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CERTAIN COMPACT GENERALIZATIONS OF WELL-KNOWN POLYNOMIAL INEQUALITIES

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ABSTRACT. In this paper, certain sharp compact generalizations of well-known Bernstien-type inequalities for polynomials, from which a variety of interesting results follow as special cases, are obtained.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let $\mathscr{P}_n$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n. A famous result known as Bernstein's inequality (for reference, see [8, p.531], [10, p.508] or [11] states that if $P \in \mathscr{P}_n$, then

$$\underset{|z|=1}{Max} |P'(z)| \leq n \underset{|z|=1}{Max} |P(z)|, \tag{1.1}$$

whereas concerning the maximum modulus of $P(z)$ on the circle $|z| = R \geq 1$, we have

$$\underset{|z|=R}{Max} |P(z)| \leq R^n \underset{|z|=1}{Max} |P(z)|, \quad R \geq 1. \tag{1.2}$$

(for reference, see [8, p.442] or [9, vol.I, p.137]).
If we restrict ourselves to the class of polynomials $P \in \mathscr{P}_n$ having no zero in $|z| < 1$, then inequalities (1.1) and (1.2) can be respectively replaced by

$$\underset{|z|=1}{Max} |P'(z)| \leq \frac{n}{2} \underset{|z|=1}{Max} |P(z)|, \tag{1.3}$$

and

$$\underset{|z|=R}{Max} |P(z)| \leq \frac{R^n + 1}{2} \underset{|z|=1}{Max} |P(z)|, \quad R \geq 1. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later verified by Lax [7], whereas inequality (1.4) is due to Ankey and Ravilin [1]. Aziz and Dawood [2] further improved inequalities (1.3) and (1.4) under the same hypothesis and proved that,

$$\underset{|z|=1}{Max} |P'(z)| \leq \frac{n}{2} \left\{ \underset{|z|=1}{Max} |P(z)| - \underset{|z|=1}{Min} |P(z)| \right\}, \tag{1.5}$$

$$\underset{|z|=R}{Max} |P(z)| \leq \frac{R^n + 1}{2} \underset{|z|=1}{Max} |P(z)| - \frac{R^n - 1}{2} \underset{|z|=1}{Min} |P(z)|, \quad R \geq 1. \tag{1.6}$$

Jain [5] generalized both the inequalities (1.3) and (1.4) and proved that if $P \in \mathscr{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$, $|z| = 1$ and $R \geq 1$,

$$\left| zP'(z) + \frac{n\beta}{2} P(z) \right| \leq \frac{n}{2} \left\{ \left| 1 + \frac{\beta}{2} \right| + \left| \frac{\beta}{2} \right| \right\} \underset{|z|=1}{Max} |P(z)|, \tag{1.7}$$

and

$$\left| P(Rz) + \beta \left(\frac{R+1}{2} \right)^n P(z) \right| \leq \frac{1}{2} \left[\left| R^n + \beta \left(\frac{R+1}{2} \right)^n \right| + \left| 1 + \beta \left(\frac{R+1}{2} \right)^n \right| \right] \underset{|z|=1}{Max} |P(z)|. \tag{1.8}$$

Jain [6] obtained a result concerning minimum modulus of polynomials and proved the following result.

Theorem 1.1. *If $P \in \mathscr{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for every real of complex β with $|\beta| \leq 1$,*

$$\underset{|z|=1}{Min} \left| zP'(z) + \frac{n\beta}{2} P(z) \right| \geq n \left| 1 + \frac{\beta}{2} \right| \underset{|z|=1}{Min} |P(z)|. \tag{1.9}$$

As a refinement of inequalities (1.7) and (1.8), Jain [6] proved:

Theorem 1.2. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real of complex β with $|\beta| \le 1$ and $R \ge 1$,*

$$\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \le \frac{n}{2}\left[\left\{\left|1+\frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\}\underset{|z|=1}{Max}|P(z)| - \left\{\left|1+\frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right\}\underset{|z|=1}{Min}|P(z)|\right] \tag{1.10}$$

and

$$\underset{|z|=1}{Max}\left|P(Rz) + \beta\left(\frac{R+1}{2}\right)^n P(z)\right| \le \frac{1}{2}\left[\left\{\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| + \left|1+\beta\left(\frac{R+1}{2}\right)^n\right|\right\}\underset{|z|=1}{Max}|P(z)| - \left\{\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| - \left|1+\beta\left(\frac{R+1}{2}\right)^n\right|\right\}\underset{|z|=1}{Min}|P(z)|\right]. \tag{1.11}$$

Inequalities (1.9) and (1.10) have recently appeared in [4] also.

