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# APPLICATION OF EQUIVALENCE METHOD TO CLASSIFY MONGE-AMPÈRE EQUATIONS OF ELLIPTIC TYPE

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ABSTRACT. In this paper, we apply Cartan's equivalence method to give a local classification of Monge-Ampère equations of elliptic type. Then we find a necessary and sufficient conditions such that a Monge-Ampère equation is either contactomorphic to the Laplace equation or to an Euler-Lagrange equation.

*Key words and phrases:* Exterior Differential Systems, Equivalence Problem, Monge-Ampère Equations .

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## 1. INTRODUCTION

Cartan's method to state the equivalence problem devolopped by Elie Cartan in the years 1905-1910 and more recently in [12, 10], is a crucial tool. Hence, In their work, R. Bryant, D. Grossman and P. Griffiths in the years 1997-1998 [13] clarified the strategy of Cartan. Locally, the application of Cartan's method to study the equivalence problem to classify Monge-Ampère equations [14, 1] in two variables leads to three non-zero orbits: a negative space, a null space and a positive space, which correspond respectively to three types of Monge-Ampère equations: hyperbolic, parabolic and elliptic equations. The general classification problems for symplectic Monge-Ampère equations were studied by Lychagin, Roubtsov [2, 7, 8], Kruglikov [6], A. G. Kushner [3, 5] and others: they showed that to any differential 2-form $\omega$ in the manifold of 1-jets of functions on a 2-dimensional smooth manifold $\mathcal{M}$, $J^1\mathcal{M}$, we can associate a Monge-Ampère equation $E_\omega$ which is entirely determined by the sign of the Pfaffian function $\mathrm{Pf}(\omega)$ at each point of $\mathcal{D} \subset J^1\mathcal{M}$. In [13], R. Bryant, D. Grossman and P. Griffiths applied the equivalence method to classify Monge-Ampère equations of hyperbolic type: Given a 5-dimensional contact manifold $(\mathcal{M}, I)$, they applied this method to some Monge-Ampère system $\varepsilon$, locally algebraically generated as

$$\varepsilon = \{\theta, d\theta, \Psi\},$$

for $\theta \in \Gamma(I)$ where $\theta$ is non zero contact form generating $I$, $\Gamma(I)$ being the module generated by contact form $I$, and $\Psi$ is a 2-form in $\Omega^2(\mathcal{M})$.

Our aim is to apply the equivalence method to classify Monge-Ampère equations of elliptic type. We determined the necessary and sufficient conditions for an elliptic Monge-Ampère system to be locally equivalent to the Monge-Ampère system of the linear homogeneous Laplace equations and the necessary and sufficient conditions when it is locally equivalent to an Euler-Lagrange system. A work of A. Kushner - V. Lychagin and V. Roubtsov in [9, 4] contain results on equivalence of Monge-Ampère equation to homogeneous Laplace equation. Those results are formulated in terms of the number of coefficients in the Monge-Ampère equation and can be explicity verified with just finite number of usual algebraic operations and Partial Differentiations.

## 2. MONGE-AMPÈRE EQUATIONS

Let $\mathcal{M}$ be a 2-dimensional smooth manifold and let $J^1\mathcal{M}$ be the manifold of 1-jet of smooth function on $\mathcal{M}$ given in [7] by

$$J^1\mathcal{M} := \{(a, L) \mid a \in \mathcal{M} \times \mathbb{R}, L \in T_a(\mathcal{M} \times \mathbb{R}) \ \dim L = 2, \ \ L \text{ is plane}\}.$$

For all smooth function $u \in \mathcal{C}^\infty(U)$ with $U \subset \mathcal{M}$, we associate the graph of $u$ by $\Sigma := j^1u(U) \subset J^1\mathcal{M}$ with

$$j^1u : U \to J^1\mathcal{M}.$$

Denote $a = (x^1, x^2, z)$ and $L = (p_1, p_2)$ with $x^1, x^2$ are the local coordinates on $\mathcal{M}$. We have $z \circ (j^1(u)) = u$ and for $1 \leq a \leq 2$, $p_a \circ (j^1(u)) = \frac{\partial u}{\partial x^a}$, then

$$\Sigma := \left\{ \left( x = (x^1, x^2), u(x), \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2} \right), x \in \mathcal{M} \right\},$$

is a smooth submanifold of $J^1\mathcal{M}$ which is endowed with a natural contact structure given by the 1-form not closed $\theta$

$$\theta = dz - p_1 dx^1 - p_2 dx^2.$$

If $\Psi$ is a two form over $J^1\mathcal{M}$, then

$$\Psi = \Psi_{p_1 p_2} dp_1 \wedge dp_2 + \Psi_{p_1 x^2} dp_1 \wedge dx^2 + \Psi_{p_2 x^2} dp_2 \wedge dx^2 + \Psi_{p_1 x^1} dp_1 \wedge dx^1$$

$$+\Psi_{p_2x^1}dp_2 \wedge dx^1 + \Psi_{x^1x^2}dx^1 \wedge dx^2 \mod(\theta).$$

A Monge-Ampère equations read

$$\begin{cases} \Psi|_\Sigma = 0, \\ \theta|_\Sigma = 0 \ (\Rightarrow d\theta|_\Sigma = 0), \\ dx^1 \wedge dx^2|_\Sigma \neq 0, \end{cases}$$

Along of $\Sigma$, we have

$$\Psi_{p_1p_2}\left[\frac{\partial^2 u}{(\partial x^1)^2}\frac{\partial^2 u}{(\partial x^2)^2} - \left(\frac{\partial^2 u}{\partial x^1 \partial x^2}\right)^2\right] + \Psi_{p_1x^2}\frac{\partial^2 u}{(\partial x^1)^2} + \Psi_{p_2x^1}\frac{\partial^2 u}{(\partial x^2)^2}$$
$$+(\Psi_{p_1x^1} + \Psi_{p_2x^2})\frac{\partial^2 u}{\partial x^1 \partial x^2} + \Psi_{x^1x^2}\left(x^a, u(x), \frac{\partial u}{\partial x^a}\right) = 0.$$

