The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 11, Issue 1, Article 1, pp. 1-17, 2014

SOME OPERATOR ORDER INEQUALITIES FOR CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

S.S. DRAGOMIR[1,2] AND C.E.M. PEARCE[3,✠]

Received 30 August, 2012; accepted 8 June, 2013; published 19 February, 2014.

[1]MATHEMATICS, SCHOOL OF ENGINEERING & SCIENCE, VICTORIA UNIVERSITY, PO BOX 14428, MELBOURNE CITY, MC 8001, AUSTRALIA.
sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]SCHOOL OF COMPUTATIONAL & APPLIED MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, JOHANNESBURG 2050, SOUTH AFRICA

[3]SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF ADELAIDE, ADELAIDE, AUSTRALIA

ABSTRACT. Various bounds in the operator order for the following operator transform

$$\tilde{f}(A) := \frac{1}{2}\left[f(A) - f((m+M)1_H - A)\right],$$

where A is a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ and $f : [m, M] \to \mathbb{C}$ is a continuous function on $[m, M]$ are given. Applications for the power and logaritmic functions are provided as well.

Key words and phrases: Selfadjoint operators, Functions of Selfadjoint operators, Spectral representation, Inequalities for selfadjoint operators.

2000 *Mathematics Subject Classification.* 47A63; 47A99.

ISSN (electronic): 1449-5910

✠This paper was started when the first author visited the University of Adelaide in October 2011 and is dedicated to Professor C.E.M. Pearce who passed away in 8'th of June 2012 in a tragic accident.

1. INTRODUCTION

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle .,. \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [12, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t\in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, $i.e.$ $f(A)$ is a *positive operator* on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$\text{(P)} \qquad f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A)$$

in the operator order of $B(H)$. We recall that $A \geq B$ in the operator order of $B(H)$ if $\langle Ax, x\rangle \geq \langle Bx, x\rangle$ for any $x \in H$.

For a recent monograph devoted to various inequalities for continuous functions of selfadjoint operators, see [12] and the references therein.

For other recent results see the research papers [2], [3], [4], [13], [14], [15], [16] and the survey papers [1], [9] and [10].

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family*. Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$\langle f(U)x, y\rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y\rangle), \tag{1.1}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y\rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$.

2. TRAPEZOIDAL AND OSTROWSKI TYPE INEQUALITIES IN THE OPERATOR ORDER

Utilising scalar trapezoidal type inequalities, Dragomir obtained in [8] the following results:

Theorem 2.1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$.*

1. If $f:[m,M]\to\mathbb{R}$ is continuous on $[m,M]$, then

$$\left|\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}-f(A)\right| \leq \left[\max_{t\in[m,M]} f(t)-\min_{t\in[m,M]} f(t)\right]1_H. \tag{2.1}$$

2. If $f:[m,M]\to\mathbb{C}$ is continuous and of bounded variation on $[m,M]$, then

$$\begin{aligned}\left|\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}-f(A)\right| &\leq \frac{M1_H-A}{M-m}\bigvee_m^A(f)+\frac{A-m1_H}{M-m}\bigvee_A^M(f)\\ &\leq \left[\frac{1}{2}+\frac{\left|A-\frac{m+M}{2}1_H\right|}{M-m}\right]\bigvee_m^M(f),\end{aligned} \tag{2.2}$$

where $\bigvee_m^A(f)$ denotes the operator generated by the scalar function $[m,M]\ni t\longmapsto\bigvee_m^t(f)\in\mathbb{R}$. The same notation applies for $\bigvee_A^M(f)$.

3. If $f:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[m,M]$, then

$$\begin{aligned}\left|\frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}-f(A)\right| &\leq \frac{M1_H-A}{M-m}\left|f(A)-f(m)1_H\right|+\frac{A-m1_H}{M-m}\left|f(M)1_H-f(A)\right|\\ &\leq \frac{1}{2}(M-m)L1_H.\end{aligned} \tag{2.3}$$

4. If $f:[m,M]\to\mathbb{R}$ is continuous convex on $[m,M]$ with finite lateral derivatives $f'_-(M)$ and $f'_+(m)$, then we have the inequalities:

$$\begin{aligned}0 &\leq \frac{f(m)(M1_H-A)+f(M)(A-m1_H)}{M-m}-f(A)\\ &\leq \frac{(M1_H-A)(A-m1_H)}{M-m}\left[f'_-(M)-f'_+(m)\right]\\ &\leq \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]1_H.\end{aligned} \tag{2.4}$$

When more information is available on the derivative of the function, the following inequalities may be stated as well, see [8]:

Theorem 2.2. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$. Assume that the function $f:I\to\mathbb{C}$ with $[m,M]\subset\mathring{I}$ (the interior of I) is differentiable on $\mathring{I}$.*

1. If the derivative f' is continuous and of bounded variation on $[m,M]$, then we have the inequality

$$\left|\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} - f(A)\right| \leq \frac{(A-m1_H)(M1_H - A)}{M-m}\bigvee_m^M (f') \leq \frac{1}{4}(M-m)\bigvee_m^M (f')\,1_H. \tag{2.5}$$

2. If the derivative f' is Lipschitzian with the constant $K>0$ on $[m,M]$, then we have the inequality

$$\left|\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} - f(A)\right| \leq \frac{1}{2}(M-m)(A-m1_H)(M1_H - A)K \leq \frac{1}{8}(M-m)^2 K1_H. \tag{2.6}$$

The dual case that provides Ostrowski type inequalities in the operator order have been obtained in [7]:

Theorem 2.3. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m,M]$ for some real numbers $m<M$.*

(1) *If $f:[m,M]\to\mathbb{C}$ is a continuous function of bounded variation on $[m,M]$, then we have the inequality*

$$\left|f(A) - \left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)\cdot 1_H\right| \leq \left[\frac{1}{2}1_H + \left|\frac{A - \frac{m+M}{2}1_H}{M-m}\right|\right]\bigvee_m^M (f) \tag{2.7}$$

where $\bigvee_m^M (f)$ denotes the total variation of f on $[m,M]$. The constant $\frac{1}{2}$ is best possible in (2.7).

