete. ∎

By Lemma 3.3, we see that

$$\begin{aligned} M^\phi(t, x) &\equiv \langle x(t), \phi\rangle - \langle u_0, \phi\rangle \\ &\quad - \mu \int_0^t \langle x(s), \Delta\phi\rangle ds - \int_0^t \langle (x(s) \cdot \nabla)\phi, x(s)\rangle ds, \end{aligned} \tag{3.22}$$

and

$$M^\phi(t, x)^2 - \int_0^t (-Gx(u))^*\phi \cdot ((-Gx(u))^*\phi)^* du,$$

are local martingales. Namely, M^ϕ is a local martingale whose quadratic variation is given by

$$\langle\langle M^\phi, M^\phi\rangle\rangle(t) = \int_0^t (-Gx(u))^*\phi \cdot ((-Gx(u))^*\phi)^* du.$$

By (3.5) and (3.7), we obtain

$$\sup_{k\geq 1} \mathbf{E}^{P^k} \left\{ \sup_{t\in[0,T]} |x(t)|^2_{\mathbf{H}_{B_R}} \right\} < \infty, \tag{3.23}$$

$$\sup_{k\geq 1}\mathbf{E}^{P^k}\left\{\int_0^T ||x(t)||^2_{\mathbf{V}_{B_R}}dt\right\}<\infty, \tag{3.24}$$

for a large $R>0$. By lower semicontinuity of

$$x\mapsto \sup_{t\in[0,T]}|x(t)|^2_{\mathbf{H}_{B_R}},\quad x\mapsto \int_0^T ||x(t)||^2_{\mathbf{V}_{B_R}}dt,$$

we obtain

$$\mathbf{E}^{\bar{P}}\left\{\sup_{t\in[0,T]}|x(t)|^2_{\mathbf{H}_{B_R}}+\int_0^T ||x(t)||^2_{\mathbf{V}_{B_R}}dt\right\}<\infty, \tag{3.25}$$

from (3.23) and (3.24). From this, we have

$$\mathbf{E}^{\bar{P}}\left\{\int_0^t |(-Gx(u))^*\phi\cdot((-Gx(u))^*\phi)^*|du\right\}<\infty,$$

for each $t\geq 0$. Thus, M is a martingale. By applying the representation theorem of martingale (see e.g. [7], Theorem 8.2), there exist a probability space $(\Omega',\mathcal{F}',\mathbb{P}')$ with a filtration $\{\mathcal{F}'_t\}_{t\geq 0}$ and a two-dimensional $\mathcal{F}''_t=\mathcal{B}_t\times\mathcal{F}'_t$-Brownian motion $\widetilde{B}$ defined on $(\Omega''=\mathbb{W}_T\times\Omega',\mathcal{F}''=\mathcal{B}\times\mathcal{F}',\mathbf{P}''=\bar{P}\times\mathbb{P}')$ such that for any $\phi\in\mathbf{C}^\infty_{\sigma,0}$ and $t\in[0,T]$, we have that $\mathbf{P}''$-a.s.,

$$M^\phi(t,x,\omega')=\int_0^t(-Gx(u,\omega'))^*\phi\, d\widetilde{B}(u,x,\omega'), \tag{3.26}$$

where $M^\phi(t,x,\omega')=M^\phi(t,x)$ and $x(t,\omega')=x(t)$. Set $X(t,x,\omega')=x(t,\omega')$. Then from (3.22) and (3.26), for each ϕ, with probability one,

$$\begin{aligned}&\langle X(t),\phi\rangle-\langle u_0,\phi\rangle-\mu\int_0^t\langle X(s),\Delta\phi\rangle ds\\ &-\int_0^t\langle (X(s)\cdot\nabla)\phi, X(s)\rangle ds=\int_0^t(-GX(s))^*\phi\, d\widetilde{B}(s),\end{aligned} \tag{3.27}$$

holds. Namely, $\{X(t),\widetilde{B}(t)\}_{t\geq 0}$ on $(\Omega'',\mathcal{F}'',\{\mathcal{F}''_t\}_{t\geq 0},\mathbf{P}'')$ satisfies the properties 3 and 4 of Definition 2.1. The properties 1 and 2 of Definition 2.1 are checked as follows. It is clear that $X(t)$ is an $\mathcal{F}''_t$-adapted process. Furthermore, (3.25) implies $X\in L^\infty(0,T;\mathbf{H}_{B_R})\cap L^2(0,T;\mathbf{V}_{B_R})$, a.s. The proof of Theorem 2.1 is complete. ∎

4. EXISTENCE OF WEAK SOLUTIONS OF (3.1)

In this section, we will give proofs about the a priori estimate (3.5) and Lemma 4.1 appearing in Theorem 2.1. Although the following lemma is similar to [8], we give the proof here for the reader's convenience.