Assuming $\Psi_{p_1x^1} = \Psi_{p_2x^2}$, a classical Monge-Ampère equation has the following form

$$Au_{x^1x^1} + 2BAu_{x^1x^2} + Cu_{x^2x^2} + D(u_{x^1x^1}u_{x^2x^2} - u_{x^1x^2}) + E = 0; \tag{2.1}$$

where $A$, $B$, $C$, $D$ and $E$ are a functions of independant variables $x^1$ and $x^2$, an unknown function $u = u(x^1, x^2) := z$ and its first partial derivatives $u_{x^1} := p$ and $u_{x^2} := q$. It's known that the left-hand side of Monge-Ampère equation (2.1) corresponds to the differential 2-form

$$Adx^1 \wedge dx^2 + Bdp \wedge dx^2 + C(dx^1 \wedge dp + dq \wedge dx^2) + Ddx^1 \wedge dq + Edp \wedge dq.$$

**Definition 2.1.** [13] For a contact manifold $(\mathcal{M}, I)$ with Lagrangian $\Lambda \in \Omega^n(\mathcal{M})$, the unique form $\Pi := \theta \wedge \Psi$ is called *Poincaré-Cartan form* of $\Lambda$, or equivalently, $\Pi \equiv 0 \bmod(I)$.

**Definition 2.2.** The Euler-Lagrange system of the Lagrangian $\Lambda$ is the differential ideal generated algebraically as

$$\varepsilon_\Lambda = \{\theta, d\theta, \Psi\} \in \Omega^*(\mathcal{M}).$$

A Monge-Ampère system locally generated by $, \varepsilon = \{\theta, d\theta, \Psi\}$, the generator $\Psi$ may be uniquely chosen modulo $\{I\}$ and modulo multiplication by functions. By the condition of primitivity [1] we assume

$$d\theta \wedge \Psi = 0 \mod(\theta).$$

Hence, locally, Monge-Ampère equations read

$$\begin{cases} \theta \in \Omega^1(\mathcal{M}) \text{ and } \Psi \in \Omega^2(\mathcal{M}), \\ \Psi \wedge d\theta = 0 \bmod(\theta), \\ \theta \wedge d\theta \wedge d\theta \neq 0. \end{cases}$$

**Theorem 2.3.** *A Monge-Ampère system $\varepsilon = \{\theta, d\theta, \Psi\}$ on a $2n+1$-dimensional contact manifold $(\mathcal{M}, I)$ where $\Psi$ is assumed to be primitive modulo $\{I\}$ is locally equivalent to an Euler-Lagrange system $\varepsilon_\Lambda$ if and only if it satisfies*

$$d\Pi := d(\theta \wedge \Psi) = \varphi \wedge \Pi,$$

*with $d\varphi \equiv 0$ mod$\{\theta, d\theta\}$.*

*Proof.* See [13] page 16-19. ∎

**Remark 2.4.** Along $\Sigma$, Euler-Lagrange equations of the action $\int_\Sigma \Lambda$, where $\theta|_\Sigma = 0$, are given by:

$$\frac{\partial L}{\partial z} - \frac{d}{dx^\imath}\left(\frac{\partial L}{\partial p_\imath}\right) = 0.$$

[1] This primitivity form is non-zero every where, locally $\Psi_x \notin \{\theta, d\theta\}_x$. The explaination is in [13] page 18.

**Theorem 2.5.** *(Darboux) Let* $(\Omega_1, \mathcal{M}_1)$ *and* $(\Omega_2, \mathcal{M}_2)$ *be symplectic manifolds of the same dimension. Then for any points* $a \in \mathcal{M}_1$ *and* $b \in \mathcal{M}_2$ *there are a neighborhoods* $\mathcal{O}_1 \ni a$ *and* $\mathcal{O}_2 \ni b$ *and a diffeomorphism* $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ *such that* $\varphi(a) = b$ *and* $\varphi^*(\mathcal{O}_2) = \mathcal{O}_1$ *in* $\mathcal{O}_1$*.*

**Corollary 2.6.** [9]. Let $(\Omega, \mathcal{M})$ be a $2n$-dimensional symplectic manifold. Then for point $a \in \mathcal{M}$ there are local canonical coordinates $(q^1, ..., q^n, p_1, ..., p_n)$ suth that $q^\imath(a) = p_\imath(a) = 0$ for $\imath = 1, ..., n$ and $\Omega$ has the following canonical form:

$$\Omega = \sum_{\imath=1}^{n} dq^\imath \wedge dp_\imath.$$

## 3. EQUIVALENCE PROBLEM

In this section, We apply locally the equivalence problem to classify the Monge-Ampère system $\varepsilon = (\mathcal{M}, \theta, \Psi)$ satisfying

$$\begin{cases} \theta \wedge d\theta \wedge d\theta \neq 0, \\ \Psi \wedge d\theta = 0 \operatorname{mod}(\theta), \end{cases}$$

by comparing it with another system $\tilde{\varepsilon} = (\tilde{\mathcal{M}}, \tilde{\theta}, \tilde{\Psi})$, by looking at a local diffeomorphism $\varphi : \mathcal{M} \to \tilde{\mathcal{M}}$ such that

$$\begin{cases} \varphi^* \tilde{\theta} = \theta, \\ \varphi^* \tilde{\Psi} = \Psi. \end{cases}$$

### 3.1. Preliminaries.

Let $\eta^0 = \alpha\theta \neq 0$, for some smooth function $\alpha \neq 0$. Denote by $\operatorname{mod}(I)$ up to a differential form contained in the differential ideal generated by $I$. Locally, by Darboux theorem, we can find 1-forms $\eta^0, \eta^1, \eta^2, \eta^3, \eta^4$ such that

$$d\eta^0 = \eta^1 \wedge \eta^2 + \eta^3 \wedge \eta^4 \quad \operatorname{mod}(I). \tag{3.1}$$

There exist functions $b_{\imath\jmath}$ such that $\Psi = \frac{1}{2} b_{\imath\jmath} \eta^\imath \wedge \eta^\jmath$, and since $\Psi \wedge d\eta^0 = 0 \operatorname{mod}(\theta)$, then

$$b_{12} + b_{34} = 0.$$

We will study the conditions imposed by $\eta = (\eta^0, \eta^1, \eta^2, \eta^3, \eta^4)$ such that (3.1) be checked. So there are three non-zero orbits which we call: negative space, null space and positive space.

