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SCOPE OF THE LOGARITHMIC MEAN

MURALI RAO AND AGNISH DEY

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, 1400 STADIUM ROAD, GAINESVILLE, FLORIDA 32611, U. S. A.
mrao@ufl.edu,
URL: http://people.clas.ufl.edu/mrao
agnish@ufl.edu,
URL: http://people.clas.ufl.edu/agnish

ABSTRACT. A number 'a' is between two numbers x and y if and only if a is a convex combination of x and y, in other words, it is a "weighted mean" of x and y. Geometric mean, arithmetic mean are well known examples of these "means". Of more recent vintage is the logarithmic mean which has been considered in many articles in the literature. In this note, we first discuss some of its properties. Then we shall introduce the L function and explore the inverse of this function and its connection with the Lambert's Omega function.

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1. INTRODUCTION

The logarithmic mean between two positive numbers x, y is defined as :

$$L(x,y) = \frac{x-y}{log(x) - log(y)}, \tag{1}$$

see ([1], [7], [9], [11]) etc. This is called a mean as it lies between x and y. The following integral representations appear in the literature:

$$L(x,y)^{-1} = \int_0^1 \frac{1}{tx + (1-t)y} dt,$$
$$L(x,y)^{-1} = \int_0^\infty \frac{1}{(x+t)(y+t)} dt,$$
$$L(x,y) = \int_0^1 x^\theta y^{1-\theta} d\theta,$$

see ([1], [6], [7], [9], [11]). In the next section we shall discuss some of the properties of the logarithmic mean. Most of these properties are available in the literature. We shall prove that logarithmic mean in n-many arguments (see [11]) is concave. An extension of Holder's inequality using logarithmic mean will be presented. We are going to use Neuman's integral representation to obtain a recursion formula for the logarithmic mean and will show that our formula agrees with the one obtained by Mustonen in [8], which involves computations using series expansions.

In the last section, we shall introduce the function $L(x) = L(x,1) = \int_0^1 x^\theta d\theta$. First, we shall discuss various properties of this function. Then the inverse of the L function and its connection with the Lambert Omega function will be discussed. Lambert's Omega function, also called the tree function, arises in the solution of the trinomial equation $x = q + x^m$, first considered by Lambert in 1758 and later in 1779, Euler transformed this into a symmetric form and it has found many applications ever since, see [2].

2. PROPERTIES

Here we list some properties:

$$min(x,y) \leq L(x,y) \leq max(x,y).$$

$$L(x,y) = yL(\frac{x}{y}, 1) = xL(\frac{y}{x}, 1). \tag{2}$$

$$L(x,y) = \begin{cases} 0 & \text{if } x \text{ or } y = 0, \\ x & \text{if } x = y. \end{cases}$$

$L(x,y)$ increases in each variable separately.

$L(x,y)$ is concave in (x,y).

More generally for n positive numbers $x_1, ..., x_n$, the logarithmic mean $L(x_1, ..., x_n)$ is defined by

$$L(x_1, ..., x_n) = \int_0^1 x_1^{\theta_1} x_2^{\theta_2} ... x_n^{\theta_n} \mu_n(d\theta),$$

where μ_n is the "uniform probability" measure on the simplex: $\Delta_n = \{(\theta_1, ..., \theta_n), \theta_1 + ... + \theta_n = 1\}$, see [7].

Theorem 2.1. *$L(x_1, ..., x_n)$ is symmetric. $L(x_1, ..., x_n)$ is concave in $x_1, ..., x_n$.*

Proof. Symmetry is trivial. Let us first prove concavity for two positive numbers x, y: Let $x, y, a, b > 0$. Here we need to show that

$$(tx + (1-t)a)^{\theta}(ty + (1-t)b)^{1-\theta} \geq tx^{\theta}y^{1-\theta} + (1-t)a^{\theta}b^{1-\theta} \ \ for \ \ all \ \ t \in (0,1).$$

We only need to prove this inequality when $t = \frac{1}{2}$, as mid-point concavity implies concavity for a continuous function [due to Sierpinski]. Thus we need to establish the following:

$$(1+c)^{\theta}(1+d)^{1-\theta} \geq 1 + c^{\theta}d^{1-\theta}, \ \ c = \frac{a}{x}, \ \ d = \frac{b}{y}.$$