(2) *If $f:[m,M]\to\mathbb{R}$ is an absolutely continuous function such that there exists the real constants γ and $\Gamma, \gamma<\Gamma$ with the property that $\gamma \leq f'(s) \leq \Gamma$ for almost every $s\in[m,M]$, then we have the following double inequality in the operator order of $B(H)$:*

$$\begin{aligned} -\frac{1}{2}\cdot\frac{\Gamma-\gamma}{M-m}&\left[\left(A - \frac{M\Gamma - m\gamma}{\Gamma-\gamma}\cdot 1_H\right)^2 - \Gamma\gamma\left(\frac{M-m}{\Gamma-\gamma}\right)^2\cdot 1_H\right] \\ &\leq f(A) - \left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)\cdot 1_H \\ &\leq \frac{1}{2}\cdot\frac{\Gamma-\gamma}{M-m}\left[\left(A - \frac{m\Gamma - M\gamma}{\Gamma-\gamma}\cdot 1_H\right)^2 - \Gamma\gamma\left(\frac{M-m}{\Gamma-\gamma}\right)^2\cdot 1_H\right]. \end{aligned}\tag{2.8}$$

(3) *If* $f:[m,M]\to\mathbb{C}$ *is an absolutely continuous function, then we have in the operator order the following inequalities*

$$\left|f(A)-\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)\cdot 1_H\right| \tag{2.9}$$
$$\leq\begin{cases}\left[\frac{1}{4}1_H+\left(\frac{A-\frac{m+M}{2}1_H}{M-m}\right)^2\right](M-m)\left\|f'\right\|_\infty & \text{if } f'\in L_\infty[m,M];\\[2ex] \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{A-m1_H}{M-m}\right)^{p+1}+\left(\frac{M1_H-A}{M-m}\right)^{p+1}\right](M-m)^{\frac{1}{q}}\left\|f'\right\|_q \\ \qquad\qquad \text{if } f'\in L_p[m,M],\ \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\[2ex] \left[\frac{1}{2}1_H+\left|\frac{A-\frac{m+M}{2}1_H}{M-m}\right|\right]\left\|f'\right\|_1.\end{cases}$$

Motivated by the above results we investigate in this paper the problem of bounding in the operator order the following operator transform

$$\tilde{f}(A):=\frac{1}{2}\left[f(A)-f((m+M)1_H-A)\right]$$

where A is a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ and $f:[m,M]\to\mathbb{C}$ is a continuous function on $[m,M]$. Some applications for power and logarithmic functions are provided as well.

The same notation

$$\tilde{f}(t):=\frac{1}{2}\left[f(t)-f(m+M-t)\right]$$

can be used for the scalar function $f:[m,M]\to\mathbb{C}$ and could be seen as a "*measure of asymmetry*" for f.

3. SOME IMMEDIATE BOUNDS FOR $\tilde{f}(A)$

The following result is a natural consequence of Theorem 2.1:

Theorem 3.1. *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A)\subseteq[m,M]$ *for some real numbers* $m<M$.

1. If $f:[m,M]\to\mathbb{R}$ *is continuous on* $[m,M]$, *then*

$$\left|\frac{f(M)-f(m)}{M-m}\left(A-\frac{m+M}{2}1_H\right)-\tilde{f}(A)\right| \tag{3.1}$$
$$\leq\left[\max_{t\in[m,M]}\tilde{f}(t)-\min_{t\in[m,M]}\tilde{f}(t)\right]1_H$$
$$\leq\left[\max_{t\in[m,M]}f(t)-\min_{t\in[m,M]}f(t)\right]1_H.$$

2. If $f:[m,M]\to\mathbb{C}$ is continuous and of bounded variation on $[m,M]$, then

$$\begin{aligned}(3.2)\qquad &\left|\frac{f(M)-f(m)}{M-m}\left(A-\frac{m+M}{2}1_H\right)-\tilde{f}(A)\right| \\ &\le\left[\frac{1}{2}+\frac{\left|A-\frac{m+M}{2}1_H\right|}{M-m}\right]\bigvee_m^M\left(\tilde{f}\right) \\ &\le\left[\frac{1}{2}+\frac{\left|A-\frac{m+M}{2}1_H\right|}{M-m}\right]\bigvee_m^M(f).\end{aligned}$$

3. If $f:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[m,M]$, then

$$\begin{aligned}(3.3)\qquad &\left|\frac{f(M)-f(m)}{M-m}\left(A-\frac{m+M}{2}1_H\right)-\tilde{f}(A)\right| \\ &\le\frac{1}{2}(M-m)L1_H.\end{aligned}$$

Proof. If we write the inequality (2.1) for $\tilde{f}$ we have

$$\begin{aligned}(3.4)\qquad &\left|\frac{\tilde{f}(m)(M1_H-A)+\tilde{f}(M)(A-m1_H)}{M-m}-\tilde{f}(A)\right| \\ &\le\left[\max_{t\in[m,M]}\tilde{f}(t)-\min_{t\in[m,M]}\tilde{f}(t)\right]1_H.\end{aligned}$$

Since

$$\tilde{f}(M)=-\tilde{f}(m)=\frac{f(M)-f(m)}{2},$$

then

$$\frac{\tilde{f}(m)(M1_H-A)+\tilde{f}(M)(A-m1_H)}{M-m}=\frac{f(M)-f(m)}{M-m}\left(A-\frac{m+M}{2}1_H\right)$$

and by (3.4) we deduce the first inequality in (3.1).