(1) If $\Psi \wedge \Psi$ is a negative multiple of $d\eta^0 \wedge d\eta^0$, then the local coframing $\eta$ may be chosen so that in addition of (3.1),

$$\Psi = \eta^1 \wedge \eta^2 - \eta^3 \wedge \eta^4 \ \operatorname{mod}(\theta),$$

for a classical variational problem, this occurs when the Euler-Lagrange PDE is hyperbolic.

(2) If $\Psi \wedge \Psi = 0$, then $\eta$ may be chosen so that

$$\Psi = \eta^1 \wedge \eta^3 \operatorname{mod}(\theta),$$

for a classical variational problem, this occurs when the Monge-Ampère PDE is parabolic.

(3) If $\Psi \wedge \Psi$ is a positive multiple of $d\eta^0 \wedge d\eta^0$, then the local coframing $\eta$ may be chosen so that in addition of (3.1),

$$\Psi = \eta^1 \wedge \eta^4 - \eta^3 \wedge \eta^2 \ \operatorname{mod}(\theta),$$

for a classical variational problem, this occurs when the Monge-Ampère PDE is elliptic.

In the following we will study the elliptic case, we look at the following system

$$\left\{\begin{array}{l} \eta^0=\alpha\theta\neq 0\\ d\eta^0=\eta^1\wedge\eta^2+\eta^3\wedge\eta^4 \ \ \mathrm{mod}(\theta),\\ \Psi=\eta^1\wedge\eta^4-\eta^3\wedge\eta^2 \ \ \mathrm{mod}(\theta).\end{array}\right. \tag{3.2}$$

3.2. **An algebra preliminary.** For $\omega:=(\omega^1,\omega^2,\omega^3,\omega^4)\in(\mathbb{R}^4)^*$, assuming that $\omega^1\wedge\omega^2\wedge\omega^3\wedge\omega^4\neq 0$, we consider the symmetric non-degenerate function

$$\begin{array}{rcl}\langle .,.\rangle:\Lambda^2(\mathbb{R}^4)^*\times\Lambda^2(\mathbb{R}^4)^* & \longrightarrow & \mathbb{R},\\ (\ \alpha\quad ,\quad \beta\ ) & \longmapsto & \langle\alpha,\beta\rangle:=\dfrac{\alpha\wedge\beta}{\omega^1\wedge\omega^2\wedge\omega^3\wedge\omega^4}.\end{array}$$

The Lie algebra $SL(4,\mathbb{R})$ acts on $\Lambda^2(\mathbb{R}^4)^*\simeq\mathbb{R}^6$ through the action $\forall\ g\in SL(4,\mathbb{R})$

$$\begin{array}{rcl} q_g:\Lambda^2(\mathbb{R}^4)^* & \longrightarrow & \Lambda^2(\mathbb{R}^4)^*,\\ \alpha & \longmapsto & q_g(\alpha):=g^*\alpha,\end{array}$$

$q_g$ is a quadratic form in $g$ and we have

$$\langle q_g(\alpha),q_g(\beta)\rangle=\langle g^*\alpha,g^*\beta\rangle=\langle\alpha,\beta\rangle.$$

We want to represent $SL(4,\mathbb{R})$ =Sp$(\mathbb{R}^{3,3})$ :=Sp$(3,3)$[1] on $\mathbb{R}^6$. Let $G:=SO(\Lambda^2(\mathbb{R}^4)^*,\langle .,.\rangle)\subset GL(6,\mathbb{R})$. Denote

$$\begin{array}{rcl}\Phi:\ SL(4,\mathbb{R}) & \longrightarrow & G,\\ g & \longmapsto & q_g.\end{array}$$

A basis $(\alpha_L^1,\alpha_L^2,\alpha_L^3,\alpha_R^1,\alpha_R^2,\alpha_R^3)$ of $\Lambda^2(\mathbb{R}^4)^*$ given by

$$\left\{\begin{array}{l}\alpha_L^1=\omega^1\wedge\omega^2+\omega^3\wedge\omega^4,\\ \alpha_L^2=\omega^1\wedge\omega^3+\omega^4\wedge\omega^2,\\ \alpha_L^3=\omega^1\wedge\omega^4+\omega^2\wedge\omega^3,\\ \alpha_R^1=\omega^1\wedge\omega^2-\omega^3\wedge\omega^4,\\ \alpha_R^2=\omega^1\wedge\omega^3-\omega^4\wedge\omega^2,\\ \alpha_R^3=\omega^1\wedge\omega^4-\omega^2\wedge\omega^3,\end{array}\right.$$

we have $\forall a,b\in\{1,2,3\}$

$$\langle\alpha_L^a,\alpha_L^b\rangle=2\delta_b^a,\ \ \langle\alpha_R^a,\alpha_R^b\rangle=-2\delta_b^a \text{ and } \langle\alpha_L^a,\alpha_R^b\rangle=0.$$

The signature of $\Phi$ is $(3,3)$ and $SO(\Lambda^2(\mathbb{R}^4)^*,\langle .,.\rangle)\subset SL(4,\mathbb{R})$ and we know that dim$SL(4,\mathbb{R})$ =dim$SO(\Lambda^2(\mathbb{R}^4)^*,\langle .,.\rangle)=15$, then

$$G=\text{Spin}(3,3).$$

The first step of the equivalence method is to find a group $G$ preserves (3.2), we define

$$G=G_{ellip}=\{g\in SL(4,\mathbb{R}),\ \ g^*\alpha_L^1=\alpha_L^1;\ \ g^*\alpha_L^3=\alpha_L^3\}.$$