In this paper, we first present the following interesting result which yields a number of well-known polynomial inequalities as special cases.

Theorem 1.3. *If $P \in \mathscr{P}_n$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \le 1$, $|\beta| \le 1$, $k \le 1$, $R > r \ge 1$ and $|z| \ge 1$,*

$$k^n\left|P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}P(rz)\right| \le |z|^n\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}r^n\right|\underset{|z|=k}{Max}|P(z)|. \tag{1.12}$$

The result is best possible and equality in (1.12) *holds for* $P(z) = az^n, a \neq 0$.
If we choose $\alpha = 0$ in Theorem 1.3, we get the following result.

Corollary 1.4. *If $P \in \mathscr{P}_n$, then for $\beta \in \mathbb{C}$ with $|\beta| \le 1$, $k \le 1$, $R > r \ge 1$ and $|z| \ge 1$,*

$$k^n\left|P(Rz) + \beta\left(\frac{R+k}{k+r}\right)^n P(rz)\right| \le |z|^n\left|R^n + \beta r^n\left(\frac{R+k}{k+r}\right)^n\right|\underset{|z|=k}{Max}|P(z)| \tag{1.13}$$

Equality in (1.13) *holds for* $P(z) = az^n, a \neq 0$.
Dividing the two sides of (1.12) by $R - r$ with $\alpha = 1$ and then letting $R \to r$, we get,

Corollary 1.5. *If $P \in \mathscr{P}_n$, then for $\beta \in \mathbb{C}$ with $|\beta| \le 1$, $k \le 1$, $R > 1$ and $|z| \ge 1$,*

$$k^n\left|zP'(rz) + \frac{n\beta}{k+r}P(rz)\right| \le nr^n|z|^n\left|\frac{1}{r} + \frac{\beta}{k+r}\right|\underset{|z|=k}{Max}|P(z)|. \tag{1.14}$$

The result is best possible and equality in (1.14) *holds for* $P(z) = az^n, a \neq 0$.
The following compact generalization of inequalities (1.1) and (1.2) immediately follows from Theorem 1.3, by taking $k = 1$ and $\beta = 0$ in (1.12).

Corollary 1.6. *If $P \in \mathscr{P}_n$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \le 1$, $R > r \ge 1$ and $|z| \ge 1$,*

$$|P(Rz) - \alpha P(rz)| \le |z|^n|R^n - \alpha r^n|\underset{|z|=1}{Max}|P(z)|. \tag{1.15}$$

The result is best possible as shown by $P(z) = az^n, a \neq 0.$

Remark 1.1. For $\alpha = 0$, (1.15) reduces to (1.2). For $\alpha = r = 1$, if we divide the two sides of (1.15) by $R-1$ and make $R \to 1$, we get inequality (1.1).

Next, we present the following result which includes Theorem 1.1 as a special case.

Theorem 1.7. *If* $P \in \mathscr{P}_n$ *and* $P(z)$ *has all its zeros in* $|z| \leq k$ *where* $k \leq 1$, *then for* $\alpha, \beta \in \mathbb{C}$ *with* $|\alpha| \leq 1, |\beta| \leq 1$ *and* $R > r \geq 1$,

$$\underset{|z|=1}{Min}\left|P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}P(rz)\right|k^n \geq \left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}r^n\right|\underset{|z|=k}{Min}|P(z)|. \tag{1.16}$$

The result is best possible as shown by $P(z) = az^n, a \neq 0.$
If we divide the two sides of inequality (1.16) by $R-r$, with $\alpha = 1$ and then making $R \to r$ we get.

