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A NONHOMOGENEOUS SUBDIFFUSION HEAT EQUATION

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ABSTRACT. In this paper we consider a nonhomogeneous subdiffusion heat equation of fractional order with Dirichlet boundary conditions.

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1. INTRODUCTION

In this paper, a nonhomogeneous initial boundary value problem for the time fractional diffusion heat equation in the interval will be studied. This problem was obtained from the nonhomogeneous diffusion heat equation by replacing the first order time derivative by a fractional derivative of order $0 < \alpha < 1$ in Caputo's sense. In this work, we solve the nonhomogeneous subdiffusion heat equation with fractional time, initial condition and Dirichlet boundary condition. This equation has been recently treated by a number of authors (See [1, 3]).

2. PRELIMINARY NOTIONS

In this section, we present some basic definitions and preliminary data that are used throughout the document.

Definition 2.1. Here we define the following functions for complex argument $z \in \mathbf{C}$, we will use later called Mittag-Leffler type functions :

$$E_{\alpha}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j+1)};$$

$$E_{\alpha,\beta}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j+\beta)};$$

$$e_{\alpha}^{\zeta z} := z^{\alpha-1} E_{\alpha,\alpha}(\zeta z^{\alpha}),$$

where $\zeta \in \mathbf{C}$, $\alpha, \beta > 0$ and $\Gamma(\cdot)$ is Euler's Gamma function defined for any complex number z as

$$\Gamma(z) := \int_0^{\infty} t^{z-1} e^{-t} dt, \ z \in \mathbf{C}.$$

Note that these functions are generalizations of the exponential function base e, as $e^z = \sum_{j=0}^{\infty} z^j/j!$ and $j! = \Gamma(j+1)$.

Definition 2.2. If $g(t)$ is a continuous function in the interval $[a, b]$ ($g(t) \in \mathrm{C}[a, b]$) and $\alpha > 0$, then its Riemann-Liouville fractional integral is defined by

$$I_{a+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} ds.$$

Definition 2.3. The Caputo-Djrbashyan fractional derivative of order $\alpha > 0$ of a continuous function $g : (a, b) \longrightarrow \mathbf{R}$ is defined by

$$\left(\frac{d}{dt}\right)^{\alpha} g(t) = I_{0+}^{n-\alpha} g^{(n)}(t),$$

where $n = [\alpha] + 1$, (the notation $[\alpha]$ denotes the largest integer not greater than α).

Lemma 2.1. [3] *Let $p, q \geq 0$, and $\phi(t)$ a function of absolute value integrable on an interval $[0, T]$ (namely, $|\phi(t)|$ is integrable on $[0, T]$ or $\phi(t) \in L_1[0, T]$). Then,*

$$I_{0+}^{p} I_{0+}^{q} \phi(t) = I_{0+}^{p+q} \phi(t) = I_{0+}^{q} I_{0+}^{p} \phi(t) \qquad (0)$$

is satisfied almost everywhere (i.e., except in a set of measure 0) on $[0, T]$. If further $\phi(t)$ is continuous in the interval ($\phi(t) \in \mathrm{C}[0, T]$), then (0) is true and

$$\left(\frac{d}{dt}\right)^{\alpha} I_{0+}^{\alpha} \phi(t) = \phi(t)$$

for all $t \in [0, T]$ and $\alpha > 0$.

Theorem 2.2. [1] *Let $\phi(t) \in L_1[0,\ T]$. Then, the integral equation*

$$\varphi(t) = \phi(t) + \frac{\gamma}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \varphi(\tau)\, d\tau$$

has a unique solution $\varphi(t)$ defined by the following formula:

$$\varphi(t) = \phi(t) + \gamma \int_0^t e_\alpha^{\gamma(t-\tau)} \phi(\tau)\, d\tau$$

where $e_\alpha^{\gamma z}$ is a Mittag-Leffler type function given in Definition 2.1.

3. TECHNICAL DEVELOPMENT MODEL

Let $W(x,t) : [0;a] \times [0;\infty) \longrightarrow \mathbf{R}$ be the temperature function at the point x and time t. We denote the intensity of heat source at point x and time t by the form $F(x,t)$. Heat in the borders is zero (homogeneous Dirichlet boundary condition). The initial temperature (in time $t=0$) in this system is denoted by $f(x)$.
Thus, we have a model of anomalous subdiffusion inhomogeneous heat conduction equation with fractional time

$$\left(\frac{\partial}{\partial t}\right)^{\alpha} W = \kappa \frac{\partial^2 W}{\partial x^2} + F(x,t), \qquad 0 < x < a, \quad t > 0 \tag{3.1}$$

(the constant $\kappa > 0$ is the thermal diffusivity), subject to the boundary condition and initial condition

$$W(0,t) = 0, \;\; W(a,t) = 0, \quad t > 0,$$
$$W(x,0) = f(x), \;\; 0 \le x \le a$$

with the fractional derivative order $\alpha \in (0,1)$ in the sense of Caputo.