Lemma 4.1. *There exists a unique solution $u_n^{l,\delta}$ of (3.1).*

Proof. We will take several steps to prove this lemma.

step 1. Let Π_n be the orthogonal projection onto the linear subspace spanned by $\{e_j^{(l)}\}_{|j|\leq n}$ Let us set $u_n^{l,\delta}=\Pi_n u^{l,\delta}$. Note that $u_n^{l,\delta}$ can be rewritten as a Fourier expansion with respect to $\{e_k^{(l)}\}_{k\in\mathbb{Z}_0^2}$, where $\mathbb{Z}_0^2=\mathbb{Z}^2\backslash\{0\}$, that is, $u_n^{l,\delta}=\sum_{|k|\leq n}u_n^{l,\delta,k}(s)e_k^{(l)}$, where $u_n^{l,\delta,k}$ stands for the Fourier coefficient: $u_n^{l,\delta,k}=\langle u_n^{l,\delta}(s),e_k^{(l)}\rangle_l$. Let us set $u_0^{(l),j}=\langle \Pi_n u_0^{(l)},e_j^{(l)}\rangle_l$, $u_n^{l,\delta,j}(t)=$

$\langle u_n^{l,\delta}(t), e_j^{(l)}\rangle_l$. Now let us consider the following finite dimensional simultaneous stochastic integral equations:

$$u_n^{l,\delta,j}(t) = u_0^{(l),j} + \int_0^t F_j(u_n^{l,\delta,1}(s), \cdots, u_n^{l,\delta,n}(s))ds + \int_0^t \sigma_j(u_n^{l,\delta,1}(s), \cdots, u_n^{l,\delta,n}(s))dB(s), \quad j = 1, \cdots, n, \tag{4.1}$$

where

$$F_j(u^1, \cdots, u^n) = \langle -A_\delta e_j^{(l)} + \Pi_n B(\sum_{|k|\le n} u^k e_k^{(l)}, e_j^{(l)}), \sum_{|k|\le n} u^k e_k^{(l)}\rangle_l$$
$$\sigma_j(u^1, \cdots, u^n) = -(\Pi_n G \sum_{|k|\le n} u^k e_k^{(l)})^* e_j^{(l)},$$

that is,

$$u(t) = u_0 + \int_0^t F(u(s))ds + \int_0^t \sigma(u(s))dB(s),$$

where

$$u(t) = (u_n^{l,\delta,1}(t), \cdots, u_n^{l,\delta,n}(t)), \quad u_0 = (u_0^{(l),1}, \cdots, u_0^{(l),n}),$$
$$F(u) = (F_1(u), \cdots, F_n(u))', \quad \sigma(u) = (\sigma_1(u), \cdots, \sigma_n(u))'.$$

Let us set

$$T_R = \begin{cases} \inf\{t; |u(t)| \le R\}, & \text{if } \{\} \text{ is not empty,} \\ \infty, & \text{otherwise.} \end{cases}$$

Then, we see that

$$\mathbf{E}\{|u(t\wedge T_R)|^2\} \le |u_0|^2 + C\int_0^t \mathbf{E}\{|u(s\wedge T_R)|^2\}ds,$$

holds for some C independent of R. Thus, we obtain $\mathbf{E}\{|u(t)|^2\} < \infty$. Furthermore,

$$|F(u) - F(v)| \le C_R|u - v|, \quad \text{for every } |u|, |v| \le R,$$
$$|\sigma(u) - \sigma(v)| \le C|u - v|,$$

holds for some constant $C, C_R > 0$. Therefore, (4.1) has a unique strong solution for each $\delta > 0$ and $n, l \in \mathbb{N}$. ∎

Lemma 4.2. *The following estimate holds:*

$$\mathbf{E}^{\mathbf{P}}\left\{||u_n^{l,\delta}(t)||_l^2\right\} \le ||u_0^{(l)}||_l^2. \tag{4.2}$$

