The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 11, Issue 1, Article 16, pp. 1-5, 2014

A_p FUNCTIONS AND MAXIMAL OPERATOR

CHUNPING XIE

Received 31 May, 2014; accepted 19 June, 2014; published 19 December, 2014.

DEPARTMENT OF MATHEMATICS, MILWAUKEE SCHOOL OF ENGINEERING, 1025 N. BROADWAY, MILWAUKEE, WISCONSIN 53202, U. S. A.
xie@msoe.edu
URL: http://www.msoe.edu/people/chunping.xie

ABSTRACT. The relationship between A_p functions and Hardy-Littlewood maximal operator on $L^{p,\lambda}(w)$, the weighted Morrey space, has been studied. Also the extropolation theorem of $L^{p,\lambda}(w)$ has been considered.

Key words and phrases: A_p functions, Extropolation theorem, Maximal operator, Weighted Morrey space.

2010 *Mathematics Subject Classification.* Primary 42B25, 47B38.

ISSN (electronic): 1449-5910

1. INTRODUCTION.

In [8], M. Rosenblum first introduced the A_p functions in a somehow different form. Then B. Muckenhoupt characterized A_p functions for $n = 1$ in [6]. In 1973, B. Muckenhoupt, R. Hunt, and R. Wheeden proved that $w \in A_p$ if and only if the Hilbert transform is bounded on $L^p(w)$ in [7]. There are just the beginning part of some work related to A_p functions. The extrapolation theorem is due to J. L. Rubio de Francia in [9]. More work in this area has been done by H. Helson, R. Coifman, C. Fefferman, G. Szegö, etc. We refer the interested readers to [3], [9], [7], and the references therein.

Let w be nonnegative, locally integrable with respect to the Lebesgue measure. $L^{p,\lambda}(w)$ is the space of all f defined on R^n such that

$$\|f\|_{L^{p,\lambda}(w)} = \sup_Q \left(\frac{1}{|Q|^\lambda} \int_Q |f(x)|^p w(x) dx \right)^{1/p} < \infty$$

where the supremum is taken over all bounded cubes of R^n.

The purpose of this paper is to investigate the properties of the weight function w such that the operators, Hardy-Littlewood maximal operator and Calderón-Zygmund operator are bounded on $L^{p,\lambda}(w)$. The extropolation theorem of $L^{p,\lambda}(w)$ is also considered.

2. PRELIMINARIES.

For our convenience, I recall some properties and necesary materials that we are going to need in the proofs of the later context. The following definitions and characterizations can be found in [3] and [5].

Let Q be a cube in R^n. The usual Hardy-Littlewood maximal operator on a locally integrable function f on R^n is defined by

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(t)| dt.$$

For $1 < p < \infty$, the A_p condition of $w(x)$ is the following property, $\forall\ Q$

$$\left(\frac{1}{|Q|} \int_Q w dx \right) \left(\frac{1}{|Q|} \int_Q \left(\frac{1}{w} \right)^{\frac{1}{p-1}} dx \right)^{p-1} \leq C$$

where the constant C is independent of cube Q.

The A_1 condition is $Mw \leq Cw$, a.e. $x \in R^n$.

All functions with A_p condition are called A_p functions or A_p weights in this note. The following properties of A_p weights are direct consequences of the definition.

Lemma 2.1. : *(i)* $A_p \subset A_q$, $1 \leq p < q$.
: *(ii)* $w \in A_p$ *if and only if* $w^{1-p'} \in A_{p'}$.
: *(iii) If* $w_0, w_1 \in A_1$, *then* $w_0 w_1^{1-p} \in A_p$.

This result below is called the reverse Hölder inequality.

Lemma 2.2. *If* $w \in A_p$, $1 < p < \infty$, *then there exists* $\epsilon > 0$, *such that* $w \in A_{p-\epsilon}$.

Now we give a constructive characterization of A_1 and A_p functions using the Hardy-Littlewood maximal functions. These lemmas next are playing crucial roles in my proofs.

Lemma 2.3. *Let* $f \in L^1_{loc}(R^n)$ *be such that* $Mf < \infty$, *a. e. If* $0 \leq \delta < 1$, *then* $w(x) = (Mf)^\delta$ *is an* A_1 *weight whose* A_1 *constant depends only on* δ.