Theorem 3.1. *Let the differential equation* (3.1) *with the initial condition. Then the solution of the problem is unique and has the form*

$$\begin{aligned} W(x,t) = \\ \sum_{m=1}^{\infty} \mathcal{A}_m E_\alpha \left(\left[-\kappa \left\{ \frac{m^2}{a^2} \right\} \pi^2 \right] t^\alpha \right) \sin \frac{m\pi x}{a} + \\ \sum_{m=1}^{\infty} \mathcal{E}_m \int_0^t e_\alpha^{\left[-\kappa\left\{\frac{m^2}{a^2}\right\}\pi^2\right](t-\tau)} f_k(\tau) d\tau \sin \frac{m\pi x}{a}, \end{aligned} \tag{3.2}$$

where functions $f_k(t)$ are given by (3.11) *for subscript m corresponding, coefficients $\mathcal{E}_m$ are given by* (3.9)*, coefficients $\mathcal{A}_m$ are given for $m \ge 1$, by*

$$\mathcal{A}_m = \sqrt{\frac{2}{a}} \int_0^a f(x) \sin \frac{m\pi x}{a}\; dx. \tag{3.3}$$

Proof. According to the method of Duchateau and Zachmann [2] which we can use due to homogeneous boundary conditions, the solution is sought $W(x,t)$ in the form of the Fourier series of functions $\{U_k\}_{k=1}^{\infty}$ of the linear differential operator $\mathcal{L}$, defined for function U and twice continuously differentiable by the expression

$$\mathcal{L}U = -\kappa \nabla^2 U, \tag{3.4}$$

where the Laplacian ∇^2 in dimension 1 is defined as

$$\nabla^2 U \equiv \frac{\partial^2 U}{\partial x^2}.$$

The operator $\mathcal{L}$ is set to some subset of the vector space $\mathbf{L}_2\left[(0;a)\right]$ of the functions $U\left(x\right)$, $x \in (0;a)$ such that the function $\left|U\left(x\right)\right|^2$is integrable on $(0;a)$. More precisely, the domain of definition $G_{\mathcal{L}}$ of the operator $\mathcal{L}$ consists of all functions $U\left(x\right) \in \mathbf{L}_2\left[(0;a)\right]$ satisfying the boundary conditions

$$U\left(0,t\right)=0,\ \ U\left(a,t\right)=0, \quad t>0, \tag{3.5}$$

and whose images $\mathcal{L}U \in \mathbf{L}_2\left[(0;a)\right]$.
The eigenvalue problem is posed as follows. You have to find the values of the parameter Λ (operator eigenvalues $\mathcal{L}$) such that the equation

$$\mathcal{L}U = \Lambda U \tag{3.6}$$

has nontrivial solutions (non-zero) in the domain $G_{\mathcal{L}}$. These functions are the functions of $\mathcal{L}$. The equation (3.6) equals the Helmholtz equation

$$\nabla^2 U \ + \ \frac{\Lambda}{\kappa} U \ = \ 0.$$

Let $\lambda^2 = \Lambda/\kappa$. So the equation is written

$$\nabla^2 U \ + \ \lambda^2 U \ = \ 0. \tag{3.7}$$

To solve the equation (3.7). We assume a nontrivial solution in the form

$$U\left(x\right) = X\left(x\right).$$

The corresponding derivatives are:

$$\frac{\partial U}{\partial x} = X'\left(x\right),$$

$$\frac{\partial^2 U}{\partial x^2} = X''\left(x\right).$$

Substituting in (3.7) we have

$$X''\left(x\right) + \lambda^2 X\left(x\right) = 0.$$

Dividing by $X\left(x\right)$ results

$$\frac{X''\left(x\right)}{X\left(x\right)} + \lambda^2 = 0$$

and then we have

$$\frac{X''\left(x\right)}{X\left(x\right)} = -\lambda^2.$$

By relying on each side of this equality, both sides must be equal to a constant, we choose this constant as $-\mu^2$, $\mu \in \mathbf{R}$. Then for equation (3.7) we have