Proof. Here we use the same notations as introduced in the proof of Lemma 4.1. By applying Itô's formula to $\langle u_n^{l,\delta}(t), e_j^{(l)}\rangle_l^2$,

$$\begin{aligned} &\langle u_n^{l,\delta}(t), e_j^{(l)}\rangle_l^2 - \langle u_n^{l,\delta}(t), e_j^{(l)}\rangle_l^2 \\ &= (2+\delta)\mu \int_0^t \langle u_n^{l,\delta}(s), e_j^{(l)}\rangle_l \langle \triangle u_n^{l,\delta}(s), e_j^{(l)}\rangle_l ds \\ &\quad + 2\int_0^t \langle \Pi_n(u_n^{l,\delta}(s)\cdot\nabla u_n^{l,\delta}(s)), e_j^{(l)}\rangle_l \langle u_n^{l,\delta}(s), e_j^{(l)}\rangle_l ds + \text{martingale} \\ &\quad + 2\mu\int_0^t \langle e_j^{(l)}, \frac{\partial u_n^{l,\delta}(s)}{\partial x_1}\rangle_l^2 + \langle e_j^{(l)}, \frac{\partial u_n^{l,\delta}(s)}{\partial x_2}\rangle_l^2 ds, \end{aligned} \tag{4.3}$$

where $u_n^{l,\delta}(s) \in \mathbf{C}^{\infty}_{per,\sigma}(l)$ and we use the integration by parts in the last term. Let us multiply (4.3) by $(\lambda_j^{(l)}\mu^{-1})^2$,

$$
\begin{aligned}
(4.4) \qquad & \langle\langle u_n^{l,\delta}(t), e_j^{(l)}\rangle\rangle_l^2 - \langle\langle u_n^{l,\delta}(t), e_j^{(l)}\rangle\rangle_l^2 \\
&= (2+\delta)\mu \int_0^t \langle\langle u_n^{l,\delta}(s), e_j^{(l)}\rangle\rangle_l \langle\langle \triangle u_n^{l,\delta}(s), e_j^{(l)}\rangle\rangle_l ds \\
&\quad + 2\int_0^t \langle\langle \Pi_n(u_n^{l,\delta}(s)\cdot\nabla u_n^{l,\delta}(s)), e_j^{(l)}\rangle\rangle_l \langle\langle u_n^{l,\delta}(s), e_j^{(l)}\rangle\rangle_l ds + \text{martingale} \\
&\quad + 2\mu \int_0^t \langle\langle e_j^{(l)}, \frac{\partial u_n^{l,\delta}(s)}{\partial x_1}\rangle\rangle_l^2 + \langle\langle e_j^{(l)}, \frac{\partial u_n^{l,\delta}(s)}{\partial x_2}\rangle\rangle_l^2 ds,
\end{aligned}
$$

holds. Since $\{(\mu\lambda_j^{(l)^{-1}})^{\frac{1}{2}} e_j^{(l)}\}_{j\in\mathbb{Z}_0^2}$ is orthonormal system in $\mathbf{V}_{per}(l)$, multiply (4.4) by $\mu\lambda_j^{(l)^{-1}}$, then sum from $j=1$ to $|n|$, we have

$$
\begin{aligned}
(4.5) \qquad & ||u_n^{l,\delta}(t)||_l^2 - ||u_n^{l,\delta}(0)||_l^2 \\
&\le (2+\delta)\mu \int_0^t \langle\langle u_n^{l,\delta}(s), \triangle u_n^{l,\delta}(s)\rangle\rangle_l ds \\
&\quad + 2\int_0^t \langle\langle \Pi_n(u_n^{l,\delta}(s)\cdot\nabla u_n^{l,\delta}(s)), u_n^{l,\delta}(s)\rangle\rangle_l ds + \text{martingale} \\
&\quad + 2\mu \int_0^t ||\frac{\partial u_n^{l,\delta}(s)}{\partial x_1}||_l^2 + ||\frac{\partial u_n^{l,\delta}(s)}{\partial x_2}||_l^2 ds,
\end{aligned}
$$

Note that the integrand of the second term of (R.H.S.) is equal to $\langle\langle (u_n^{l,\delta}(s)\cdot\nabla u_n^{l,\delta}(s)), u_n^{l,\delta}(s)\rangle\rangle_l$. As for the first term of (R.H.S.), we have

$$
\langle\langle u_n^{l,\delta}(s), \triangle u_n^{l,\delta}(s)\rangle\rangle_l = -||\frac{\partial u_n^{l,\delta}(s)}{\partial x_1}||_l^2 - ||\frac{\partial u_n^{l,\delta}(s)}{\partial x_2}||_l^2,
$$

by using the integration by parts formula. On the other hand, we see

$$
(4.6) \qquad \langle\langle \Pi_n(u_n^{l,\delta}(s)\cdot\nabla u_n^{l,\delta}(s)), u_n^{l,\delta}(s)\rangle\rangle_l = 0.
$$