Now, let $f(c) = (1+c)^{\theta}(1+d)^{1-\theta} - 1 + c^{\theta}d^{1-\theta}$, then $f'(c) = \theta(1+c)^{\theta-1}(1+d)^{1-\theta} - \theta c^{\theta-1}d^{1-\theta}$. Let $c > d$. Then, $(\frac{1+d}{d})^{1-\theta} > (\frac{1+c}{c})^{1-\theta}$, i.e $(\frac{1+d}{d})^{1-\theta} > (\frac{c}{1+c})^{\theta-1}$, i.e $(1+d)^{1-\theta}(1+c)^{\theta-1} > c^{\theta-1}d^{1-\theta}$, this implies that $\theta(1+c)^{\theta-1}(1+d)^{1-\theta} - \theta c^{\theta-1}d^{1-\theta} > 0$, hence $f'(c) > 0$, i.e f is increasing if $c > d$. When $c = d$, $f(d) = 0$, hence $f(c) \geq 0$ for $c \geq d$. Similarly, we observe that if $c < d$, $f'(c) < 0$, i.e f is decreasing here and $f(d) = 0$, hence $f(c) \geq 0$ for $c < d$ as well. Thus we have $f(c) \geq 0$ for all c, completing the proof of our claim.

Now we use induction to prove the result for n positive numbers $x_1, ..., x_n$. The case $n = 2$ has been done above.

Induction hypothesis: Let us assume that when $n = m$ we have :

$$\prod_{i=1}^{m}(tx_i + (1-t)a_i)^{\theta_i} \geq t\prod_{i=1}^{m} x_i^{\theta_i} + (1-t)\prod_{i=1}^{m} a_i^{\theta_i}.$$

Now let $S_m = \sum_{i=1}^{m} \theta_i$ and $n = m + 1$. Then,

$$\begin{aligned}
&\prod_{i=1}^{m+1}(tx_i + (1-t)a_i)^{\theta_i} \\
&= [\prod_{i=1}^{m}(tx_i + (1-t)a_i)^{\frac{\theta_i}{S_m}}]^{S_m}(tx_{m+1} + (1-t)a_{m+1})^{\theta_{m+1}} \\
&\geq [t\prod_{i=1}^{m} x_i^{\frac{\theta_i}{S_m}} + (1-t)\prod_{i=1}^{m} a_i^{\frac{\theta_i}{S_m}}]^{S_m}(tx_{m+1} + (1-t)a_{m+1})^{\theta_{m+1}} \\
&\geq t(\prod_{i=1}^{m} x_i^{\frac{\theta_i}{S_m}})^{S_m}x_{m+1}^{\theta_{m+1}} + (1-t)(\prod_{i=1}^{n} a_i^{\frac{\theta_i}{S_m}})^{S_m}a_{m+1}^{\theta_{m+1}},
\end{aligned}$$

hence the theorem is proved. ∎

Extension of Holder's Inequality

Theorem 2.2. *Let f, g be two positive integrable functions. Then $\int L(f,g) \leq L(\int f, \int g)$.*

Proof. Let $0 < \theta < 1$. Let $p = \frac{1}{\theta}$, $q = \frac{1}{1-\theta}$, then $\frac{1}{p} + \frac{1}{q} = 1$. Holder's inequality gives us: $\int(f^{\theta})(g^{1-\theta})dx \leq (\int f)^{\theta}(\int g)^{1-\theta}dx$.