Corollary 1.8. *If* $P \in \mathscr{P}_n$ *and* $P(z)$ *has all its zeros in* $|z| \leq k$ *where* $k \leq 1$ *then for* $|\beta| \leq 1$ *and* $r \geq 1$,

$$\underset{|z|=1}{Min}\left|zP'(rz) + \frac{n\beta}{k+r}P(rz)\right|k^n \geq nr^n\left|\frac{1}{r} + \frac{\beta}{k+r}\right|\underset{|z|=k}{Min}|P(z)| \tag{1.17}$$

The result is sharp.

Remark 1.2. For $k = r = 1$, inequality (1.16) reduces to Theorem (1.1).

Setting $\beta = 0$ in Theorem 1.7, we obtain:

Corollary 1.9. *If* $P \in \mathscr{P}_n$ *and* $P(z)$ *has all its zeros in* $|z| \leq k$ *where* $k \leq 1$, *then for* $\alpha \in \mathbb{C}$ *with* $|\alpha| \leq 1$ *and* $R > r \geq 1$,

$$\underset{|z|=1}{Min}|P(Rz) - \alpha P(rz)|k^n \geq |R^n - \alpha r^n|\underset{|z|=k}{Min}|P(z)|.$$

.

For polynomials $P \in \mathscr{P}_n$ having no zero in $|z| < k$, we also establish the following result which leads to a compact generalization of inequalities (1.7) and (1.8).

Theorem 1.10. *If* $P \in \mathscr{P}_n$ *and* $P(z)$ *does not vanish in the disk* $|z| < k$ *where* $k \leq 1$, *then for all* $\alpha, \beta \in \mathbb{C}$ *with* $|\alpha| \leq 1, |\beta| \leq 1$, $R > r \geq 1$ *and* $|z| \geq 1$,

$$\left|P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}P(rz)\right| \leq \frac{1}{2}\left[\left|1 - \alpha + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}\right| + \frac{|z|^n}{k^n}\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}r^n\right|\right]\underset{|z|=k}{Max}|P(z)|. \tag{1.18}$$

Remark 1.3. If we take $\alpha = k = r = 1$ in Theorem1.10 and divide the two sides of inequality (1.18) by $R-1$ and then make $R \to 1$, we get inequality (1.7), whereas inequality (1.8) follows from Theorem 1.10 when $\alpha = 0$ and $k = 1$.
For $k = 1$, Theorem 1.10 reduces to the result due to Aziz and Rather [3].

As a refinement of Theorem1.10, we finally prove the following result, which provides a compact generalization of inequalities (1.7), (1.8) and (1.10) as well.

Theorem 1.11. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in the disk $|z| < k, k \leq 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq 1$,*

$$\begin{aligned} \underset{|z|=1}{Max}\left|P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}P(rz)\right| \\ \leq \frac{1}{2}\Bigg[\Bigg\{\frac{1}{k^n}\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}r^n\right| \\ + \left|1 - \alpha + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}\right|\Bigg\}\underset{|z|=k}{Max}|P(z)| \\ -\Bigg\{\frac{1}{k^n}\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}r^n\right| \\ - \left|1 - \alpha + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}\right|\Bigg\}\underset{|z|=k}{Min}|P(z)|\Bigg]. \end{aligned} \tag{1.19}$$

If we take $\alpha = 1$ and divide the two sides of inequality (1.19) by $R - r$ then letting $R \to r$, we get:

Corollary 1.12. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| \leq k$ where $k \leq 1$, then for $|\beta| \leq 1$ and $r \geq 1$,*

$$\begin{aligned} \underset{|z|=1}{Max}\left|zP'(rz) + \frac{n\beta}{k+r}P(rz)\right| \leq & \frac{n}{2}\Bigg[\left\{\frac{r^n}{k^n}\left|\frac{1}{r} + \frac{\beta}{k+r}\right| + \left|\frac{\beta}{k+r}\right|\right\}\underset{|z|=k}{Max}|P(z)| \\ & - \left\{\frac{r^n}{k^n}\left|\frac{1}{r} + \frac{\beta}{k+r}\right| - \left|\frac{\beta}{k+r}\right|\right\}\underset{|z|=k}{Min}|P(z)|\Bigg]. \end{aligned} \tag{1.20}$$

Remark 1.4. For $k = r = 1$, inequality (1.20) reduces to Theorem 1.2.