For all $\xi$ in the Lie algebra $\mathfrak{g}_{ellip}:=\mathfrak{g}$, we have $\xi=(\xi^\imath_\jmath)_{1\leq\imath,\jmath\leq 4}\in M(4,\mathbb{R})$ tr$\xi=0$, and moreover, for $L_\xi\in G$ we have $L_\xi=I+\xi+\circ(\xi)$, then

$$\left\{\begin{array}{l}\xi\in M(4,\mathbb{R}),\ \ \mathrm{tr}\xi=0,\\ \xi^*\alpha_L^1=0;\ \ \ \xi^*\alpha_L^3=0,\end{array}\right. \tag{3.3}$$

which gives

$$\left\{\begin{array}{c}\xi_1^1+\xi_2^2+\xi_3^3+\xi_4^4=0,\\ \xi_a^1\omega^a\wedge\omega^2+\xi_a^2\omega^1\wedge\omega^a+\xi_a^3\omega^a\wedge\omega^4+\xi_a^4\omega^3\wedge\omega^a=0,\\ \xi_a^1\omega^a\wedge\omega^4+\xi_a^4\omega^1\wedge\omega^a+\xi_a^2\omega^a\wedge\omega^3+\xi_a^3\omega^2\wedge\omega^a=0,\end{array}\right.$$

[1] "Spin" is a notation used by physicists. (Spin$(1,3)=SL(2,\mathbb{C})$).

then there exist $a, b, c, d, e, f \in \mathbb{R}$ and a basis $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6$ of $\mathfrak{g}$ such as

$$\xi = a\xi_1 + b\xi_2 + c\xi_3 + d\xi_4 + e\xi_5 + f\xi_6,$$

with

$$\xi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \xi_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \xi_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xi_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \xi_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \xi_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

Denote the $\mathbb{R}$-linear map by

$$\begin{aligned} \Phi: \quad \mathfrak{g} \quad &\longrightarrow \quad sl(2,\mathbb{C}), \\ X, Y \longmapsto \Phi([X,Y]) &= [\Phi(X), \Phi(Y)]. \end{aligned}$$

A basis of $sl(2,\mathbb{C})$ is $(h_0, e_0, f_0, h_1, e_1, f_1)$ is of the form

$$h_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad f_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$h_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad e_1 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \quad f_1 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$$

such that

$$[h_0, h_1] = [e_0, e_1] = [f_0, f_1] = 0, \;\; [h_a, e_b] = -2i^{a+b}e_0,$$
$$[h_a, f_b] = -2i^{a+b}f_0 \text{ and } [e_a, f_b] = -i^{a+b}h_0.$$

We have this correspondence

$$\begin{aligned} \xi_1 &\longleftrightarrow h_0, \\ \xi_2 &\longleftrightarrow e_0, \\ \xi_3 &\longleftrightarrow f_1, \\ \xi_4 &\longleftrightarrow h_1, \\ \xi_5 &\longleftrightarrow e_1, \\ \xi_6 &\longleftrightarrow f_0, \end{aligned}$$

then this showed the lemma

**Lemma 3.1.** The $\mathbb{R}$-linear map $\Phi : \mathfrak{g} \longrightarrow sl(2,\mathbb{C})$ is a homomorphism of Lie algebra.

Denote the real-linear isomorphism

$$\begin{aligned} T: \quad \mathbb{R}^4 \quad &\longrightarrow \quad \mathbb{C}^2 \\ X = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} &\longmapsto \begin{pmatrix} x^3 + ix^1 \\ x^2 + ix^4 \end{pmatrix}, \end{aligned}$$

we can show that $\forall \;\; 1 \leq \imath \leq 6$

$$T(\xi_\imath X) = \Phi(\xi_\imath)T(X),$$

Then we can continious to work in the 2-dimensional complex.

3.3. **Back to the equivalence problem.** In this part, we will apply directly the equivalence problem for the Monge-Ampère system of elliptic type on contact manifold of dimension 5. We will give criteria in terms of the differential invariants thus obtained for a given system to be locally equivlent to the system associated to the linear homogeneous Laplace equation or to an Euler Lagrange system. On the contact manifold $\mathcal{M}$, one can locally find a coframing $\omega = g^{-1}\eta$ where $\eta$ satisfied 3.2 and $g \in \mathfrak{g}$ the Lie Algebra defined ine the previous part. The exterior derivative of this equation is

$$d\omega = g^{-1}dg \wedge \omega + g^{-1}d\eta. \tag{3.4}$$

**Definition 3.2.** Let $G \in Gl(n, \mathbb{C})$ be a subgroup and $B \cong \mathcal{M} \times G$. A $G$-structure $B \to \mathcal{M}$ is a principal subbundle of the coframe bundle $\mathcal{F}(\mathcal{M}) \to \mathcal{M}$, having structure group $G$. A pseudo-connection in the $G$-structure is a $\mathfrak{g}$-valued 1-form on $B$ whose restriction to the fiber tangent spaces $\mathcal{V}_b \subset T_bB$ equals the identification $\mathcal{V}_b \cong \mathfrak{g}$ induced by the right $G$-action on $B$.

Introducing any pseudo-connection $\varphi \in \Omega^1(B) \otimes \mathfrak{g}$ satisfies the fundamental formula for the equivalence method given by

$$d\omega = -\varphi \wedge \omega + \tau,$$

where $\tau \in \Omega^2(B)$ is the torsion of the pseudo-connection $\varphi$. A consequence of (3.4), for $0 \le \imath, \jmath, k \le 4$ we have

$$\varphi = -g^{-1}dg \text{ and } \tau = g^{-1}d\eta = \frac{1}{2}T^{\imath}_{\jmath k}\omega^{\jmath} \wedge \omega^{k}.$$