Let μ be a probability measure on $(0,1)$. Integrating the above inequality with respect to $\mu(d\theta)$ we have:

$$\int_0^1 \mu(d\theta)\int f^{\theta}g^{1-\theta}dx \leq \int_0^1 (\int f)^{\theta}(\int g)^{1-\theta}\mu(d\theta).$$

Changing the order of integration on the left hand side of the above inequality we get:

$$\int L_{\mu}(f,g) \leq L_{\mu}(\int f, \int g).$$

If μ is uniform on $(0,1)$, then we obtain the desired inequality. ∎

Theorem 2.3. *Let $X_1, X_2, ..., X_n$ be a supermartingale, then $Y_n = L_n(X_1, ..., X_n)$ is also a supermartingale.*

Proof.

$$\begin{aligned}
&E[Y_{n+1}|F_n]\\
&= \int_{\theta_1+\theta_2+...+\theta_n\leq 1} X_1^{\theta_1}...X_n^{\theta_n} E(X_{n+1}^{1-\theta_1-...-\theta_n}|F_n)d\theta_1...d\theta_n\\
&\leq \int_{\theta_1+...+\theta_n\leq 1} X_1^{\theta_1}...X_n^{\theta_n}(E(X_{n+1}|F_n))^{1-\theta_1-...\theta_n}d\theta_1...d\theta_n\\
&\leq \int_{\theta_1+...+\theta_n\leq 1} X_1^{\theta_1}...X_n^{1-\theta_1-...-\theta_{n-1}}d\theta_1...d\theta_n\\
&= \int X_1^{\theta_1}...X_n^{1-\theta_1-...-\theta_{n-1}} \int_0^{1-\theta_1-...-\theta_{n-1}} d\theta_n\\
&\leq \int X_1^{\theta_1}X_2^{\theta_2}...X_n^{1-\theta_1-...-\theta_{n-1}} = L_n(X_1, ..., X_n),
\end{aligned}$$

thus the theorem follows. ∎

Recursion Formula for Logarithmic Mean

Let $x_1, x_2, ..., x_n$ be n positive numbers. We define $M_n(x_1, ..., x_n)$ to be:

$$M_n(x_1, ..., x_n) = n! \int_0^1 x_1^{\theta_1}d\theta_1... \int_0^{1-\theta_1-\theta_2-...-\theta_{n-1}} x_n^{\theta_n}d\theta_n, \ \sum_{i=1}^n \theta_i \leq 1.$$

We already know that the logarithmic mean of $x_1, ..., x_n$ is given by:

$$L_n(x_1, ..., x_n) = (n-1)! \int_0^1 x_1^{\theta_1}d\theta_1... \int_0^{1-\theta_1-\theta_2-...-\theta_{n-2}} x_{n-1}^{\theta_{n-1}}x_n^{1-\theta_1-...-\theta_{n-1}}d\theta_{n-1}, \ \sum_{i=1}^n \theta_i = 1.$$

Proposition 2.4. $L_{n+1}(x_1, ..., x_{n+1}) = x_{n+1}M_n(\frac{x_1}{x_{n+1}}, ..., \frac{x_n}{x_{n+1}})$.

Proof.

$$\begin{aligned}
&L_{n+1}(x_1, ..., x_{n+1})\\
&= n! \int_0^1 x_1^{\theta_1}d\theta_1... \int_0^{1-\theta_1-...-\theta_{n-1}} x_n^{\theta_n}x_{n+1}^{1-\theta_1-...-\theta_n}d\theta_n\\
&= n! \int_0^1 x_1^{\theta_1}... \int_0^{1-\theta_1-...-\theta_{n-1}} x_n^{\theta_n}\frac{x_{n+1}}{x_{n+1}^{\theta_1+...+\theta_n}}d\theta_n\\
&= x_{n+1}n! \int_0^1 (\frac{x_1}{x_{n+1}})^{\theta_1}d\theta_1... \int_0^{1-\theta_1-...-\theta_{n-1}} (\frac{x_n}{x_{n+1}})^{\theta_n}d\theta_n\\
&= x_{n+1}M_n(\frac{x_1}{x_{n+1}}, ..., \frac{x_n}{x_{n+1}}),
\end{aligned}$$

hence the claim is estabished. ∎

That is for n many arguments we have: $L_n(x_1, ..., x_n) = x_n M_{n-1}(\frac{x_1}{x_n}, ..., \frac{x_{n-1}}{x_n})$. Now,