For $\alpha = 0$, Theorem 1.11 reduces following result.

Corollary 1.13. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < k, k \leq 1$ then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $R > r \geq 1$,*

$$\begin{aligned} \underset{|z|=1}{Max}\left|P(Rz) + \beta\left(\frac{R+k}{k+r}\right)^n P(rz)\right| \\ \leq \frac{1}{2}\Bigg[\left\{\frac{1}{k^n}\left|R^n + \beta r^n\left(\frac{R+k}{k+r}\right)^n\right| + \left|1 + \beta\left(\frac{R+k}{k+r}\right)^n\right|\right\}\underset{|z|=k}{Max}|P(z)| \\ - \left\{\frac{1}{k^n}\left|R^n + \beta r^n\left(\frac{R+k}{k+r}\right)^n\right| - \left|1 + \beta\left(\frac{R+k}{k+r}\right)^n\right|\right\}\underset{|z|=k}{Min}|P(z)|\Bigg]. \end{aligned} \tag{1.21}$$

Corollary1.13 leads to a refinement of inequality (1.11) for $k = 1$.
The following result immediately follows from Theorem1.11 by taking $\beta = 0$ and $k = 1$.

Corollary 1.14. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for all $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $R > r \geq 1$,*

$$\begin{aligned}\underset{|z|=1}{Max}\,|P(Rz) - \alpha P(rz)| \leq & \left(\frac{|R^n - \alpha r^n| + |1-\alpha|}{2}\right)\underset{|z|=1}{Max}\,|P(z)| \\ & - \left(\frac{|R^n - \alpha r^n| - |1-\alpha|}{2}\right)\underset{|z|=1}{Min}\,|P(z)|\,. \end{aligned} \tag{1.22}$$

The result is sharp and extremal polynomial is $P(z) = az^n + b, |a| = |b| \neq 0.$

Remark 1.5. For $\alpha = 0,$ inequality (1.22) reduces to inequality (1.6). Also if we take $r = 1$ and divide the two sides if inequality (1.22) by $R - 1$ with $\alpha = 1$ and let $R \to 1,$ we get inequality (1.5).

2. LEMMAS

For the proof of these theorems, we need the following Lemmas.

Lemma 2.1. *If $P \in \mathscr{P}_n$ and $P(z)$ have all its zeros in $|z| \leq k$ where $k \leq 1$, then for every $R \geq r \geq 1$ and $|z| = 1$,*

$$|P(Rz)| \geq \left(\frac{R+k}{r+k}\right)^n |P(rz)|\,.$$

Proof. Since all the zeros of $P(z)$ lie in $|z| \leq k$, $k \leq 1$ we write

$$P(z) = C\prod_{j=1}^{n}\left(z - r_j e^{i\theta_j}\right),$$

where $r_j \leq k \leq 1$. Now for $0 \leq \theta < 2\pi$, $R > r \geq 1$, we have

$$\begin{aligned}\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}}\right| &= \left\{\frac{R^2 + r_j^2 - 2Rr_j\cos(\theta - \theta_j)}{r^2 + r_j^2 - 2rr_j\cos(\theta-\theta_j)}\right\}^{1/2}, \\ &\geq \left\{\frac{R+r_j}{r+r_j}\right\}, \\ &\geq \left\{\frac{R+k}{r+k}\right\}, \text{for } \ j = 1, 2, \cdots, n.\end{aligned}$$