$$X''\left(x\right) + \ \mu^2 X\left(x\right) \ = \ 0. \tag{3.8}$$

The solution corresponding to (3.8) can be expressed as
$X(x) \ = \ A\cos\mu x \ + \ B\mathrm{sin}\mu x.$ In terms of the variables separated boundary conditions become

$$X\left(0\right) = X\left(a\right) = 0.$$

Then, to obtain a nontrivial solution X of the equation (3.8) must be

$X(0) = 0,\ X(a) = 0$, respectively so that we have $A = 0$ and

$$\sin\mu a = 0,\ \ B \neq 0.$$

The latter gives
$\mu = \frac{m\pi}{a},\ m = 1,\ 2, \ldots$. Accordingly,

$$X_m(x) = B_m \sin\frac{m\pi x}{a},\ \ m = 1,\ 2, \ldots.$$

Recalling that $\lambda^2 = \mu^2$, the solutions of the equation (3.7) can be written as

$$U_m(x) = \mathcal{E}_m \sin\frac{m\pi x}{a},$$
$$m = 1,\ 2, \ldots$$

for each of the corresponding eigenvalues

$$\lambda_m^2 = \left(\frac{m^2}{a^2}\right)\pi^2,$$

for which the equation (3.6) is expressed as

$$\Lambda_m = \kappa\lambda_m^2$$

Thus we define $\Lambda_k \equiv \Lambda_m$, $U_k \equiv U_m$ and $\mathcal{E}_k \equiv \mathcal{E}_m$.
So, with this redefining the numbering we have

$$\mathcal{L}U_k = \Lambda_k U_k, \quad U_k \in G_{\mathcal{L}}, \quad k = 1,\ 2,\ \ldots.$$

These eigenfunctions of $\mathcal{L}$ can be chosen orthonormal with

$$\mathcal{E}_k = \sqrt{\tfrac{2}{a}}. \tag{3.9}$$

Thereby

$$\langle U_k, U_l\rangle \equiv \int_0^a U_k(x)\, U_l(x) dx =$$
$$= \left\{ \begin{array}{c} \frac{2}{a}\int_0^a \sin\frac{m_k\pi x}{a} \sin\frac{m_l\pi x}{a}\, dx = \delta_{kl}, \\ m_k,\ m_l = 1, 2, \ldots, \end{array} \right.$$

where the sub-subscript k, l of m correspond to the respective eigenfunction.
$\{U_k\}$ is a complete set of $\mathbf{L}_2\left[(0; a)\right]$ and each function $u(x) \in G_{\mathcal{L}}$ can be represented as a series

$$u(x) = \sum_{k=1}^{\infty} \langle u, U_k\rangle\, U_k(x).$$

For $t > 0$ solving the problem of anomalous diffusion equation of heat (3.1) that satisfies the initial and boundary conditions prescribed can be written as

$$W(x,t) = \sum_{k=1}^{\infty} U_k(x) T_k(t), \tag{3.10}$$

where $T_k(t) = \langle W, U_k\rangle$. To find the fractional differential equation for functions $T_k(t)$, solution (3.10) is substituted into the equation (3.1)

$$\sum_{l=1}^{\infty} U_l(x) \left(\frac{d}{dt}\right)^{\alpha} T_l(t) = -\sum_{l=1}^{\infty} T_l(t) \cdot \mathcal{L}U_l(x) + F(x,t)$$

$$= -\sum_{l=1}^{\infty} T_l(t) \cdot \Lambda_l U_l(x) + F(x,t).$$

After taking the scalar product of this equation for the eigenfunction U_k,

$$\sum_{l=1}^{\infty} \langle U_k, U_l \rangle \left(\frac{d}{dt}\right)^{\alpha} T_l(t) = -\sum_{l=1}^{\infty} T_l(t) \cdot \Lambda_l \langle U_k, U_l \rangle + \langle U_k, F \rangle$$

and using the orthonormality of eigenfunctions, we obtain the equations

$$\left(\frac{d}{dt}\right)^{\alpha} T_k(t) + \Lambda_k T_k(t) = f_k(t), \tag{3.11}$$

with $f_k(t) \equiv \langle U_k, F \rangle, \quad k = 1, 2, \ldots$. Due to the initial condition of the equation (3.1), of (3.10) we have

$$W(x,0) = f(x) = \sum_{k=1}^{\infty} U_k(x) T_k(0),$$

$$T_k(0) = \langle W|_{t=0}, U_k \rangle = \langle f, U_k \rangle. \tag{3.12}$$

For the initial condition $T_k(0)$ note that the solution of the corresponding homogeneous problem (3.1) (i.e., with $F(x,t) \equiv 0$) has the form

$$W_{\mathrm{H}}(x,t) = \sum_{k=1}^{\infty} U_k(x) T_{\mathrm{H},k}(t),$$

where

$$T_{\mathrm{H},k}(t) = A_{\mathrm{H},k} E_{\alpha}(-\Lambda_k t^{\alpha}), \quad k = 1, 2, \ldots$$

is the general solution of the homogeneous equation corresponding to (3.11) (since $f_k(t) \equiv 0$ if $F = 0$) for each Λ_k [4].