$$\begin{aligned}
&M_n(x_1, ..., x_n) \\
&= n! \int_0^1 x_1^{\theta_1} d\theta_1 ... \int_0^{1-\theta_1-...-\theta_{n-1}} x_n^{\theta_n} d\theta_n \\
&= \frac{x_n n!}{ln(x_n)} \int_0^1 (\frac{x_1}{x_n})^{\theta_1} d\theta_1 ... \int_0^{1-\theta_1-...-\theta_{n-2}} (\frac{x_{n-1}}{x_n})^{\theta_{n-1}} d\theta_{n-1} \\
&\quad - \frac{n!}{ln(x_n)} \int_0^1 x_1^{\theta_1} d\theta_1 ... \int_0^{1-\theta_1-...-\theta_{n-2}} x_{n-1}^{\theta_{n-1}} d\theta_{n-1} \\
&= \frac{n x_n}{ln(x_n)} M_{n-1}(\frac{x_1}{x_n}, ..., \frac{x_{n-1}}{x_n}) - \frac{n}{ln(x_n)} M_{n-1}(x_1, ..., x_{n-1}).
\end{aligned}$$

Thus we have:

$$\begin{aligned}
&L_n(x_1, ..., x_n) = x_n M_{n-1}(\frac{x_1}{x_n}, ..., \frac{x_{n-1}}{x_n}) \\
&= x_n[\frac{x_{n-1}}{x_n} \frac{(n-1)}{ln(\frac{x_{n-1}}{x_n})} M_{n-2}(\frac{x_1}{x_{n-1}}, ..., \frac{x_{n-2}}{x_{n-1}}) - \frac{(n-1)}{ln(\frac{x_{n-1}}{x_n})} M_{n-2}(\frac{x_1}{x_n}, ..., \frac{x_{n-2}}{x_n})] \\
&= \frac{(n-1)}{ln(\frac{x_{n-1}}{x_n})} x_{n-1} M_{n-2}(\frac{x_1}{x_{n-1}}, ..., \frac{x_{n-2}}{x_{n-1}}) - \frac{(n-1)}{ln(\frac{x_{n-1}}{x_n})} x_n M_{n-2}(\frac{x_1}{x_n}, ..., \frac{x_{n-2}}{x_n}) \\
&= \frac{(n-1)}{ln(\frac{x_{n-1}}{x_n})} [L_{n-1}(x_1, ..., x_{n-1}) - L_{n-1}(x_1, ..., x_{n-2}, x_n)].
\end{aligned}$$

Now we prove that our formula agrees with [8].

Since $L_n(x_1, ..., x_n)$ is symmetric, we have :

$L_n(x_1, ..., x_n) = (n-1)! \int_0^1 x_2^{\theta_1} d\theta_1 ... \int_0^{1-\theta_1-...-\theta_{n-2}} x_n^{\theta_{n-1}} x_1^{1-\theta_1-...-\theta_{n-1}} d\theta_{n-1}$ and we write $M_n(x_2, ..., x_n, x_1) = n! \int_0^1 x_2^{\theta_1} d\theta_1 ... \int_0^{1-\theta_1-...-\theta_{n-1}} x_1^{\theta_n} d\theta_n$. Then the above proposition takes the form

$$L_n(x_1, ..., x_n) = x_1 M_{n-1}(\frac{x_2}{x_1}, ..., \frac{x_n}{x_1}).$$

By proceeding in similar manner we obtain:

$$\begin{aligned}
&M_n(x_2, ..., x_n, x_1) = n! \int_0^1 x_2^{\theta_1} d\theta_1 ... \int_0^{1-\theta_1-...-\theta_{n-2}} x_n^{\theta_{n-1}} d\theta_{n-1} \frac{x_1^{1-\theta_1-...-\theta_{n-1}} - 1}{ln(x_1)} \\
&= \frac{x_1 n!}{ln(x_1)} \int_0^1 (\frac{x_2}{x_1})^{\theta_1} d\theta_1 ... \int_0^{1-\theta_1-...-\theta_{n-2}} (\frac{x_n}{x_1})^{\theta_{n-1}} d\theta_{n-1} \\
&\quad - \frac{n!}{ln(x_1)} \int_0^1 x_2^{\theta_1} d\theta_1 ... \int_0^{1-\theta_1-...-\theta_{n-2}} x_n^{\theta_{n-1}} d\theta_{n-1} \\
&= \frac{x_1 n}{ln(x_1)} M_{n-1}(\frac{x_2}{x_1}, ..., \frac{x_n}{x_1}) - \frac{n}{ln(x_1)} M_{n-1}(x_2, ..., x_n).
\end{aligned}$$

