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END-POINT AND TRANSVERSALITY CONDITIONS IN THE CALCULUS OF VARIATIONS: DERIVATIONS THROUGH DIRECT REASONING

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ABSTRACT. We offer an intuitive explanation of the end-point and transversality conditions that complement the Euler equation in the calculus of variations. Our reasoning is based upon the fact that any variation given to an optimal function must entail a zero net gain to the functional, all consequences of implied changes in its derivative being fully taken into account.

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1. INTRODUCTION.

While necessary conditions for determining extremals of a differentiable function defined over an open interval are obvious (its gradient should be equal to zero), optimizing functionals requires rules that seem much less obvious. For instance, for most people who are not teaching the calculus of variations, it definitely requires some effort to remember that first order conditions of maximizing the functionals $\int_{x_0}^{x_1} F[x, y(x), y'(x)]\, dx$ or $\int\int_D G[x, y, z(x, y), z_x, z_y] dxdy$ where the domain of integration is fixed are the Euler equation $F_y - \frac{d}{dx}F_{y'} = 0$ and the Euler-Ostrogradski equation $G_z - \frac{\partial}{\partial x}G_p - \frac{\partial}{\partial y}G_q = 0$, where $p \equiv z_x$ and $q \equiv z_y$.

In [1] we showed that these equations were not as arcane as they looked, and that on the contrary they could be intuitively derived at one stroke. Our reasoning was based upon the geometrical approach used by Euler and the fact that if optimal functions $y(x)$ or $z(x, y)$ existed, they must be such that at any of their points any small variation imparted to them should entail a zero net advantage to the functional, taking into account the incidence of that variation on the slopes of y or z.

We show here that this reasoning can be extended to obtain directly the classical end-point and transversality conditions, which apparently offer little intuitive sense, particularly if the end value of y is fixed or if $y(x_1)$ can move along a certain curve $g(x)$ at the teminal point.

1. A reminder of terminal point conditions.

Suppose that we wish to maximize $\int_{x_0}^{x_1} F[x, y(x), y'(x)]\, dx$; assume the initial point $(x_{0,}y_0)$ is fixed – the case where the initial point is movable could be treated in an analogous way. Besides the Euler equation, necessary conditions for $y(x)$ to be an optimal solution are the summarized in the following table:

Table 1.1: Conditions to be met at $x = x_1$ according to the nature of the end point, additional to the Euler equation.

Nature of end point	Condition additional to $F_y - \frac{d}{dx}F_{y'} = 0$ at $x = x_1$
a) both x_1 and y_1 are free	$F = 0$ and $F_{y'} = 0$
b) x_1 is fixed and y_1 is free	$F_{y'} = 0$
c) y_1 is fixed and x_1 is free	$F - y'F_{y'} = 0$
d) $y(x_{1)} = g(x_1)$, $g(x)$ fixed	$F + (g' - y')F_{y'} = 0$

2. The logic behind those conditions.

In all treatises, those conditions are demonstrated either through the differential or the derivative approach (see for instance [2], [3], [4]). We now offer an intuitive explanation of each of those conditions. For that purpose, we suppose without loss of generality that the partial derivatives F_y and $F_{y'}$ are positive.

2.1. Consider first case a) where both x_1 and y_1 are free and suppose that the optimal point (x_1, y_1) has been found. Any increase imparted to x_1 or to y_1 imply the following. First, all consequences, direct and indirect, of a change in y are taken care in the Euler equation $F_y - \frac{d}{dx}F_{y'} = 0$ (analysed in $[1]$).We now have to take into account the fact that x_1 is not fixed any more but that it can move by dx_1. This entails two consequences: first, a benefit in the direct increase of the functional measured by the additional infinitesimal element $F[x_1, y(x_1), y'(x_1)]dx_1$; second, a further benefit due to the possible change of y' at the end point; the accrued value for the functional is $F_{y'}$. Therefore both impacts should be zero, and we must have, at $x = x_1$, $F = 0$ and $F_{y'} = 0$ in addition to $F_y - \frac{d}{dx}F_{y'} = 0$.

2.2. In the case b) (x_1 fixed, y_1 free) there is no possible increase of the integrand due to a change in x_1, but the benefit of a free value of y entailing a change in the slope y' is maintained. This only implies $F_{y'} = 0$ at $x = x_1$ in addition to the Euler equation.

Cases c) and d) require special attention; they correspond to the constraint defined by a fixed terminal value y_1 (case c) and by the fact that y_1 can move along a given curve $g(x)$ (case d).

2.3. Consider first case c) where y_1 is fixed and x_1 is free; a change dx_1 normally would impart at the terminal point a slope equal to $y'(x_1)dx_1$, carrying a gain for the functional measured by $F_{y'}y'dx_1$ at x_1; but since y cannot change at the terminal point, this advantage *cannot be counted any more*: it now must be considered as a cost, to be balanced against the gain $F[x_1, y(x_1), y'(x_1)]dx_1$ mentioned in 2.1. Hence the equality $Fdx_1 = F_{y'}y'dx_1$, equivalent to $F - y'F_{y'} = 0$ at $x = x_1$, the condition additional to $F_y - \frac{d}{dx}F_{y'} = 0$ over $[x_0, x_1]$.

2.4. The final case d) where $y(x)$ can move along a curve $g(x)$ at the terminal point can be treated as a direct extension of the preceding one. While the cost of not benefitting of the own slope of the curve $y(x)$ has still to be borne, we can now account for the gain generated by the slope of $g(x)$, equal to $F_{y'}g'dx_1$ at x_1. This implies the additional condition $F + (g' - y')F_{y'} = 0$ at $x = x_1$.

REFERENCES

[1] OLIVIER DE LA GRANDVILLE, A One-Line Proof of the Euler and Ostrogradski Equations, *The Australian Journal of Mathematical Analysis with Applications*, Volume **9**, Issue *2*, Article 8, pp. 1-6, 2012.

[2] ARNOLD M. ARTHURS, *Complementary Variational Principles*, Clarendon Press, 1980.

[3] LEV ELSGOLC, *Calculus of Variations*, International Series of Monographs in Pure and Applied Mathematics, Pergamon Press, Reading, MA, 1962.

[4] I. GELFAND, and S. Fomin, *Calculus of variations*, Englewood Cliffs, NJ, Prentice Hall, 1963.