If we denote $\delta=\min_{t\in[m,M]}f(t)$ and $\Delta=\max_{t\in[m,M]}f(t)$ then $\delta\le f(t)\le\Delta$ and $-\Delta\le -f(m+M-t)\le-\delta$ which gives that

$$-\frac{1}{2}(\Delta-\delta)\le\tilde{f}(t)\le\frac{1}{2}(\Delta-\delta)$$

therefore

$$\max_{t\in[m,M]}\tilde{f}(t)\le\frac{1}{2}(\Delta-\delta)\ \text{ and }\ -\frac{1}{2}(\Delta-\delta)\le\min_{t\in[m,M]}\tilde{f}(t)$$

which implies that

$$\max_{t\in[m,M]}\tilde{f}(t)-\min_{t\in[m,M]}\tilde{f}(t)\le\Delta-\delta$$

and the second inequality in (3.1) is proved.

The first inequality in (3.2) follows from (2.2).

If f is of bounded variation, then obviously $\tilde{f}$ is of bounded variation and

$$\bigvee_m^M\left(\tilde{f}\right)\le\frac{1}{2}\left[\bigvee_m^M(f)+\bigvee_m^M(f(m+M-\cdot))\right]=\bigvee_m^M(f).$$

This proves the last part of (3.2).

Now, if f is Lipschitzian with the constant $L>0$ then $\tilde{f}$ is also Lipschitzian with at least the same constant L and by (2.3) we deduce the desired result (3.3). ∎

We need the following notation

$$\widehat{g}(s) := \frac{1}{2}\left[g(s) + g(m+M-s)\right], s \in [m, M]$$

where $g : [m, M] \to \mathbb{C}$.

Theorem 3.2. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$.*

(1) *If $f : [m, M] \to \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality*

$$\left|\tilde{f}(A)\right| \leq \left[\frac{1}{2}1_H + \left|\frac{A - \frac{m+M}{2}1_H}{M-m}\right|\right] \bigvee_m^M \left(\tilde{f}\right) \leq \left[\frac{1}{2}1_H + \left|\frac{A - \frac{m+M}{2}1_H}{M-m}\right|\right] \bigvee_m^M (f). \tag{3.5}$$

(2) *If $f : [m, M] \to \mathbb{R}$ is an absolutely continuous function such that there exists the real constants γ and $\Gamma, \gamma < \Gamma$ with the property that $\gamma \leq f'(s) \leq \Gamma$ for almost every $s \in [m, M]$, then we have the following double inequality in the operator order of $B(H)$:*

$$\begin{aligned} &-\frac{1}{2} \cdot \frac{\Gamma-\gamma}{M-m}\left[\left(A - \frac{M\Gamma - m\gamma}{\Gamma-\gamma} \cdot 1_H\right)^2 - \Gamma\gamma\left(\frac{M-m}{\Gamma-\gamma}\right)^2 \cdot 1_H\right] \\ &\leq \tilde{f}(A) \\ &\leq \frac{1}{2} \cdot \frac{\Gamma-\gamma}{M-m}\left[\left(A - \frac{m\Gamma - M\gamma}{\Gamma-\gamma} \cdot 1_H\right)^2 - \Gamma\gamma\left(\frac{M-m}{\Gamma-\gamma}\right)^2 \cdot 1_H\right]. \end{aligned} \tag{3.6}$$

(3) *If $f : [m, M] \to \mathbb{C}$ is an absolutely continuous function, then we have in the operator order the following inequalities*

$$\begin{aligned} &\left|\tilde{f}(A)\right| \\ &\leq \begin{cases} \left[\frac{1}{4}1_H + \left(\frac{A - \frac{m+M}{2}1_H}{M-m}\right)^2\right](M-m)\left\|\widehat{(f')}\right\|_\infty & \text{if } f' \in L_\infty[m, M]; \\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{A - m1_H}{M-m}\right)^{p+1} + \left(\frac{M1_H - A}{M-m}\right)^{p+1}\right](M-m)^{\frac{1}{q}}\left\|\widehat{(f')}\right\|_q & \\ \qquad \text{if } f' \in L_p[m, M],\ \frac{1}{p} + \frac{1}{q} = 1,\ p > 1; & \\ \left[\frac{1}{2}1_H + \left|\frac{A - \frac{m+M}{2}1_H}{M-m}\right|\right]\left\|\widehat{(f')}\right\|_1 & \end{cases} \\ &\leq \begin{cases} \left[\frac{1}{4}1_H + \left(\frac{A - \frac{m+M}{2}1_H}{M-m}\right)^2\right](M-m)\left\|f'\right\|_\infty & \text{if } f' \in L_\infty[m, M]; \\ \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{A - m1_H}{M-m}\right)^{p+1} + \left(\frac{M1_H - A}{M-m}\right)^{p+1}\right](M-m)^{\frac{1}{q}}\left\|f'\right\|_q & \\ \qquad \text{if } f' \in L_p[m, M],\ \frac{1}{p} + \frac{1}{q} = 1,\ p > 1; & \\ \left[\frac{1}{2}1_H + \left|\frac{A - \frac{m+M}{2}1_H}{M-m}\right|\right]\left\|f'\right\|_1. & \end{cases} \end{aligned} \tag{3.7}$$