Hence

$$\begin{aligned}\left|\frac{P(Re^{i\theta})}{P(re^{i\theta})}\right| &= \prod_{j=1}^{n}\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}}\right|, \\ &\geq \prod_{j=1}^{n}\left(\frac{R+k}{r+k}\right), \\ &= \left(\frac{R+k}{r+k}\right)^n,\end{aligned}$$

for $0 \leq \theta < 2\pi$. This implies for $|z| = 1$ and $R > r \geq 1$,

$$|P(Rz)| \geq \left(\frac{R+k}{r+k}\right)^n |P(rz)|\,,$$

which completes the proof of Lemma 2.1. ∎

Lemma 2.2. *If $F \in \mathscr{P}_n$ and $F(z)$ has all its zeros in the disk $|z| \leq k$ where $k \leq 1$ and $P(z)$ is a polynomial of degree at most n such that*

$$|P(z)| \leq |F(z)| \quad for \quad |z| = k,$$

then for arbitrary real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,

$$\left| P(Rz) - \alpha P(rz) + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} P(rz) \right|$$

$$\leq \left| F(Rz) - \alpha F(rz) + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} F(rz) \right|. \tag{2.1}$$

Proof. Since polynomial $F(z)$ of degree n has all its zeros in $|z| \leq k$ and $P(z)$ is a polynomial of degree at most n such that

$$|P(z)| \leq |F(z)| \quad \text{for} \quad |z| = k, \tag{2.2}$$

therefore, if $F(z)$ has a zero of multiplicity s at $z = ke^{i\theta_0}$, then $P(z)$ has a zero of multiplicity at least s at $z = ke^{i\theta_0}$. If $P(z)/F(z)$ is a constant, then inequality (2.1) is obvious. We now assume that $P(z)/F(z)$ is not a constant, so that by the maximum modulus principle, it follows that

$$|P(z)| < |F(z)| \ \text{ for } |z| > k \ .$$

Suppose $F(z)$ has m zeros on $|z| = k$ where $0 \leq m < n$, so that we can write

$$F(z) = F_1(z)F_2(z)$$

where $F_1(z)$ is a polynomial of degree m whose all zeros lie on $|z| = k$ and $F_2(z)$ is a polynomial of degree exactly $n - m$ having all its zeros in $|z| < k$. This implies with the help of inequality (2.2) that

$$P(z) = P_1(z)F_1(z)$$

where $P_1(z)$ is a polynomial of degree at most $n - m$. Again, from inequality (2.2), we have

$$|P_1(z)| \leq |F_2(z)| \ \text{ for } |z| = k$$

where $F_2(z) \neq 0 \ for \ |z| = k$. Therefore for every real or complex number λ with $|\lambda| > 1$, a direct application of Rouche's Theorem shows that the zeros of the polynomial $P_1(z) - \lambda F_2(z)$ of degree $n - m \geq 1$ lie in $|z| < k$. Hence the polynomial

$$G(z) = F_1(z)\left(P_1(z) - \lambda F_2(z)\right) = P(z) - \lambda F(z)$$

has all its zeros in $|z| \leq k$ with at least one zero in $|z| < k$, so that we can write

$$f(z) = (z - te^{i\delta})H(z)$$

where $t < k$ and $H(z)$ is a polynomial of degree $n-1$ having all its zeros in $|z| \leq k$. Applying Lemma 2.1 to the polynomial $H(z)$, we obtain for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned}
|G(Re^{i\theta})| &= |Re^{i\theta} - te^{i\delta}||H(Re^{i\theta})| \\
&\geq |Re^{i\theta} - te^{i\delta}| \left(\frac{R+k}{k+r} \right)^{n-1} |H(re^{i\theta})|, \\
&= \left(\frac{R+k}{k+r} \right)^{n-1} \frac{|Re^{i\theta} - te^{i\delta}|}{|re^{i\theta} - te^{i\delta}|} |(re^{i\theta} - te^{i\delta})H(re^{i\theta})|, \\
&\geq \left(\frac{R+k}{k+r} \right)^{n-1} \left(\frac{R+t}{r+t} \right) |G(re^{i\theta})|.
\end{aligned}$$

This implies for $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$\left(\frac{r+t}{R+t}\right)|G(Re^{i\theta})| \geq \left(\frac{R+k}{k+r}\right)^{n-1}|G(re^{i\theta})|. \tag{2.3}$$

Since $R > r \geq 1 > t$ so that $G(Re^{i\theta}) \neq 0$ for $0 \leq \theta < 2\pi$ and $\frac{r+k}{k+R} > \frac{r+t}{R+t}$, from inequality (2.3), we obtain

$$|G(Re^{i\theta})| > \left(\frac{R+k}{k+r}\right)^{n}|G(re^{i\theta})|, \quad R > 1 \ \text{ and } \ 0 \leq \theta < 2\pi. \tag{2.4}$$