Lemma 2.4. *If* $1 < p < \infty$, *then* M *is bounded on* $L^p(w)$ *if and only if* $w \in A_p$.

Here the space $L^p(w)$ is the set of all functions f defined on R^n such that

$$\|f\|_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x)dx\right)^{1/p} < \infty.$$

The main results of this note are as follows.

Theorem 2.5. *For $1 < p < \infty$, if M is bounded on $L^{p,\lambda}(w)$, then $w \in A_{p+1}$. Moreover if $w \in A_p$, then M is bounded on $L^{p,\lambda}(w)$.*

We now state the extrapolation theorem of $L^{p,\lambda}(w)$.

Theorem 2.6. *For a fixed r, $1 < r < \infty$, if T is a bounded operator on $L^{r,\lambda}(w)$ for any $w \in A_r$, with operator norm depending only on the A_r constant of w, then T is bounded on $L^{p,\lambda}(w)$, $1 < p < \infty$, for any $w \in A_p$.*

Throughout this paper, the letter C denotes a positive constant which may vary at each occurrence but is independent of the essential variables and quantities.

3. PROOFS.

After all preparations we did in the last section, now we are going to give all proofs of all theorems.

Proof of Theorem 2.5. Suppose that M is bounded on $L^{p,\lambda}(w)$. For a fixed cube Q_0, let $f(x) = w^{-\frac{1}{p}}\chi_{Q_0}$. Then

$$\|f\|_{L^{p,\lambda}(w)} = |Q_0|^{\frac{1-\lambda}{p}}$$

that is, $f \in L^{p,\lambda}(w)$

Also for $x \in Q_0$, we have

$$Mf(x) \geq \frac{1}{|Q_0|}\int_{Q_0} |f(t)|dt = \frac{1}{|Q_0|}\int_{Q_0} w(t)^{-\frac{1}{p}}dt.$$

So

$$\begin{aligned} & \int_{Q_0} w(x)dx \left(\frac{1}{|Q_0|}\int_{Q_0} w(x)^{-\frac{1}{p}}dx\right)^p \\ \leq \; & \int_{Q_0} |Mf(x)|^p w(x)dx \\ \leq \; & |Q_0|^\lambda C\|f\|^p_{L^{p,\lambda}(w)} = C|Q_0|, \end{aligned}$$

that is

$$\frac{1}{|Q_0|}\int_{Q_0} w(x)dx \left(\frac{1}{|Q_0|}\int_{Q_0} w(x)^{-\frac{1}{p}}dx\right)^p \leq C,$$

which means that $w \in A_{p+1}$.

Conversely, suppose that $w \in A_p$ ($p > 1$). Then for any cube Q,

$$\frac{1}{|Q|}\int_Q |f(x)|dx \leq C\left(\frac{1}{\int_Q w(x)dx}\int_Q |f(x)|^p w(x)dx\right)^{\frac{1}{p}}$$

So if we write $d\mu = w(x)dx$, then

$$Mf(x) \leq C\left(M_\mu |f|^p\right)^{\frac{1}{p}}.$$

Since $w \in A_p$, we know that $d\mu$ is a doubling measure and similar to the proof of [1], we can prove that M_μ is bounded on $L^{r,\lambda}(d\mu)$, the space of all functions such that

$$\|f\|_{L^{r,\lambda}(d\mu)} = \sup_Q \left(\frac{1}{|\mu(Q)|^\lambda} \int_Q |f(x)|^r d\mu \right)^{\frac{1}{r}} < \infty.$$

Again since $w \in A_p$, by the Lemma 2.2, for some small $\epsilon > 0$, $w \in A_{p-\epsilon}$.
Choosing $r = \frac{p}{p-\epsilon}$, we have

$$\begin{aligned} \|Mf\|_{L^{p,\lambda}} &\leq \|M_\mu |f|^{p-\epsilon}\|_{L^{p,\lambda}}^{\frac{1}{p-\epsilon}} \\ &\leq C\|f\|_{L^{p,\lambda}}. \end{aligned}$$