Proof. Follows by Theorem 2.3 applied for $\tilde{f}$ and observing that

$$\frac{1}{M-m}\int_m^M \tilde{f}(t)\,dt = 0$$

and the fact that if $\gamma \le f'(s) \le \Gamma$ for almost every $s \in [m, M]$, then

$$\begin{aligned} \left(\tilde{f}(s)\right)' &= \frac{1}{2}\left[f(s) - f(m+M-s)\right]' \\ &= \frac{1}{2}\left[f'(s) + f'(m+M-s)\right] \\ &= \widehat{(f')}(s) \in [\gamma, \Gamma] \end{aligned}$$

for almost every $s \in [m, M]$, where we have used the notation

$$\widehat{g}(s) := \frac{1}{2}\left[g(s) + g(m+M-s)\right], s \in [m, M].$$

The last part from (3.7) follows from the fact that

$$\begin{aligned} \|\widehat{g}\|_q &\le \frac{1}{2}\left[\|g\|_q + \|g(m+M-\cdot)\|_q\right] \\ &= \|g\|_q \end{aligned}$$

for any $q \in [1, \infty]$. ∎

Finally, we can state the following result as well:

Theorem 3.3. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. Assume that the function $f : I \to \mathbb{C}$ with $[m, M] \subset \mathring{I}$ (the interior of I) is differentiable on $\mathring{I}$.*

1. If the derivative f' is continuous and of bounded variation on $[m, M]$, then we have the inequality

$$\begin{aligned} &\left|\frac{f(M)-f(m)}{M-m}\left(A - \frac{m+M}{2}1_H\right) - \tilde{f}(A)\right| \\ &\le \frac{(A - m1_H)(M1_H - A)}{M-m}\bigvee_m^M\left(\widehat{(f')}\right) \\ &\le \frac{1}{4}(M-m)\bigvee_m^M\left(\widehat{(f')}\right)1_H. \end{aligned} \tag{3.8}$$

2. If the derivative f' is Lipschitzian with the constant $K > 0$ on $[m, M]$, then we have the inequality

$$\begin{aligned} &\left|\frac{f(M)-f(m)}{M-m}\left(A - \frac{m+M}{2}1_H\right) - \tilde{f}(A)\right| \\ &\le \frac{1}{2}(M-m)(A - m1_H)(M1_H - A)K \\ &\le \frac{1}{8}(M-m)^2 K1_H. \end{aligned} \tag{3.9}$$

This is a direct consequence of Theorem 2.2 and the details are omitted.

4. OTHER BOUNDS

The following simple bounds for the operator $\left|\tilde{f}(A)\right|$ hold:

Theorem 4.1. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$.*

(1) *If the function $f : [m, M] \to \mathbb{C}$ is continuous, then*

$$\left|\tilde{f}(A)\right| \le \frac{1}{2}\left[\max_{t\in[m,M]} f(t) - \min_{t\in[m,M]} f(t)\right] 1_H. \tag{4.1}$$

(2) *If the function $f : [m, M] \to \mathbb{C}$ is continuous and of bounded variation, then*

$$\left|\tilde{f}(A)\right| \le \frac{1}{2}\bigvee_m^M \left(\tilde{f}\right) 1_H \le \frac{1}{2}\bigvee_m^M (f) 1_H. \tag{4.2}$$

(3) *If the function $f : [m, M] \to \mathbb{C}$ is $r-H$-Hölder continuous, i.e. for fixed $r \in (0, 1]$ and $H > 0$ we have*

$$|f(t) - f(s)| \le |t - s| \text{ for any } t, s \in [m, M],$$

then

$$\left|\tilde{f}(A)\right| \le \frac{1}{2^{1-r}} H \left|A - \frac{m+M}{2} 1_H\right|^r. \tag{4.3}$$

(4) *If the function $f : [m, M] \to \mathbb{C}$ is absolutely continuous on $[m, M]$, then*

$$\left|\tilde{f}(A)\right| \le \begin{cases} \left|A - \frac{m+M}{2} 1_H\right| \|f'\|_\infty & \text{if } f' \in L_\infty[m, M] \\ \\ \frac{1}{2^{1-1/q}}\left|A - \frac{m+M}{2} 1_H\right|^{1/q} \|f'\|_p & \text{if } f' \in L_\infty[m, M], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases} \tag{4.4}$$

Proof. 1. As above, if we denote $\delta = \min_{t\in[m,M]} f(t)$ and $\Delta = \max_{t\in[m,M]} f(t)$ then $\delta \le f(t) \le \Delta$ and $-\Delta \le -f(m + M - t) \le -\delta$ which gives that

$$\left|\tilde{f}(t)\right| \le \frac{1}{2}(\Delta - \delta)$$

for any $t \in [m, M]$.

Applying the property (P) we deduce the desired result.

2. Since $\tilde{f}(M) = -\tilde{f}(m)$, then we have

$$\begin{aligned} \left|\tilde{f}(t)\right| &= \left|\tilde{f}(t) - \frac{\tilde{f}(M) + \tilde{f}(m)}{2}\right| \\ &\le \frac{\left|\tilde{f}(t) - \tilde{f}(m)\right| + \left|\tilde{f}(M) - \tilde{f}(t)\right|}{2} \le \frac{1}{2}\bigvee_m^M \left(\tilde{f}\right), \end{aligned}$$

for any $t \in [m, M]$.

Applying the property (P) we deduce the first inequality in (4.2). The second part was proven before.