Denote by $\pi$ the vector valued 1-forms such that, $\omega = P\pi$, where $P \in M(5, \mathbb{C})$ for

$$\begin{cases} \pi^0 = \omega^0, \\ \pi^1 = \omega^3 + i\omega^1, \\ \pi^2 = \omega^2 + i\omega^4, \\ \bar{\pi}^1 = \omega^3 - i\omega^1, \\ \bar{\pi}^2 = \omega^2 - i\omega^4, \end{cases}$$

then

$$d\pi = -\psi \wedge \pi + P^{-1}\tau, \tag{3.5}$$

where $\psi = P^{-1}\varphi P$. Consider $\forall M_\natural = (a^{\imath}_{\jmath})_{1 \le \imath, \jmath \le 4} \in \mathfrak{g}$, we note $M \in M(5, \mathbb{C})$ by

$$M = \begin{pmatrix} a^0_0 & 0 \\ 0 & M_\natural \end{pmatrix},$$

if $M_\natural = (a^{\imath}_{\jmath})_{1 \le \imath, \jmath \le 4} \in \mathfrak{g}$, then

$$P^{-1}MP = \begin{pmatrix} a^0_0 & 0 & 0 & 0 & 0 \\ 0 & a^1_1 + ia^1_3 & a^1_4 + ia^1_2 & 0 & 0 \\ 0 & a^2_3 - ia^2_1 & a^2_2 - ia^2_4 & 0 & 0 \\ 0 & 0 & 0 & a^3_3 + ia^3_1 & a^3_2 + ia^3_4 \\ 0 & 0 & 0 & a^4_1 - ia^4_3 & a^4_4 - ia^4_2 \end{pmatrix}.$$

**Proposition 3.3.** *Let $(\mathcal{M}^5, \varepsilon)$ an elliptic Monge-Ampère system. An adapted coframe is the sections of $G-$structure on $\mathcal{M}$, where $G$ is the smallest subgroup generated by all matrices of size (1,2,2) of the form*

$$\begin{pmatrix} a^0_0 & 0 & 0 \\ C & A & 0 \\ \bar{C} & 0 & \bar{A} \end{pmatrix}, \tag{3.6}$$

*where $A \in$sl$(2, \mathbb{C})$ and det$A = a^0_0 \neq 0$.*

*Proof.* The sections of $G-$structure adapted (3.2) are of the form (3.6). ∎

The second step of equivalence problem is "Calculation of the structure equations", we have $\psi = P^{-1}\varphi P$, thus it is of the form

$$\psi = \begin{pmatrix} \psi_0^0 & 0 & 0 & 0 & 0 \\ \psi_0^1 & \psi_1^1 & \psi_2^1 & 0 & 0 \\ \psi_0^2 & \psi_1^2 & \psi_2^2 & 0 & 0 \\ \bar{\psi}_0^1 & 0 & 0 & \bar{\psi}_1^1 & \bar{\psi}_2^1 \\ \bar{\psi}_0^2 & 0 & 0 & \bar{\psi}_1^2 & \bar{\psi}_2^2 \end{pmatrix}, \tag{3.7}$$

where $\psi_1^1 + \psi_2^2 = \bar{\psi}_1^1 + \bar{\psi}_2^2 = \psi_0^0$ .

We assume $P^{-1}\tau := \begin{pmatrix} \tau^0 \\ \tau^1 \\ \tau^2 \\ \bar{\tau}^1 \\ \bar{\tau}^2 \end{pmatrix}$, where

$$\tau^0 := d\pi^0 + \psi_0^0 \wedge \pi^0 = \frac{i}{2}(\bar{\pi}^1 \wedge \bar{\pi}^2 - \pi^1 \wedge \pi^2),$$

and for $\imath = 1, 2, 3, 4$, we assume

$$\tau^\imath = T_{12}^\imath \pi^1 \wedge \pi^2 + T_{1\bar{1}}^\imath \pi^1 \wedge \bar{\pi}^1 + T_{1\bar{2}}^\imath \pi^1 \wedge \bar{\pi}^2 + T_{2\bar{1}}^\imath \pi^2 \wedge \bar{\pi}^1 + T_{2\bar{2}}^\imath \pi^2 \wedge \bar{\pi}^2 + T_{\bar{1}\bar{2}}^\imath \bar{\pi}^1 \wedge \bar{\pi}^2$$
$$+ T_{01}^\imath \pi^0 \wedge \pi^1 + T_{02}^\imath \pi^0 \wedge \pi^2 + T_{0\bar{1}}^\imath \pi^0 \wedge \bar{\pi}^1 + T_{0\bar{2}}^\imath \pi^0 \wedge \bar{\pi}^2.$$

This produces the structure equations

$$\begin{cases} d\pi^0 = -\psi_0^0 \wedge \pi^0 + \frac{i}{2}(\bar{\pi}^1 \wedge \bar{\pi}^2 - \pi^1 \wedge \pi^2), \\ d\pi^1 = -\psi_0^1 \wedge \pi^0 - \psi_1^1 \wedge \pi^1 - \psi_2^1 \wedge \pi^2 + \tau^1, \\ d\pi^2 = -\psi_0^2 \wedge \pi^0 - \psi_1^2 \wedge \pi^1 - \psi_2^2 \wedge \pi^2 + \tau^2, \\ d\bar{\pi}^1 = -\bar{\psi}_0^1 \wedge \pi^0 - \bar{\psi}_1^1 \wedge \bar{\pi}^1 - \bar{\psi}_2^1 \wedge \bar{\pi}^2 + \bar{\tau}^1, \\ d\bar{\pi}^2 = -\bar{\psi}_0^2 \wedge \pi^0 - \bar{\psi}_1^2 \wedge \bar{\pi}^1 - \bar{\psi}_2^2 \wedge \bar{\pi}^2 + \bar{\tau}^2, \end{cases} \tag{3.8}$$

Now we go to the next step which allows us to absorb the maximum of torsion in (3.8) respecting $\psi_1^1 + \psi_2^2 = \bar{\psi}_1^1 + \bar{\psi}_2^2 = \psi_0^0$. First, by change the form $\psi_0^\imath \leftarrow \psi_0^\imath - T_{0*}^\imath \pi^*$ we can consider [1].

$$T_{0*}^\imath = 0.$$

By a change of $\psi_2^1$ and $\psi_1^2$, we can write

$$T_{2\bar{1}}^1 = T_{2\bar{2}}^1 = T_{12}^1 = T_{1\bar{1}}^2 = T_{1\bar{2}}^2 = T_{12}^2 = 0,$$