Indeed, in the case of two dimensional torus, there exists a stream function $\phi(s)$ satisfying $u_n^{l,\delta}(s) = \nabla^{\perp}\phi(s)$. (4.6) is shown by using such ϕ (see [6] Proposition 6.3). However, it does not hold in the case of higher dimension in general. As a result,

$$
(4.7) \qquad \mathbf{E}^P\left\{\left\|u_n^{l,\delta}(t)\right\|_l^2\right\} \le ||u_0^{(l)}||_l^2.
$$

The proof is complete. ∎

Lemma 4.3. *The following estimates hold:*

$$
(4.8) \qquad \sup_{n,l} \mathbf{E}\left\{\sup_{t\in[0,T]} |u_n^{l,\delta}(t)|_l^2 + \delta\mu\int_0^T ||u_n^{l,\delta}||_l^2\right\} < \infty.
$$

and

$$
(4.9) \qquad \sup_{n,l} \mathbf{E}\left\{\sup_{t\in[0,T]} |u_n^{l,\delta}(t)|_l^p\right\} < \infty, \qquad \textit{for } p \in [2, \frac{2+\delta}{2}].
$$

Furthermore, in particular if $p=2$, the following estimate holds:

$$
(4.10) \qquad \sup_{n\ge 1,\delta>0} \mathbf{E}\left\{\sup_{t\in[0,T]} |u_n^{l,\delta}(t)|_l^2\right\} \le C_1||u_0^{(l)}||_l^2,
$$

for $C_1 > 0$ independent of n and δ. In addition, let $K \subset \mathbb{R}^2$ be a compact set. For an integer l and $R > 0$ satisfying $(-l, l)^2 \supset B_{4R} \cup \Omega_K \cup$ supp u_0. Then,

$$\sup_{n\geq 1,\delta>0,R>0,l\in\mathbb{N}} \mathbf{E}\left\{\sup_{t\in[0,T]} |u_n^{l,\delta,R}(t)|^2\right\} \leq C_2||u_0||^2, \tag{4.11}$$

holds for some constant $C_2 > 0$ independent of n, δ, l and R.

Proof. By Itô's formula applied to $|u_n^{l,\delta}(t)|_l^p$, $p > 2$, it is easy to see that

$$\begin{aligned}
&|u_n^{l,\delta}(t)|_l^p \leq |\Pi_n u_0^{(l)}|_l^p \\
&+ \mu p \left(-\frac{2+\delta}{2} + p - 1\right) \int_0^t |u_n^{l,\delta}(s)|_l^{p-2} ||u_n^{l,\delta}(s)||_l^2 ds \\
&+ p\int_0^t |u_n^{l,\delta}(s)|_l^{p-2} (\Pi_n G u_n^{l,\delta}(s))^* u_n^{l,\delta}(s) dB(s),
\end{aligned}$$

Then, $\sup_n \mathbf{E}\left\{|u_n^{l,\delta}(t)|_l^p\right\} < \infty$ holds if $\mathbf{E}\left\{|u_0^{(l)}|_l^p\right\} < \infty$ and

$$-\frac{2+\delta}{2} + p - 1 \leq 0, \qquad \text{that is, } p \in [0, 2 + \frac{\delta}{2}].$$

Clearly, this condition ensures that

$$\sup_n \mathbf{E} \int_0^t |u_n^{l,\delta}(s)|_l^{p-2} ||u_n^{l,\delta}(s)||_l^2 ds < \infty,$$

holds. Note that the following trivial inequality holds:

$$|Gu|^2_{\mathrm{L}_{\mathrm{H.S}}(\mathbb{R}^2;\mathbf{H}_{per}(l))} \leq 2\mu ||u||_l^2 + \lambda |u|_l^2, \quad u \in \mathbf{V}_{per}(l), \lambda > 0.$$

Then the stochastic term can be estimated as follows:

$$\begin{aligned}
&\mathbf{E}\left\{\sup_{s\in[0,t]} \left|\int_0^s p|u_n^{l,\delta}(s')|_l^{p-2} (\Pi_n G u_n^{l,\delta}(s'))^* u_n^{l,\delta}(s') dB(s')\right|\right\} \\
\leq &C\mathbf{E}\left\{\left(\int_0^t p^2 |u_n^{l,\delta}(s')|_l^{2p-4} |(\Pi_n G u_n^{l,\delta}(s'))^*|^2_{\mathrm{L}_{\mathrm{H.S}}} |u_n^{l,\delta}(s')|_l^2 ds'\right)^{\frac{1}{2}}\right\},
\end{aligned}$$

where we have used the Burkholder's inequality. The right hand is bounded from above by