3. Utilising the definition, we have

$$\begin{aligned} \left|\tilde{f}(t)\right| &= \frac{1}{2}|f(t) - f(m + M - t)| \\ &\le \frac{1}{2} H |2t - (m + M)|^r = \frac{1}{2^{1-r}} H \left|t - \frac{m+M}{2}\right|^r \end{aligned}$$

for any $t \in [m, M]$.

Applying the property (P) we deduce the desired inequality in (4.3).

4. Since f is absolutely continuous on $[m, M]$, then

$$\left|\tilde{f}(t)\right| = \frac{1}{2}\left|f(t) - f(m + M - t)\right| = \frac{1}{2}\left|\int_t^{m+M-t} f'(s)\, ds\right|$$

for any $t \in [m, M]$.

Utilising the integral Hölder's inequality we have

$$\begin{aligned}
\left|\tilde{f}(t)\right| &\leq \frac{1}{2}\left|\int_t^{m+M-t} \left|f'(s)\right| ds\right| \\
&\leq \frac{1}{2} \times \begin{cases} \left|2t - (m + M)\right| \left\|f'\right\|_\infty & \text{if } f' \in L_\infty[m, M] \\ \\ \left|2t - (m + M)\right|^{1/q} \left\|f'\right\|_p & \begin{array}{c}\text{if } f' \in L_\infty[m, M], \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1.\end{array} \end{cases} \\
&= \begin{cases} \left|t - \frac{m+M}{2}\right| \left\|f'\right\|_\infty & \text{if } f' \in L_\infty[m, M] \\ \\ \frac{1}{2^{1-1/q}}\left|t - \frac{m+M}{2}\right|^{1/q} \left\|f'\right\|_p & \begin{array}{c}\text{if } f' \in L_\infty[m, M], \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1.\end{array} \end{cases}
\end{aligned}$$

for any $t \in [m, M]$.

Applying the property (P) we deduce the desired inequality in (4.4). ∎

The following result the provided upper and lower bounds for $\tilde{f}(A)$ in the operator order of $B(H)$ also holds:

Theorem 4.2. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. Assume that the function $f : I \to \mathbb{C}$ with $[m, M] \subset \mathring{I}$ (the interior of I) is differentiable on $\mathring{I}$. If the derivative f' is continuous and convex on $[m, M]$ then*

$$\begin{aligned}
&\frac{1}{2}\left[f(M) - f(m)\right] 1_H - \frac{f'(m) + f'(M)}{2}\left(M 1_H - A\right) \\
&\leq \tilde{f}(A) \\
&\leq \frac{1}{2}\left[f(M) - f(m)\right] 1_H - f'\left(\frac{m + M}{2}\right)\left(M 1_H - A\right)
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
&f'\left(\frac{m + M}{2}\right)\left(A - m 1_H\right) - \frac{1}{2}\left[f(M) - f(m)\right] 1_H \\
&\leq \tilde{f}(A) \\
&\leq \frac{f'(m) + f'(M)}{2}\left(A - m 1_H\right) - \frac{1}{2}\left[f(M) - f(m)\right] 1_H.
\end{aligned} \tag{4.6}$$

We also have the inequality

$$\begin{aligned}\tag{4.7} &\frac{1}{2}\left[f'\left(\frac{m+M}{2}\right)(A-m1_H)-\frac{f'(m)+f'(M)}{2}(M1_H-A)\right]\\ &\leq \tilde{f}(A)\\ &\leq \frac{1}{2}\left[\frac{f'(m)+f'(M)}{2}(A-m1_H)-f'\left(\frac{m+M}{2}\right)(M1_H-A)\right].\end{aligned}$$

Proof. Let $\{E_\lambda\}_\lambda$ be the *spectral family* of the operator A. For $x \in H, \|x\| = 1$, consider the function $g : [m, M] \to \mathbb{R}$,

$$g(\lambda) := \left\langle \frac{1}{2}(E_\lambda + E_{m+M-\lambda})x, x\right\rangle.$$

Then $g(\lambda) = g(m + M - \lambda)$ for any $\lambda \in [m, M]$, i.e., g is symmetrical on $[m, M]$ and $g(\lambda) \geq 0$ for any $\lambda \in [m, M]$.

By the spectral representation (1.1) we also have that

$$\begin{aligned}\int_{m-0}^{M} g(\lambda)\,d\lambda &= \int_{m-0}^{M}\left\langle \frac{1}{2}(E_\lambda + E_{m+M-\lambda})x, x\right\rangle d\lambda\\ &= \int_{m-0}^{M}\langle E_\lambda x, x\rangle\,d\lambda\\ &= \langle E_\lambda x, x\rangle\,\lambda\big|_{m-0}^{M} - \int_{m-0}^{M}\lambda d\langle E_\lambda x, x\rangle\\ &= \langle (M1_H - A)x, x\rangle\end{aligned}$$

for any $x \in H, \|x\| = 1$.

We use Fejér's inequality, see for instance [11, pp. 1-2], which says that if $h : [a, b] \to \mathbb{R}$ is convex and g is symmetrical on $[a, b]$ and nonnegative, then

$$h\left(\frac{a+b}{2}\right)\int_a^b g(\lambda)\,d\lambda \leq \int_a^b h(\lambda)\,g(\lambda)\,d\lambda \leq \frac{h(a)+h(b)}{2}\int_a^b g(\lambda)\,d\lambda.$$

By writing this inequality for $h = f'$, we can state that

$$\begin{aligned}\tag{4.8} f'\left(\frac{m+M}{2}\right)\int_{m-0}^{M} g(\lambda)\,d\lambda &\leq \int_{m-0}^{M} f'(\lambda)\,g(\lambda)\,d\lambda\\ &\leq \frac{f'(m)+f'(M)}{2}\int_{m-0}^{M} g(\lambda)\,d\lambda,\end{aligned}$$

for any $x \in H, \|x\| = 1$.