Hence we have:

$$\begin{aligned}
&L_n(x_1, ...x_n) = x_1 M_{n-1}(\frac{x_2}{x_1}, ..., \frac{x_n}{x_1}) \\
&= x_1[\frac{x_n}{x_1} \frac{(n-1)}{ln(\frac{x_n}{x_1})} M_{n-2}(\frac{x_2}{x_n}, ..., \frac{x_{n-1}}{x_n}) - \frac{(n-1)}{ln(\frac{x_n}{x_1})} M_{n-2}(\frac{x_2}{x_1}, ..., \frac{x_{n-1}}{x_1})]
\end{aligned}$$

$$= \frac{(n-1)}{ln(\frac{x_n}{x_1})}[x_n M_{n-2}(\frac{x_2}{x_n},...,\frac{x_{n-1}}{x_n}) - x_1 M_{n-2}(\frac{x_2}{x_1},...,\frac{x_{n-1}}{x_1})]$$
$$= \frac{(n-1)}{ln(\frac{x_n}{x_1})}[L_{n-1}(x_2,...,x_n) - L_{n-1}(x_1,...,x_{n-1})] \ for \ n = 2,3,...;$$

-which is exactly the formula obtained in [8]. □

3. THE L FUNCTION

In this note, we write

$$L(x) = L(x,1) = \int_0^1 x^\theta d\theta. \tag{3}$$

Properties: • L is continuous, strictly increasing and maps $[0,1]$ onto $[0,1]$ and $(1,\infty)$ onto $(1,\infty)$.

For $0 \leq x \leq 1$, $0 \leq x \leq L(x) \leq 1$.
For $1 \leq x \leq \infty$, $1 \leq L(x) < x < \infty$.

$L(x)$ increasing and concave on $[0,\infty)$.

•

$$(L(x))^p \leq L(x^p) \text{ (Holder)}, p \geq 1,$$
$$(L(x))^p \geq L(x^p) \ \ 0 < p \leq 1,$$
$$|L(x) - L(y)| \leq L(|x-y|).$$

Indeed, $|x^\theta - y^\theta| \leq |x-y|^\theta$, for $0 \leq \theta \leq 1$.

•

$$L(x+y) \leq L(x) + L(y) \ \ as(\ (x+y)^\theta \leq x^\theta + y^\theta, \text{for } 0 < \theta < 1),$$
$$(\lambda \wedge 1)L(x) \leq L(\lambda x) \leq (\lambda \vee 1)L(x) \ \text{ for } \lambda > 0.$$

Let $L^{(n)}$ denote the n-th iterate of L: $L^{(n+1)}(x) = L[L^{(n)}(x)]$. Then

$$1 - L^{(n)}(x) \leq \frac{1}{2^n x} \ \ 0 < x < 1,$$
$$L^{(n)}(x) \leq 1 + \frac{x}{2^n} \ \ 1 \leq x < \infty.$$

Indeed, $L(x) \geq \sqrt{x}$, and by using induction we get $L^{(n)}(x) \geq x^{2^{-n}}$, so $1 - L^{(n)}(x) \leq 1 - x^{2^{-n}} \leq \frac{1}{2^n x}$ [in fact, $1 - x^\theta = \theta \int_x^1 y^{\theta-1} dy \leq \frac{\theta}{x} \int_x^1 dy \leq \frac{\theta}{x}$]. If $x > 1$, we use $L(x) \leq \frac{1+x}{2} \leq 1 + \frac{x}{2}$ and induction to see $L^{(n)}(x) \leq 1 + \frac{x}{2^n}$, so that $L^{(n)}(x) - 1 \leq \frac{x}{2^n}$ for $x > 1$.

Relationship between $L(x,y)$ and $L(x), L(y)$

We observe the following:

when $x < 1$, x^θ decreases, when $y > 1$, $y^{1-\theta}$ decreases.
when $x > 1$, x^θ increases, when $y < 1$, $y^{1-\theta}$ increases.
when both $x, y < 1$, x^θ decreases, $y^{1-\theta}$ increases.
when both $x, y > 1$, x^θ increases, $y^{1-\theta}$ decreases.