Respecting $\psi_1^1 + \psi_2^2 = \bar{\psi}_1^1 + \bar{\psi}_2^2 = \psi_0^0$, we can write

$$T_{1\bar{1}}^1 = T_{2\bar{1}}^2 = V_1 \text{ and } T_{2\bar{2}}^2 = T_{1\bar{2}}^1 = V_2,$$

thus (3.8) becomes

$$\begin{cases} d\pi^0 = -\psi_0^0 \wedge \pi^0 + \frac{i}{2}(\bar{\pi}^1 \wedge \bar{\pi}^2 - \pi^1 \wedge \pi^2), \\ d\pi^1 = -\psi_0^1 \wedge \pi^0 - \psi_1^1 \wedge \pi^1 - \psi_2^1 \wedge \pi^2 + V_1 \pi^1 \wedge \bar{\pi}^1 + V_2 \pi^1 \wedge \bar{\pi}^2 + U_1 \bar{\pi}^1 \wedge \bar{\pi}^2, \\ d\pi^2 = -\psi_0^2 \wedge \pi^0 - \psi_1^2 \wedge \pi^1 - \psi_2^2 \wedge \pi^2 + V_1 \pi^2 \wedge \bar{\pi}^1 + V_2 \pi^2 \wedge \bar{\pi}^2 + U_2 \bar{\pi}^1 \wedge \bar{\pi}^2, \\ d\bar{\pi}^1 = -\bar{\psi}_0^1 \wedge \pi^0 - \bar{\psi}_1^1 \wedge \bar{\pi}^1 - \bar{\psi}_2^1 \wedge \bar{\pi}^2 + \bar{V}_1 \bar{\pi}^1 \wedge \pi^1 + \bar{V}_2 \bar{\pi}^1 \wedge \pi^2 + \bar{U}_1 \pi^1 \wedge \pi^2, \\ d\bar{\pi}^2 = -\bar{\psi}_0^2 \wedge \pi^0 - \bar{\psi}_1^2 \wedge \bar{\pi}^1 - \bar{\psi}_2^2 \wedge \bar{\pi}^2 + \bar{V}_1 \bar{\pi}^2 \wedge \pi^1 + \bar{V}_2 \bar{\pi}^2 \wedge \pi^2 + \bar{U}_2 \pi^1 \wedge \pi^2, \end{cases} \tag{3.9}$$

[1] $*, \star \in \{1, 2, \bar{1}, \bar{2}\}$

here $V_\imath$ and $U_\imath$ are the new coefficients of torsion which are expressed in terms of $T^\imath_{\star\star}$.
After calculating $0 \equiv d(d\pi^0)$, we have

$$U_1 = -2\bar{V}_2, \;\; U_2 = 2\bar{V}_1.$$

We calculate $d(d\pi^1) \equiv 0$ and $d(d\pi^2) \equiv 0$, thus we have the relation mod $\{\pi^0, \pi^1, \pi^2\}$.

$$0 \equiv d \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} + \frac{i}{2} \begin{pmatrix} \psi^1_0 \\ \psi^2_0 \end{pmatrix} + \begin{pmatrix} \psi^1_1 & \psi^1_2 \\ \psi^2_1 & \psi^2_2 \end{pmatrix} . \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} - \psi^0_0 \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}. \tag{3.10}$$

Let $G_1$-structure $B_1 \subset B$ in which $\tau^1 = \tau^2 = 0$ and $\varphi^1_0$, $\varphi^2_0$ are semi-basic, we consider the projector $\Phi : B \to B_1$ such that, for $x \in B$ we associate $x.g_0$ is a submersion which respects fibers. Thus $G_1$ is a sub-group acting over $B_1$ generated by matrices of the form

$$g_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \bar{A} \end{pmatrix}. \tag{3.11}$$

Denote by

$$\psi^\imath_0 = P^\imath_0 \pi^0 + P^\imath_* \pi^*, \text{ and } \bar{\psi}^\imath_0 = \bar{P}^\imath_0 \pi^0 + \bar{P}^\imath_* \pi^*,$$

then the structure equations read

$$\begin{cases} d\pi^0 = -\psi^0_0 \wedge \pi^0 + \frac{i}{2}(\bar{\pi}^1 \wedge \bar{\pi}^2 - \pi^1 \wedge \pi^2), \\ d\pi^1 = -\psi^1_1 \wedge \pi^1 - \psi^1_2 \wedge \pi^2 - P^1_* \pi^* \wedge \pi^0, \\ d\pi^2 = -\psi^2_1 \wedge \pi^1 - \psi^2_2 \wedge \pi^2 + P^2_* \pi^* \wedge \pi^0, \end{cases} \tag{3.12}$$

the same way as previous, we absorb the torsion, respecting the condition $\psi^1_1 + \psi^2_2 = \psi^0_0$

$$\begin{cases} d\pi^0 = -\psi^0_0 \wedge \pi^0 + \frac{i}{2}(\bar{\pi}^1 \wedge \bar{\pi}^2 - \pi^1 \wedge \pi^2), \\ d\pi^1 = -\psi^1_1 \wedge \pi^1 - \psi^1_2 \wedge \pi^2 - P\pi^1 \wedge \pi^0 - P^1_{\bar{1}} \bar{\pi}^1 \wedge \pi^0 - P^1_{\bar{2}} \bar{\pi}^2 \wedge \pi^0, \\ d\pi^2 = -\psi^2_1 \wedge \pi^1 - \psi^2_2 \wedge \pi^2 - P\pi^2 \wedge \pi^0 - P^2_{\bar{1}} \bar{\pi}^2 \wedge \pi^0 - P^2_{\bar{2}} \bar{\pi}^2 \wedge \pi^0, \end{cases} \tag{3.13}$$