Integrating by parts, we observe that

$$\begin{aligned}\tag{4.9} I &:= \int_{m-0}^{M} f'(\lambda)\,g(\lambda)\,d\lambda = f(\lambda)\,g(\lambda)\big|_{m-0}^{M} - \int_{m-0}^{M} f(\lambda)\,dg(\lambda)\\ &= \frac{1}{2}[f(M) - f(m)]\\ &- \frac{1}{2}\left[\int_{m-0}^{M} f(\lambda)\,d(\langle E_\lambda x, x\rangle) + \int_{m-0}^{M} f(\lambda)\,d(\langle E_{m+M-\lambda}x, x\rangle)\right].\end{aligned}$$

Utilising the change of variable $t = m + M - \lambda$ and the spectral representation (1.1), we get that

$$\int_{m-0}^{M} f(\lambda)\, d\left(\langle E_{m+M-\lambda} x, x\rangle\right) = -\langle f((m+M)1_H - A)x, x\rangle$$

for any $x \in H$, $\|x\| = 1$ and since

$$\int_{m-0}^{M} f(\lambda)\, d\left(\langle E_{\lambda} x, x\rangle\right) = \langle f(A)x, x\rangle$$

for any $x \in H$, $\|x\| = 1$, then by (4.9) we obtain

$$I = \frac{1}{2}[f(M) - f(m)] - \frac{1}{2}\langle [f(A) - f((m+M)1_H - A)]x, x\rangle,$$

for any $x \in H$, $\|x\| = 1$.

On making use of the inequality (4.8) we can state that

$$\begin{aligned} & f'\left(\frac{m+M}{2}\right)\langle (M1_H - A)x, x\rangle \\ & \le \frac{1}{2}[f(M) - f(m)] - \frac{1}{2}\langle [f(A) - f((m+M)1_H - A)]x, x\rangle \\ & \le \frac{f'(m) + f'(M)}{2}\langle (M1_H - A)x, x\rangle, \end{aligned}$$

for any $x \in H$, $\|x\| = 1$, which is equivalent with (4.5).

Now, if we replace in the inequality (4.5) the operator A with the operator $(m+M)1_H - A$, then we get the inequality

$$\begin{aligned} & f'\left(\frac{m+M}{2}\right)(A - m1_H) \\ & \le \frac{1}{2}[f(M) - f(m)]1_H + \frac{1}{2}[f(A) - f((m+M)1_H - A)] \\ & \le \frac{f'(m) + f'(M)}{2}(A - m1_H), \end{aligned}$$

which is equivalent with (4.6).

Finally, we observe that the inequality (4.7) is obtained by adding the inequalities (4.5) with (4.6). ∎

The following result may be stated as well:

Theorem 4.3. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. Assume that the function $f : I \to \mathbb{C}$ with $[m, M] \subset \mathring{I}$ (the interior of I) is differentiable on $\mathring{I}$.*

(1) *If $|f'|$ is convex on $[m, M]$, then*

$$\left|\tilde{f}(A)\right| \le (M - m)\left[\frac{|f'(m)| + |f'(M)|}{2}\right]\left[\frac{1}{4} + \left(\frac{A - \frac{m+M}{2}1_H}{M - m}\right)^2\right]. \tag{4.10}$$

(2) *If $|f'|$ is concave on $[m, M]$, then*

$$\left|\tilde{f}(A)\right| \le (M - m)\left|f'\left(\frac{m+M}{2}\right)\right|\left[\frac{1}{4} + \left(\frac{A - \frac{m+M}{2}1_H}{M - m}\right)^2\right]. \tag{4.11}$$

(3) *If* $|f'|$ *is quasiconvex on* $[m,M]$, *then*

$$\left|\tilde{f}(A)\right| \leq (M-m)\max\{|f'(m)|,|f'(M)|\}\left[\frac{1}{4}+\left(\frac{A-\frac{m+M}{2}1_H}{M-m}\right)^2\right]. \tag{4.12}$$

Proof. Integrating by parts in the Riemann integral, we get the following representation:

$$\begin{aligned}\tilde{f}(t) &= \tilde{f}(t) - \frac{1}{M-m}\int_m^M \tilde{f}(s)\,ds \\ &= \frac{1}{M-m}\int_m^t (s-m)\left(\tilde{f}(s)\right)'ds + \frac{1}{M-m}\int_t^M (s-M)\left(\tilde{f}(s)\right)'ds \\ &= \frac{1}{M-m}\int_m^t (s-m)\widehat{(f')}(s)\,ds + \frac{1}{M-m}\int_t^M (s-M)\widehat{(f')}(s)\,ds\end{aligned} \tag{4.13}$$

for any $t\in[m,M]$.

Taking the modulus in (4.14) we get

$$\begin{aligned}&\left|\tilde{f}(t)\right| \\ &\leq \frac{1}{M-m}\int_m^t (s-m)\left|\widehat{(f')}(s)\right|ds + \frac{1}{M-m}\int_t^M (M-s)\left|\widehat{(f')}(s)\right|ds \\ &\leq \frac{1}{M-m}\int_m^t (s-m)\widehat{|f'|}(s)\,ds + \frac{1}{M-m}\int_t^M (M-s)\widehat{|f'|}(s)\,ds\end{aligned} \tag{4.14}$$

for any $t\in[m,M]$.