Thus, $x, y < 1$ implies $\int_0^1 x^\theta y^{1-\theta} d\theta \le (\int_0^1 x^\theta d\theta)(\int_0^1 y^{1-\theta} d\theta)$, i.e $L(x,y) \le L(x)L(y)$.
Similarly $x, y > 1$ implies $L(x,y) \le L(x)L(y)$.
When one of them is less than 1 and the other one is bigger than 1
$\int_0^1 x^\theta y^{1-\theta} d\theta \ge (\int_0^1 x^\theta d\theta)(\int_0^1 y^{1-\theta} d\theta)$, i.e $L(x,y) \ge L(x)L(y)$.

Lambert's Omega Function

In 1758, Lambert solved the trinomial equation $x = q + x^m$ by developing a series for x in powers of q, see [5]. In [3], Euler transformed Lambert's equation into the more symmetric form

$$x^\alpha - x^\beta = (\alpha - \beta) v x^{\alpha+\beta} \tag{4}$$

by replacing $x^{-\beta}$ for x and setting $m = \frac{\alpha}{\beta}$ and $q = (\alpha - \beta)v$. In [2], Corless argues that this led Euler to the following:

$$x^n = 1 + nv + \frac{1}{2}n(n+\alpha+\beta)v^2 + \frac{1}{6}n(n+\alpha+2\beta)(n+2\alpha+\beta)v^3 + etc. \tag{5}$$

After deriving this series, Euler observed the special cases, starting with $\alpha = \beta$. To see its implication in the original trinomial equation, Corless suggests us to divide (4) by $(\alpha - \beta)$ and then as $\beta \to \alpha$ we get

$$log(x) = v x^\alpha. \tag{6}$$

If we can solve equation (6) for $\alpha = 1$, then we can solve it for any $\alpha \neq 0$. To see this, multiply equation (6) by α, simplify $\alpha log(x)$ to $log x^\alpha$, put $z = x^\alpha$ and $u = \alpha v$. Thus we end up getting $log(z) = uz$, which is just equation (6) with $\alpha = 1$. To solve this equation using (5), Euler put $\alpha = \beta = 1$ and had a series $\frac{(x^n-1)}{n}$. Next setting $n = 0$ he obtained $log(x)$ on the left hand side and a series on the right hand side:

$$log(x) = v + \frac{2^1}{2!}v^2 + \frac{3^2}{3!}v^3 + \frac{4^3}{4!}v^4 + ...$$

This series, which converges for $|v| < \frac{1}{e}$, defines a function $T(v)$ called the tree function. It equals $-\Omega(-v)$, where $\Omega(z)$ is defined to be the function satisfying $\Omega(z)e^{\Omega(z)} = z$. This function has many applications - in the enumeration of trees, in the calculation of water wave heights, in general relativity and quantum mechanics etc. Now here we would like to relate this very important function to our L function:

We start with $L(x,y) = \lambda$ iff $L(\frac{x}{\lambda}, \frac{y}{\lambda}) = 1$. And $L(x,y) = 1$ iff $xe^{-x} = ye^{-y}$ (i.e $yL(\frac{x}{y}, 1) = 1$).

Inverse of L and Lambert Omega function

Let $L(x) = y$. Then, $\frac{x-1}{log(x)} = y$ iff $\frac{x-1}{y} = log(x)$ iff $\frac{x}{y}e^{-\frac{x}{y}} = \frac{1}{y}e^{-\frac{1}{y}}$ iff $\rho(\frac{x}{y}) = \rho(\frac{1}{y})$, where ρ is defined as $\rho(\theta) = \theta e^{-(\theta-1)}$. We observe that ρ maps $[0,\infty)$ onto $[0,1]$, ρ is strictly increasing on $[0,1]$ and strictly decreasing on $[1,\infty)$ and $\rho(0) = 0$, $\rho(1) = 1$, $\rho(\infty) = 0$.