Equivalently,

$$|G(Rz)| > \left(\frac{R+k}{k+r}\right)^{n}|G(rz)|$$

for $|z| = 1$ and $R > r \geq 1$. Hence for every real or complex number α with $|\alpha| \leq 1$ and $R > r \geq 1$, we have

$$\begin{aligned} |G(Rz) - \alpha G(rz)| &\geq |G(Rz)| - |\alpha|\,|G(rz)| \\ &> \left\{\left(\frac{R+k}{k+r}\right)^{n} - |\alpha|\right\}|G(rz)|, \ \text{ for } \ |z| = 1. \end{aligned} \tag{2.5}$$

Also, inequality (2.4) can be written in the form

$$|G(re^{i\theta})| < \left(\frac{k+r}{R+k}\right)^{n}|G(Re^{i\theta})| \tag{2.6}$$

for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$. Since $G(Re^{i\theta}) \neq 0$ and $\left(\frac{k+r}{R+k}\right)^n < 1$, from inequality (2.6), we obtain for $0 \leq \theta < 2\pi$ and $R > r \geq 1$,

$$|G(re^{i\theta})| < |G(Re^{i\theta})|.$$

That is,

$$|G(rz)| < |G(Rz)| \ \text{ for } \ |z| = 1.$$

Since all the zeros of $G(Rz)$ lie in $|z| \leq (k/R) < 1$, a direct application of Rouche's Theorem shows that the polynomial $G(Rz) - \alpha G(rz)$ has all its zeros in $|z| < 1$ for every real or complex number α with $|\alpha| \leq 1$. Applying Rouche's Theorem again, it follows from (2.5) that for arbitrary real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq 1$, all the zeros of the polynomial

$$\begin{aligned} T(z) &= G(Rz) - \alpha G(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^{n} - |\alpha|\right\}G(rz) \\ &= \left[P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^{n} - |\alpha|\right\}P(rz)\right] \\ &\qquad - \lambda\left[F(Rz) - \alpha F(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^{n} - |\alpha|\right\}F(rz)\right] \end{aligned}$$

lie in $|z| < 1$. This implies

$$\begin{aligned} &\left|P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^{n} - |\alpha|\right\}P(rz)\right| \\ &\qquad \leq \left|F(Rz) - \alpha F(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^{n} - |\alpha|\right\}F(rz)\right| \end{aligned} \tag{2.7}$$

for $|z| \geq 1$ and $R > r \geq 1$. If inequality (2.7) is not true, then there a point $z = z_0$ with $|z_0| \geq 1$ such that

$$\left|P(Rz_0) - \alpha P(rz_0) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}P(rz_0)\right| > \left|F(Rz_0) - \alpha F(rz_0) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}F(rz_0)\right|$$

But all the zeros of $F(Rz)$ lie in $|z| < (k/R) < 1$, therefore, it follows (as in case of $G(z)$) that all the zeros of $F(Rz) - \alpha F(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}F(rz)$ lie in $|z| < 1$. Hence

$$F(Rz_0) - \alpha F(rz_0) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}F(rz_0) \neq 0$$

with $|z_0| \geq 1$.We take

$$\lambda = \frac{P(Rz_0) - \alpha P(rz_0) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}P(rz_0)}{F(Rz_0) - \alpha F(rz_0) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}F(rz_0)},$$

then λ is a well defined real or complex number with $|\lambda| > 1$ and with this choice of λ, we obtain $T(z_0) = 0$ where $|z_0| \geq 1$. This contradicts the fact that all the zeros of $T(z)$ lie in $|z| < 1$. Thus (2.7) holds for $|\alpha| \leq 1$, $|\beta| \leq 1$, $|z| \geq 1$, and $R > r \geq 1$. ∎

Lemma 2.3. *If $P \in \mathscr{P}_n$ and $P(z)$ have no zero in $|z| < k, k \leq 1$, then for $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and $|z| \geq 1$*

$$\left|P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}P(rz)\right| \leq k^n\left|Q\left(Rz/k^2\right) - \alpha Q\left(rz/k^2\right) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}Q\left(rz/k^2\right)\right| \tag{2.8}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.