This completes the proof of Theorem 2.5. ∎

Proof of Theorem 2.6. First we show that when $1 < q < r$ and $w \in A_1$, T is bounded on $L^{q,\lambda}$. By Lemma 2.3, the function

$$(Mf)^{\frac{r-q}{r-1}} \in A_1 \quad \text{since} \quad r - q < r - 1$$

and by Lemma 2.1, $w(Mf)^{q-r}$ is an A_r weight.
Therefore, for any cube Q, we have

$$\begin{aligned} & \frac{1}{|Q|^\lambda} \int_Q |Tf(x)|^q w(x)dx \\ = & \frac{1}{|Q|^\lambda} \int_Q |Tf(x)|^q \, (Mf)^{-\frac{(r-q)q}{r}} \, (Mf)^{\frac{(r-q)q}{r}} \, w(x)dx \\ \leq & \|Tf\|^q_{L^{r,\lambda}(w(Mf)^{q-r})} \cdot \|Mf\|^{\frac{r-q}{r}q}_{L^{q,\lambda}(w)} \\ \leq & C\|f\|^q_{L^{r,\lambda}(w(Mf)^{q-r})} \cdot \|Mf\|^{\frac{r-q}{r}q}_{L^{q,\lambda}(w)} \end{aligned}$$

Since $|f| \leq Mf$, a.e. and $q - r < 0$, so $(Mf)^{q-r} \leq |f|^{q-r}$, a.e. and $w \in A_1 \subset A_q$ and Theorem 5, we have

$$\frac{1}{|Q|^\lambda} \int_Q |Tf(x)|^q w(x)dx \leq C\|f\|^q_{L^{q,\lambda}}$$

That is,

$$\|Tf\|^q_{L^{q,\lambda}(w)} \leq C\|f\|^q_{L^{q,\lambda}(w)}$$

Now let us show that given any p, $1 < p < \infty$, and q, $1 < q < \min(p, r)$, T is bounded on $L^{q,\lambda}(w)$ if $w \in A_{p/q}$.

Let Q_0 be a fixed cube and assume $w \in A_{p/q}$. Then by duality, there exists nonnegative $g_{Q_0} \in L^{p/(p-q)}_{Q_0}(w)$ with norm 1 such that

$$\begin{aligned} & \left(\frac{1}{|Q_0|^\lambda} \int_{Q_0} |Tf(x)|^p w(x)dx \right)^{q/p} \\ = & \frac{1}{|Q_0|^{\lambda q/p}} \int_{Q_0} |Tf(x)|^q g(x) w(x)dx \end{aligned}$$

where

$$g(x) = \begin{cases} g_{Q_0}(x) & \text{if } x \in Q_0 \,, \\ 0 & \text{otherwise.} \end{cases}$$

For any $s > 1$, $gw \leq (M(gw)^s)^{1/s} \in A_1$ by Lemma 2.3. So we have

$$\begin{aligned} & \frac{1}{|Q_0|^{\lambda q/p}} \int_{Q_0} |Tf(x)|^q g(x) w(x) dx \\ \leq & \frac{1}{|Q_0|^{\lambda q/p}} \int_{Q_0} |Tf(x)|^q \left(M(gw)^s\right)^{1/s} dx \\ \leq & \|Tf\|^q_{L^{q,\lambda q/p}(M(gw)^s)^{1/s}}. \end{aligned}$$

By the first part of the proof, we have

$$\begin{aligned} & \frac{1}{|Q_0|^{\lambda q/p}} \int_{Q_0} |Tf(x)|^q g(x) w(x) dx \\ \leq & C\|f\|^q_{L^{q,\lambda q/p}(M(gw)^s)^{1/s}}. \end{aligned}$$

where C is depending on s only. So we have

$$\begin{aligned} & \|f\|^q_{L^{q,\lambda q/p}(M(gw)^s)^{1/s}} \\ \leq & C\|f\|^q_{L^{p,\lambda}(w)}. \end{aligned}$$

Therefore

$$\|Tf\|_{L^{p,\lambda}(w)} \leq C\|f\|_{L^{p,\lambda}(w)}$$

At this point the desired result follows immediately from this: for $w \in A_p$, by Lemma 2.2, there exsits $q > 1$ such that $w \in A_{p/q}$ and so T is bounded on $L^{p,\lambda}(w)$. ∎

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