1. If $|f'|$ is convex on $[m,M]$, then

$$|f'(s)| \leq \frac{(s-m)|f'(M)|+(M-s)|f'(m)|}{M-m}$$

and

$$|f'(m+M-s)| \leq \frac{(s-m)|f'(m)|+(M-s)|f'(M)|}{M-m}$$

for any $s\in[m,M]$.

If we add the above two inequalities and divide by 2, then we get

$$\widehat{|f'|}(s) \leq \frac{1}{2}\left[|f'(m)|+|f'(M)|\right] \tag{4.15}$$

for any $s\in[m,M]$.

On making use of (4.14) and (4.15) we deduce

$$\begin{aligned}
\left|\tilde{f}(t)\right| &\leq \frac{1}{2}\frac{1}{M-m}\left[\left|f'(m)\right|+\left|f'(M)\right|\right] \\
&\quad\times\left[\int_m^t (s-m)\,ds+\int_t^M (M-s)\,ds\right] \\
&= \frac{1}{2}\frac{1}{M-m}\left[\left|f'(m)\right|+\left|f'(M)\right|\right] \\
&\quad\times\left[\frac{(t-m)^2+(M-t)^2}{2}\right] \\
&= \frac{1}{2}\left[\left|f'(m)\right|+\left|f'(M)\right|\right]\left[\frac{1}{4}+\left(\frac{t-\frac{m+M}{2}}{M-m}\right)^2\right](M-m)
\end{aligned} \tag{4.16}$$

for any $t \in [m, M]$.

Applying the property (P) we deduce the desired inequality in (4.10).

2. If $|f'|$ is concave on $[m, M]$, then

$$\widehat{|f'|}(s) = \frac{1}{2}\left[\left|f'(s)\right|+\left|f'(m+M-s)\right|\right] \leq \left|f'\left(\frac{m+M}{2}\right)\right|$$

for any $s \in [m, M]$ and by (4.14) we deduce

$$\begin{aligned}
\left|\tilde{f}(t)\right| &\leq \frac{1}{M-m}\left|f'\left(\frac{m+M}{2}\right)\right| \\
&\quad\times\left[\int_m^t (s-m)\,ds+\int_t^M (M-s)\,ds\right] \\
&= \left|f'\left(\frac{m+M}{2}\right)\right|\left[\frac{1}{4}+\left(\frac{t-\frac{m+M}{2}}{M-m}\right)^2\right](M-m)
\end{aligned} \tag{4.17}$$

for any $t \in [m, M]$.

Applying the property (P) we deduce the desired inequality in (4.11).

3. If $|f'|$ is quasiconvex on $[m, M]$, then

$$\widehat{|f'|}(s) = \frac{1}{2}\left[\left|f'(s)\right|+\left|f'(m+M-s)\right|\right] \leq \max\left\{\left|f'(m)\right|,\left|f'(M)\right|\right\}$$

for any $s \in [m, M]$ from where we similarly get the desired result (4.12). ∎

5. APPLICATIONS

Consider the function $f : [m, M] \to \mathbb{R}$ with $[m, M] \subset (0, \infty)$ given by $f(t) = \ln t$. Then $f'(t) = \frac{1}{t}$ is convex and on making use of Theorem 4.2 we get for any A a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ that

$$
\begin{aligned}
(5.1)\qquad &\ln\sqrt{\frac{M}{m}}1_H-\frac{m+M}{2mM}\left(M1_H-A\right)\\
&\le\frac{1}{2}\left\{\ln A-\ln\left[\left(m+M\right)1_H-A\right]\right\}\\
&\le\ln\sqrt{\frac{M}{m}}1_H-\frac{2}{m+M}\left(M1_H-A\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(5.2)\qquad &\frac{2}{m+M}\left(A-m1_H\right)-\ln\sqrt{\frac{M}{m}}1_H\\
&\le\frac{1}{2}\left\{\ln A-\ln\left[\left(m+M\right)1_H-A\right]\right\}\\
&\le\frac{m+M}{2mM}\left(A-m1_H\right)-\ln\sqrt{\frac{M}{m}}1_H.
\end{aligned}
$$

We also have the inequality

$$
\begin{aligned}
(5.3)\qquad &\frac{1}{2}\left[\frac{2}{m+M}\left(A-m1_H\right)-\frac{m+M}{2mM}\left(M1_H-A\right)\right]\\
&\le\frac{1}{2}\left\{\ln A-\ln\left[\left(m+M\right)1_H-A\right]\right\}\\
&\le\frac{1}{2}\left[\frac{m+M}{2mM}\left(A-m1_H\right)-\frac{2}{m+M}\left(M1_H-A\right)\right].
\end{aligned}
$$

Now, if we use the first statement in Theorem 4.3, then we get

$$
\begin{aligned}
(5.4)\qquad &\frac{1}{2}\left|\ln A-\ln\left[\left(m+M\right)1_H-A\right]\right|\\
&\le\left(M-m\right)\frac{m+M}{2mM}\left[\frac{1}{4}+\left(\frac{A-\frac{m+M}{2}1_H}{M-m}\right)^2\right].
\end{aligned}
$$