Respecting $\psi^1_1 + \psi^2_2 = \bar{\psi}^1_1 + \bar{\psi}^2_2 = \psi^0_0$, we can show

$$P + \bar{P} = 0. \tag{3.14}$$

We have

$$0 = -d\psi^0_0 \wedge \pi^0 + \frac{i}{2}\psi^0_0 \wedge \bar{\pi}^1 \wedge \bar{\pi}^2 - \frac{i}{2}\psi^0_0 \wedge \pi^1 \wedge \pi^2$$
$$+\frac{i}{2} d\bar{\pi}^1 \wedge \bar{\pi}^2 - \frac{i}{2}\bar{\pi}^1 \wedge d\bar{\pi}^2 - \frac{i}{2} d\pi^1 \wedge \pi^2 + \frac{i}{2}\pi^1 \wedge d\pi^2,$$

thus

$$2i d\psi^0_0 \wedge \pi^0 = (2\bar{P}\bar{\pi}^1 \wedge \bar{\pi}^2 - 2P\pi^1 \wedge \pi^2 - (P^2_{\bar{1}} + \bar{P}^2_{\bar{1}})\pi^1 \wedge \bar{\pi}^1 + (\bar{P}^1_{\bar{1}} - P^2_{\bar{2}})\pi^1 \wedge \bar{\pi}^2$$
$$+(P^1_{\bar{1}} - \bar{P}^2_{\bar{2}})\pi^2 \wedge \bar{\pi}^1 + (\bar{P}^1_{\bar{2}} - P^1_{\bar{2}})\pi^2 \wedge \bar{\pi}^2) \wedge \pi^0,$$

thus

$$P - \bar{P} = 0,$$

for (3.14), then we have

$$P = 0,$$

then (3.13) reads

$$\begin{cases} d\pi^0 = -\psi^0_0 \wedge \pi^0 + \frac{i}{2}(\bar{\pi}^1 \wedge \bar{\pi}^2 - \pi^1 \wedge \pi^2), \\ d\pi^1 = -\psi^1_1 \wedge \pi^1 - \psi^1_2 \wedge \pi^2 - P^1_{\bar{1}} \bar{\pi}^1 \wedge \pi^0 - P^1_{\bar{2}} \bar{\pi}^2 \wedge \pi^0, \\ d\pi^2 = -\psi^2_1 \wedge \pi^1 - \psi^2_2 \wedge \pi^2 - P^2_{\bar{1}} \bar{\pi}^2 \wedge \pi^0 - P^2_{\bar{2}} \bar{\pi}^2 \wedge \pi^0, \end{cases} \tag{3.15}$$

in particular

$$
\begin{aligned}
2id\psi_0^0 &= -(P_{\bar{1}}^2+\bar{P}_{\bar{1}}^2)\pi^1\wedge\bar{\pi}^1+(\bar{P}_1^1-P_{\bar{2}}^2)\pi^1\wedge\bar{\pi}^2\\
&+(P_{\bar{1}}^1-\bar{P}_{\bar{2}}^2)\pi^2\wedge\bar{\pi}^1+(\bar{P}_{\bar{2}}^1-P_{\bar{2}}^1)\pi^2\wedge\bar{\pi}^2. \qquad (3.16)
\end{aligned}
$$

We define a pair of $2\times 2$ matrix-valued functions on $B_1$ by

$$
S_1=\begin{pmatrix} P_{\bar{1}}^1+\bar{P}_{\bar{2}}^2 & \bar{P}_{\bar{2}}^1+P_{\bar{2}}^1\\ P_{\bar{1}}^2-\bar{P}_{\bar{1}}^2 & \bar{P}_1^1+P_{\bar{2}}^2\end{pmatrix},\quad S_2=\begin{pmatrix} P_{\bar{1}}^1-\bar{P}_{\bar{2}}^2 & \bar{P}_{\bar{2}}^1-P_{\bar{2}}^1\\ P_{\bar{1}}^2+\bar{P}_{\bar{1}}^2 & \bar{P}_1^1-P_{\bar{2}}^2\end{pmatrix}.
$$

**Theorem 3.4.** *An elliptic Monge-Ampère system $(\mathcal{M},\varepsilon)$ satisfies $S_1=S_2=0$ if and only if it is locally equivalent to the Monge-Ampère system for the linear homogeneous Laplace equations.*

*Proof.* If $S_2=0$, then $\psi_0^0$ is closed, from (3.16), $S_2=0$ if and only if for some 1-form $\alpha$ we have

$$d\psi_0^0=\alpha\wedge\pi^0.$$

But $dd=0$, hence $0\equiv-\alpha\wedge d\pi^0$, which gives

$$\alpha\equiv 0 \ \text{mo}\{\pi^0\}$$

Conversely, if $d\psi_0^0=0$, then $S_2=0$. In case $S_1=S_2=0$, then $d\psi_0^0=0$, thus we can locally find a function $\lambda>0$ such that

$$\psi_0^0=\lambda^{-1}d\lambda.$$

In case $S_1=S_2=0$ we can find

$$d(\pi^1\wedge\pi^2)=-\psi_0^0\pi^1\wedge\pi^2,$$

hence, we can write

$$d(\lambda\omega^1\wedge\omega^4)=d(\lambda\omega^3\wedge\omega^2)=d(\lambda\omega^1\wedge\omega^2)=d(\lambda\omega^3\wedge\omega^4)=0.$$

Then locally by (2.5) there exist a functions $x,y,p$ and $q$ such that

$$
\begin{aligned}
-dp\wedge dx &= \lambda\omega^1\wedge\omega^2,\\
-dq\wedge dy &= \lambda\omega^3\wedge\omega^4,\\
-dp\wedge dy &= \lambda\omega^1\wedge\omega^4,\\
-dq\wedge dx &= \lambda\omega^3\wedge\omega^2,
\end{aligned}
$$

Not that

$$d(\lambda\pi^0)=d(\lambda\omega^0)=\frac{i}{2}(\bar{\pi}^1\wedge\bar{\pi}^2-\pi^1\wedge\pi^2)=\lambda(\omega^1\wedge\omega^2+\omega^3\wedge\omega^4)=-dp\wedge dx-dq\wedge dy.$$

By Poincaré lemma, locally there is exist a function $z$, such that

$$\lambda\omega^0=dz-pdx-qdy.$$

Then, in local coordinates, the elliptic Monge-Ampère system is

$$
\begin{aligned}
\varepsilon &= \{\omega^0,\omega^1\wedge\omega^2+\omega^3\wedge\omega^4,\omega^1\wedge\omega^4-\omega^3\wedge\omega^2\}\\
&= \{dz-pdx-qdy,-dp\wedge dx-dq\wedge dy,-dp\wedge dy+dq\wedge dx\}.
\end{aligned}
$$

∎

It's natural to ask about the situation in which $S_2=0$, but possibly $S_1\neq 0$.