Each $A_{\mathrm{H},k}$ an arbitrary constant which is determined by applying the homogeneous initial condition, which is the same as for the non-homogeneous equation ($W_{\mathrm{H}}(x,0) = W(x,0) = f(x)$),

$$W_{\mathrm{H}}(x,0) = \sum_{l=1}^{\infty} U_l(x) A_{\mathrm{H},l} = W(x,0) = f(x), \tag{3.13}$$

from which we get by taking the dot product of f given by (3.13) by U_k and considering (3.12),

$$T_k(0) = \langle U_k, f \rangle = A_{\mathrm{H},k}. \tag{3.14}$$

That is for $m_k \geq 1$,

$$A_{\mathrm{H},k} = \sqrt{\frac{2}{a}} \int_0^a f(x) \sin \frac{m_k \pi x}{a} \, dx.$$

To find the solution of the Cauchy problem for the equation (3.11) with the initial condition (3.14) consider the following. By Lemma 2.1, we have

$$\left(\frac{d}{dt}\right)^{\alpha} T_k(t) = I_{0+}^{1-\alpha} T_k'(t).$$

Substituting this result in equation (3.11), the following equation is obtained:

$$I_{0+}^{1-\alpha} T_k'(t) + \Lambda_k T_k(t) = f_k(t).$$

Applying the operator I_{0+}^{α} to this equation, we obtain the following Volterra integral equation of the second kind:

$$T_k(t) = I_{0+}^{\alpha} f_k(t) + T_k(0) - \frac{\Lambda_k}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} T_k(\tau) \, d\tau. \tag{3.15}$$

According to the Theorem 3.1, and using the formulas [1], [5],

$$\frac{1}{\Gamma(\xi)}\int_0^z \tau^{\beta-1}E_{\alpha,\beta}(\zeta\tau^{\alpha})(z-\tau)^{\xi-1}d\tau = z^{\beta+\xi-1}E_{\alpha,\beta+\xi}(\zeta z^{\alpha});$$

$$\frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z) = E_{\alpha,\beta}(z),$$

the integral equation (3.15), considering the initial condition (3.14), has a solution $T_k(t)$ defined only by the following formula

$$T_k(t) = A_{\mathrm{H},k}E_{\alpha}(-\Lambda_k t^{\alpha}) + \int_0^t e_{\alpha}^{-\Lambda_k(t-\tau)}f_k(\tau)d\tau$$

Substituting this in the series (3.10), we obtain the formal solution of the problem given by the equation of heat subdiffusion (3.1) that satisfies the initial and boundary conditions given here:

$$W(x,t) = \sum_{k=1}^{\infty} U_k(x)\left[A_{\mathrm{H},k}E_{\alpha}(-\Lambda_k t^{\alpha}) + \int_0^t e_{\alpha}^{-\Lambda_k(t-\tau)}f_k(\tau)d\tau\right].$$

So, as the solution of the equation of anomalous subdiffusion heat that meets the prescribed boundary conditions can be written as

$$W(x,t) = \sum_{m=1}^{\infty}\mathcal{A}_m E_{\alpha}\left(\left[-\kappa\left\{\frac{m^2}{a^2}\right\}\pi^2\right]t^{\alpha}\right)\sin\frac{m\pi x}{a} + \sum_{m=1}^{\infty}\mathcal{E}_m\int_0^t e_{\alpha}^{\left[-\kappa\left\{\frac{m^2}{a^2}\right\}\pi^2\right](t-\tau)}f_k(\tau)d\tau\sin\frac{m\pi x}{a}, \tag{3.16}$$

where functions $f_k(t)$ are given by (3.11) for subscript m corresponding, coefficients $\mathcal{E}_m$ are given by (3.9), coefficients $\mathcal{A}_m$ are given for $m \geq 1$, by

$$\mathcal{A}_m = \sqrt{\frac{2}{a}}\int_0^a f(x)\sin\frac{m\pi x}{a}\,dx. \tag{3.17}$$

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