Further, if we consider the power function $f:[m,M]\subset(0,\infty)\to\mathbb{R}$, $f(t)=t^p, p>0$ then $f'(t)=pt^{p-1}$and for $p\ge 2$ we have that f' is convex and by Theorem 4.2 we have for any A a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ that

$$
\begin{aligned}
(5.5)\qquad &\frac{1}{2}\left(M^p-m^p\right)1_H-p\frac{m^{p-1}+M^{p-1}}{2}\left(M1_H-A\right)\\
&\le\frac{1}{2}\left[A^p-\left(\left(m+M\right)1_H-A\right)^p\right]\\
&\le\frac{1}{2}\left(M^p-m^p\right)1_H-p\left(\frac{m+M}{2}\right)^{p-1}\left(M1_H-A\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(5.6)\qquad &p\left(\frac{m+M}{2}\right)^{p-1}\left(A-m1_H\right)-\frac{1}{2}\left(M^p-m^p\right)1_H\\
&\le\frac{1}{2}\left[A^p-\left(\left(m+M\right)1_H-A\right)^p\right]\\
&\le p\frac{m^{p-1}+M^{p-1}}{2}\left(A-m1_H\right)-\frac{1}{2}\left(M^p-m^p\right)1_H.
\end{aligned}
$$

We also have the inequality

$$\begin{aligned}(5.7)\qquad &\frac{1}{2}p\left[\left(\frac{m+M}{2}\right)^{p-1}(A-m1_H)-\frac{m^{p-1}+M^{p-1}}{2}(M1_H-A)\right]\\ &\le\frac{1}{2}\left[A^p-((m+M)1_H-A)^p\right]\\ &\le\frac{1}{2}p\left[\frac{m^{p-1}+M^{p-1}}{2}(A-m1_H)-\left(\frac{m+M}{2}\right)^{p-1}(M1_H-A)\right].\end{aligned}$$

Now, if we apply the first statement from Theorem 4.3, then we get for $p\ge 2$ that

$$\begin{aligned}(5.8)\qquad &\frac{1}{2}\left|A^p-((m+M)1_H-A)^p\right|\\ &\le p(M-m)\frac{m^{p-1}+M^{p-1}}{2}\left[\frac{1}{4}+\left(\frac{A-\frac{m+M}{2}1_H}{M-m}\right)^2\right].\end{aligned}$$

By the second statement of the same theorem we also have for $1\le p<2$ that

$$\begin{aligned}(5.9)\qquad &\frac{1}{2}\left|A^p-((m+M)1_H-A)^p\right|\\ &\le p(M-m)\left(\frac{m+M}{2}\right)^{p-1}\left[\frac{1}{4}+\left(\frac{A-\frac{m+M}{2}1_H}{M-m}\right)^2\right].\end{aligned}$$

REFERENCES

[1] R.P. AGARWAL and S.S. DRAGOMIR, A survey of Jensen type inequalities for functions of selfadjoint operators in Hilbert spaces, *Computers and Mathematics with Applications* **59** (2010), pp. 3785-3812.

[2] S.S. DRAGOMIR, Čebyšev's type inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 9. [ONLINE: `http://www.staff.vu.edu.au/RGMIA/v11(E).asp`].

[3] S.S. DRAGOMIR, Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 11. [ONLINE: `http://www.staff.vu.edu.au/RGMIA/v11(E).asp`].

[4] S.S. DRAGOMIR, Inequalities for the Čebyšev functional of two functions of selfadjoint operators in Hilbert spaces, Prerpint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. . [ONLINE: `http://www.staff.vu.edu.au/RGMIA/v11(E).asp`].

[5] S.S. DRAGOMIR, Some trapezoidal vector inequalities for continuous functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **13** (2010), No. 2, Art. 14. [ONLINE: `http://rgmia.org/v13n2.php`].

[6] S.S. DRAGOMIR, Some generalized trapezoidal vector inequalities for continuous functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **13** (2010), Supplement, Art. 14. [ONLINE: `http://rgmia.org/v13(E).php`].

[7] S.S. DRAGOMIR, Ostrowski's type inequalities for some classes of continuous functions of selfadjoint operators in Hilbert spaces, *RGMIA Res. Rep. Coll.* **13** (2010), No. 2, Art. 9. [ONLINE:`http://rgmia.org/v13n2.php`].

[8] S.S. DRAGOMIR, Vector and operator trapezoidal type inequalities for continuous functions of selfadjoint operators in Hilbert spaces, *Electronic Journal of Linear Algebra*, Volume **22**, pp.

161-178, March 2011. Preprint, *RGMIA Res. Rep. Coll.* **14** (2011), Art. 10.[ONLINE: `http://rgmia.org/v14.php`].

[9] S.S. DRAGOMIR, Inequalities for the Čebyšev functional of two functions of selfadjoint operators in Hilbert spaces, *Austral. J. Math. Anal. Appl.*, Volume **6**, Issue 1, Article 7, pp. 1-58, 2009.

[10] S.S. DRAGOMIR, Ostrowski's type inequalities for continuous functions of selfadjoint operators on Hilbert spaces: a survey of recent results, *Ann. Funct. Anal.* **2** (2011), no. 1, pp. 139–205.

[11] S.S. DRAGOMIR and C.E.M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. (ONLINE:`http://rgmia.org/monographs/hermite_hadamard.html`).

[12] T. FURUTA, J. MIĆIĆ HOT, J. PEČARIĆ and Y. SEO, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[13] A. MATKOVIĆ, J. PEČARIĆ and I. PERIĆ, A variant of Jensen's inequality of Mercer's type for operators with applications. *Linear Algebra Appl.* **418** (2006), No. 2-3, pp. 551–564.

[14] B. MOND and J. PEČARIĆ, Convex inequalities in Hilbert spaces, *Houston J. Math.*, **19**(1993), pp. 405-420.

[15] B. MOND and J. PEČARIĆ, Classical inequalities for matrix functions, *Utilitas Math.*, **46**(1994), pp. 155-166.

[16] J. PEČARIĆ, J. MIĆIĆ and Y. SEO, Inequalities between operator means based on the Mond-Pečarić method. *Houston J. Math.* **30** (2004), no. 1, pp. 191–207.