Now let us consider Ω defined on $[0,1]$ by $\Omega(z) = \sum_1^\infty \frac{n^{n-1}}{n!}\frac{1}{e^n}z^n$, see [12]. It can be shown that : Ω is one to one, onto on $[0,1]$, $\Omega(0) = 0$, $\Omega(1) = 1$, $\Omega(\rho(\theta)) = \theta$, $\rho(\Omega(\theta)) = \theta$, i.e Ω and ρ are inverses on $[0,1]$.

$\rho(x) = \theta$ has two roots as we have observed before and the unique root in $(0,1)$ is given by $\Omega(\theta)$. The solution of the equation $\rho(x) = \theta$ for $x \in (1,\infty)$ is quite hard to get and we cannot find in the literature any approach to find this root. Here we show that the inverse of L helps solve this problem.

Let us again assume $L(x) = y$. This is equivalent to $\frac{x}{y}e^{-\frac{x}{y}} = \frac{1}{y}e^{-\frac{1}{y}}$, i.e $\rho(\frac{x}{y}) = \rho(\frac{1}{y})$. Recall that $x < y$ iff $y < 1$ and $x > y$ if y(and hence x) > 1. Thus,

$$\begin{aligned}
&L(x) = y,\ \ x < 1\\
&\text{iff } \rho(\frac{x}{y}) = \rho(\frac{1}{y}) = \frac{1}{y}e^{-(\frac{1}{y}-1)}\\
&\text{i.e } \frac{x}{y} = \Omega[\frac{1}{y}e^{-(\frac{1}{y}-1)}]\\
&\text{i.e } x = \Omega[\frac{1}{y}e^{-(\frac{1}{y}-1)}] = \Omega(\rho(\frac{1}{y})).
\end{aligned}$$

Thus, on $[0, 1]$, the inverse of L is $\Omega(\rho(\frac{1}{y})) = \sum_1^\infty \frac{n^{n-1}}{n!}(\frac{1}{y}e^{-\frac{1}{y}})^n$. Thus we have proved that

$\rho(a) = \rho(b)$ iff $ae^{-a} = be^{-b}$ iff $L(a, b) = 1$ iff $bL(\frac{a}{b}, 1) = 1$ iff $L(\frac{a}{b}) = \frac{1}{b}$. And if $b < 1$ (or $\frac{a}{b} < 1$), $\frac{a}{b} = \Omega(\rho(\frac{1}{b}))$.

Recall that $L(x) = y$ iff $\frac{x}{y}e^{-\frac{x}{y}} = \frac{1}{y}e^{-\frac{1}{y}}$, i.e $\rho(\frac{x}{y}) = \rho(\frac{1}{y})$. As observed before, given $\theta \in [0, 1]$ has exactly two roots: $\rho(z_1) = \rho(z_2) = \theta$, one in $[0, 1]$, and the other one in $[1, \infty)$. We know $z_1 \wedge z_2 = \Omega(\theta)$, i.e the smaller root is $\Omega(\theta)$.

Theorem 3.1. *Let $\theta \in [0, 1]$. Then $\Omega(\theta) \in [0, 1]$. Let x be such that $L(x) = \frac{1}{\Omega(\theta)}$. Then $x\Omega(\theta)$ is the larger root.*

Proof. We write $y = \frac{1}{\Omega(\theta)}$. Then since $L(x) = y$, $\rho(\frac{x}{y}) = \rho(\frac{1}{y}) = \rho(\Omega(\theta)) = \theta$. Since, $y > 1$, $x > 1$, and $L(x) = y$ implies $x > y$, so $\frac{x}{y} > 1$ and hence $\frac{x}{y}$ is the largest root, concluding the proof of the theorem. ∎

The above theorem shows that finding the inverse of L on $[1, \infty)$ finds the larger root of $\rho(z) = \theta$. Of-course $\Omega(\theta)$ is the smaller root. Thus the inverse of L is of interest.