**Theorem 3.5.** *An elliptic Monge-Ampère system $(\mathcal{M},\varepsilon)$ satisfies $S_2=0$ if and only if it is locally equivalent to an Euler-Lagrange system.*

*Proof.* The condition for $\varepsilon$ to contain a Poincaré-Cartan form

$$\begin{aligned}\Pi &= \frac{1}{2}\lambda\pi^0 \wedge (\bar{\pi}^1 \wedge \bar{\pi}^2 + \pi^1 \wedge \pi^2) \\ &= \lambda\omega^0 \wedge (\omega^1 \wedge \omega^4 - \omega^3 \wedge \omega^2).\end{aligned}$$

We can assume that $\Pi$ to be closed for some $\lambda > 0$ on $B_1$. By differentiating then

$$0 = (d\lambda - 2\lambda\psi_0^0) \wedge \omega^0 \wedge (\omega^1 \wedge \omega^4 - \omega^3 \wedge \omega^2).$$

Exterior algebra, for some function $\mu$, say

$$d\lambda - 2\lambda\psi_0^0 = \mu\lambda\omega^0.$$

In other words,

$$d(\log\lambda) - 2\psi_0^0 = \mu\omega^0.$$

Hence

$$d\psi_0^0 \equiv 0 \bmod \{\omega^0\}.$$

But we know that

$$d\omega^0 = \omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4.$$

Then (3.16), gives $S_2 = 0$. ∎

3.4. **Remark in Cartan's test.**

**Definition 3.6.** If $(\pi^0, \pi^1, \pi^2)$ be a lifted coframe, then the associated Exterior Differential System, with equivalence condition $\pi^0 \wedge \pi^1 \wedge \pi^2 \wedge \bar{\pi}^1 \wedge \bar{\pi}^2 \neq 0$, it is involutive if and only if it satisfies the test of Cartan.

To apply equivalence method to some problem there are several steps, one important is Cartan's test. If the problem is involutive we can conclude, if this is not the case, it is necessary to extend the system to continue. We begin, for example, to test the involution in the elliptic case. To find this, it is a process to follow [12, 10], in (3.15), we have $r = 5$ and $n = 3$; to find the reduced characters of Cartan, replace in (3.15) $\psi_\jmath^\imath$ by $z_{\jmath 0}^\imath\pi^0 + z_{\jmath 1}^\imath\pi^1 + z_{\jmath 2}^\imath\pi^2$, we can show

$$(3.17) \qquad \begin{cases} z_{0\jmath}^0 = 0 \quad \jmath = 0, ..., 2, \\ z_{10}^1 = 0, \;\; z_{11}^1 = z_{12}^2, \\ z_{20}^1 = 0, \;\; z_{12}^1 = z_{21}^1, z_{22}^1, \\ z_{10}^2 = 0, \;\; z_{11}^2, \end{cases}$$

The four parameters $z_{11}^1, z_{12}^1, z_{22}^1, z_{11}^2$ can be chosen arbitrarily, thus the degree of indeterminancy $r^{(1)}$ of a lifted coframe is the number of free variables in the solution to the associaled linear absorption system

$$r^{(1)} = 4.$$

Let be $X = (x^0, ..., x^2) \in \mathbb{R}^3$ and the matrix $M$ of size $3 \times 4$ define by

$$M(X) := M_k^{\imath l}(X) := \sum_{\jmath=0}^{2} A_{\jmath k}^{\imath l} x^\jmath, \;\; \imath = 0, ..., 2, \;\; \binom{l}{k} \in \left(\binom{0}{0}, \binom{1}{1}, \binom{1}{2}, \binom{2}{1}\right),$$

where $A_{\jmath k}^{\imath l}$ are a coefficients define in (3.15). In other words

$$M(X) = \left(A_{0k}^{\imath l}(x^0) + A_{1k}^{\imath l}(x^1) + A_{2k}^{\imath l}(x^2)\right)_{\substack{0 \le \imath \le 2 \\ \binom{l}{k} \in \left(\binom{0}{0}, \binom{1}{1}, \binom{1}{2}, \binom{2}{1}\right)}}.$$

Thus

$$M(X) = \begin{pmatrix} -x^0 & 0 & 0 & 0 \\ 0 & -x^1 & -x^2 & 0 \\ -x^2 & x^2 & 0 & -x^1 \end{pmatrix}.$$

For $X=(-1,-1,0)$, thus

$$M=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we denote by $s'_1, ..., s'_3$ the reduced characters Cartan then

$$s'_1 = 3.$$

Now, for $X=(x^0,...,x^2)$ and $Y=(y^0,...,y^2)$, we have

$$\begin{pmatrix} M(X) \\ M(Y) \end{pmatrix} = \begin{pmatrix} -x^0 & 0 & 0 & 0 \\ 0 & -x^1 & -x^2 & 0 \\ -x^2 & x^2 & 0 & -x^1 \\ -y^0 & 0 & 0 & 0 \\ 0 & -y^1 & -y^2 & 0 \\ -y^2 & y^2 & 0 & -y^1 \end{pmatrix}.$$

For $X=(-1,-1,0)$ and $Y=(0,0,-1)$, we have $s'_1+s'_2=4$, thus

$$s'_2 = 1.$$

Or we have $s'_1+s'_2+s'_3=r=4$, then

$$s'_3 = 0,$$

thus we have

$$s'_1 + 2s'_2 + 3s'_3 = 5 > r^{(1)} = 4.$$

Hence the system (3.15) is not satisfies Cartan's test, thus it's necessary to extend the system to continue. Note that before the step of normalizing, the system (3.9) satisfies Cartan's test, I gave a proof of this in my thesis [11]. This leads to further investigations.

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