$$\begin{aligned}
&C\mathbf{E}\left\{\left(\int_0^t p^2 |u_n^{l,\delta}(s')|_l^{2p-2} \left(2\mu ||u_n^{l,\delta}(s')||_l^2 + \lambda |u|_l^2\right) ds'\right)^{\frac{1}{2}}\right\} \\
\leq &C\mathbf{E}\{(\int_0^t (\sup_{s\in[0,t]} |u_n^{l,\delta}(s)|_l^p)(2\mu p^2 ||u_n^{l,\delta}(s')||_l^2 |u_n^{l,\delta}(s')|_l^{p-2} \\
&+ \lambda p^2 (\sup_{\sigma\in[0,s']} |u_n^{l,\delta}(\sigma)|_l^p) ds')^{\frac{1}{2}}\}.
\end{aligned}$$

Furthermore, the right hand is bounded from above by

$$\begin{aligned}
&\frac{1}{2}\mathbf{E}\left\{\sup_{s\in[0,t]} |u_n^{l,\delta}(s)|_l^p\right\} + C^2 \mu p^2 \mathbf{E}\left\{\int_0^t ||u_n^{l,\delta}(s')||_l^2 |u_n^{l,\delta}(s')|_l^{p-2} ds'\right\} \\
&+ \frac{C^2}{2} \lambda p^2 \int_0^t \mathbf{E}\left\{\sup_{\sigma\in[0,s']} |u_n^{l,\delta}(\sigma)|_l^p\right\} ds',
\end{aligned}$$

As a result, by Gronwall's lemma, (4.9) follows for $p \in [2, 2 + \frac{\delta}{2}]$. As for (4.10), it is easily obtained by using (4.7) in Lemma 4.2. Finally, concerning (4.11), it is obtained by noting l is chosen as $(-l, l)^2$ contains both the support of u_0 and B_{4R}. The proof is complete. ∎

REFERENCES

[1] A. BASSON, Spatially homogeneous solutions of 3D stochastic Navier-Stokes equations and local energy inequality, *Stochastic Process. Appl.*, **118** (2008), pp. 417–451.

[2] Z. BRZEŹNIAK and E. MOTYL, Existence of a martingale solution of the stochastic Navier-Stokes equations in unbounded 2D and 3D domains, *J. Differential Equations*, **254** (2013), pp. 1627–1685.

[3] M. CAPINSKI and N. J. CUTLAND, Stochastic Euler equations on the torus, *Ann. Appl. Probab.*, **9** (1999), pp. 688–705.

[4] F. CIPRIANO and A. B. CRUZEIRO, Navier-Stokes equation and diffusions on the group of homeomorphisms of the torus, *Comm. Math. Phys.*, **275** (2007), pp. 255–269.

[5] A. B. CRUZEIRO and I. TORRECILLA, On a 2D stochastic Euler equation of transport type: existence and energy transfer, *Stoch. Dyn.*, online.

[6] G. DA PRATO, *Kolmogorov Equations for Stochastic PDEs*, Birkhäuser Verlag, Basel, 2004.

[7] G. DA PRATO and J. ZABCZYK, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[8] F. FLANDOLI and D. GATAREK, Martingale and stationary solutions for stochastic Navier-Stokes equations, *Probab. Theory Related Fields*, **102** (1995), pp. 367–391.

[9] A. INOUE and T. FUNAKI, On a new derivation of the Navier-Stokes equation, *Comm. Math. Phys.*, **65** (1979), pp. 83–90.

[10] B. KUKSIN, The Eulerian limit for 2D statistical hydrodynamics, *J. Statist. Phys.*, **115** (2004), pp. 469–492.

[11] R. MIKULEVICIUS and B. L. ROZOVSKII, Stochastic Navier-Stokes equations for turbulent flows, *SIAM J. Math. Anal.*, **35** (2004), pp. 1250–1310.

[12] R. MIKULEVICIUS and B. L. ROZOVSKII, Global L_2-solutions of stochastic Navier-Stokes equations, *Ann. Probab.*, **33** (2005), pp. 137–176.

[13] S. PESZAT and J. ZABCZYK, *Stochastic Partial Differential Equations with Levy Noise*, Cambridge University Press, Cambridge, 2007.

[14] M. J. VIŠIK and A. V. FURSIKOV, *Mathematical Problems of Statistical Hydromechanics*, Akademische Verlagsgesellschaft, Leipzig, 1986.

[15] S. YOKOYAMA, Construction of weak solutions of a certain stochastic Navier-Stokes equation, *Stochastics*, **86** (2014), pp. 573–593.