Theorem 3.2. *If $L(x) = y$ and $y < 1$, then $x = y\Omega(\theta)$ where $\theta = \rho(\frac{1}{y}) = \frac{1}{y}e^{-(\frac{1}{y}-1)}$.*

Proof. $L(x) = y$ iff $\frac{x}{y}e^{-(\frac{x}{y}-1)} = \frac{1}{y}e^{-(\frac{1}{y}-1)} = \theta$. Say $\theta \in [0, 1]$. If $y < 1$, then $x < y$, so $\frac{x}{y} \in [0, 1]$. Thus $\Omega(\theta) = \frac{x}{y}$, since $\rho(\frac{x}{y}) = \theta$, and $\frac{x}{y} < 1$. Hence, $x = y\Omega(\theta)$ and the theorem follows. ∎

We do not know a formula for $L^{-1}(x)$ if $x > 1$. Now, we give an iterative procedure for obtaining L^{-1}.

Let $x > 1$: $L(x) = y$ means $\frac{x-1}{log(x)} = y$, or $x - 1 = ylog(x)$. We write $x - 1 = z$ to get $z = ylog(1 + z)$, $z \geq 0$. That is z is a fixed point of the function $f(z) = ylog(1 + z)$, $z \geq 0$.

Claim 1. *$f(z)$ is strictly increasing in $[0, \infty)$ and maps the interval $[y - 1, y^2]$ into itself.*

Proof. (1) $f(z)$ is strictly increasing is clear.

(2) $f(y - 1) = ylog(y) > y - 1$ is seen by differentiation.

(3) It is seen by differentiation that $\sqrt{\xi} - log(1 + \xi)$ is increasing in $[0, \infty)$. It follows that $log(1 + \xi) \leq \sqrt{\xi}$, hence $f(z) \leq y^2$ if $z \leq y^2$. Thus, f maps $[y - 1, y^2]$ into itself. ∎

Since f is increasing and f maps $[y-1, y^2]$ into itself, the iterates $x_n = f^{(n)}(y-1)$ increases to a fixed point (with $x_0 = y - 1$).

Claim 2. *f has a unique fixed point in $[y - 1, \infty]$.*

Proof. Let $\rho(z) = z - f(z)$. If $\rho(z)$ has two zeros $z_1, z_2 \in [y-1, \infty)$, with $z_1 < z_2$, then by Rolle's Theorem $\rho'(z) = 0$ for some $z \in (z_1, z_2)$. But $\rho'(z) = 0$ if and only if $z = y - 1$. ∎

Convergence Rate: We saw above that the iterates x_n converges to a fixed point. We have:

$$\begin{aligned}
&x_{n+1} - x_n = ylog(1 + x_n) - ylog(1 + x_{n-1})\\
&= ylog[1 + \frac{x_n - x_{n-1}}{1 + x_{n-1}}]\\
&\leq y\frac{x_n - x_{n-1}}{1 + x_{n-1}} \text{ (because } log(1+z) \leq z)\\
&\leq y\frac{x_n - x_{n-1}}{1 + x_1} \text{ (as } x_n \text{ is increasing)}\\
&= \frac{y}{1 + ylog(y)}(x_n - x_{n-1}) \text{ (as } x_1 = ylog(1 + x_0) = ylog(1 + y - 1) = ylogy)\\
&= \alpha(x_n - x_{n-1}) \text{ (where } \alpha = \frac{y}{1 + ylog(y)} < 1 \text{ if } y > 1).
\end{aligned}$$

Thus, the convergence rate is exponential.

We can also use the fixed point method to find the inverse of L in $(0, 1)$. Let us consider $y \in [0, 1]$. We want to find x such that $L(x) = y$. Let $f(x) = \frac{x}{L(x)}y$, $0 \leq x \leq y$. If $x < 1$, $L(x) > x$, so $f(x) < y$. Thus f maps continuously $[0, y]$ into itself. So it has a fixed point. We know it is unique as L is strictly increasing.

f is increasing. $f'(x) = y[\frac{1}{L(x)} - \frac{x}{L^2(x)}L'(x)] = \frac{y}{L^2(x)}[L(x) - xL'(x)]$.

Claim: $L(x) - xL'(x) > 0$ in $(0, 1)$.

In fact, $L'(x) = \int_0^1 \theta x^{\theta-1} d\theta$, so $xL'(x) = \int_0^1 \theta x^\theta d\theta < \int_0^1 x^\theta d\theta = L(x)$. Hence, $f^{(n)}(x)$ converges to the fixed point for any $0 < x < y$.

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