Proof. By hypothesis, the polynomial $P(z) \neq 0$ in $|z| < k$, where $k \leq 1$. Therefore, all the zeros of polynomial $Q(z/k^2)$ lie in $|z| < k \leq 1$. As

$$|k^nQ(z/k^2)| = |P(z)| \quad \text{for} \quad |z| = k,$$

Applying Lemma2.2 with $F(z)$ replaced by $k^nQ(z/k^2)$, we get for arbitrary real or complex numbers α, β with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,

$$\left|P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}P(rz)\right| \leq k^n\left|Q\left(Rz/k^2\right) - \alpha Q\left(rz/k^2\right) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}Q\left(rz/k^2\right)\right|$$

This proves the Lemma 2.3. ∎

Lemma 2.4. *If $P \in \mathscr{P}_n$ and $Q(z) = z^n\overline{P(1/\overline{z})}$ then for $\alpha, \beta \in \mathbb{C}$,with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$, $k \leq 1$ and $|z| \geq 1$,*

$$\left|P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}P(rz)\right| + k^n\left|Q(Rz/k^2) - \alpha Q(rz/k^2) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}Q(rz/k^2)\right|$$
$$\leq \left[\frac{|z|^n}{k^n}\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}r^n\right| + \left|1 - \alpha + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}\right|\right]\underset{|z|=k}{Max}|P(z)|. \tag{2.9}$$

Proof. Let $M = Max_{|z|=k}|P(z)|$, then by Rouche's Theorem, the polynomial $F(z) = P(z) - \mu M$ does not vanish in $|z| < k$ for every $\mu \in \mathbb{C}$ with $|\mu| > 1$. Applying Lemma 2.3 to polynomial $F(z)$, we get for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| \geq 1$,

$$\left|F(Rz) - \alpha F(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}F(rz)\right| \leq k^n\left|H(Rz/k^2) - \alpha H(rz/k^2) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}H(rz/k^2)\right|$$

where $H(z) = z^n\overline{F(1/\overline{z})} = z^n\overline{P(1/\overline{z})} - \overline{\mu}Mz^n$. Replacing $F(z)$ by $P(z) - \mu M$ and $H(z)$ by $Q(z) - \overline{\mu}Mz^n$, we have for $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| \geq 1$,

$$\left|P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}P(rz) - \mu\left\{1 - \alpha + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}\right\}M\right|$$
$$\leq k^n\left|Q(Rz/k^2) - \alpha Q(rz/k^2) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}Q(rz/k^2) - \frac{\overline{\mu}}{k^{2n}}\left[R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}r^n\right]Mz^n\right|, \tag{2.10}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$. Choosing argument of λ in the right hand side of inequality (2.10) such that

$$k^n\left|Q(Rz/k^2) - \alpha Q(rz/k^2) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}Q(rz/k^2) - \frac{\overline{\mu}}{k^{2n}}\left[R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}r^n\right]Mz^n\right|$$
$$= \frac{|\overline{\mu}z^n|M}{k^n}\left|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}r^n\right| - k^n\left|Q(Rz/k^2) - \alpha Q(rz/k^2) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}Q(rz/k^2)\right|, \tag{2.11}$$

which is possible by applying Theorem 1.3 to the polynomial $Q(z/k^2)$ and using the fact that $Max_{|z|=k}|Q(z/k^2)| = M/k^n$, we get for $|\alpha| \le 1, |\beta| \le 1$ and $|z| \ge 1$,

$$\begin{aligned}\Bigg| P(Rz) - \alpha P(rz) + \beta \left\{ \left(\frac{R+k}{k+r}\right)^n - |\alpha| \right\} P(rz) \Bigg| \\ - |\mu| \left| \left\{ 1 - \alpha + \beta \left\{ \left(\frac{R+k}{k+r}\right)^n - |\alpha| \right\} \right\} M \right| \\ \le \frac{|\overline{\mu} z^n| M}{k^n} \left| R^n - \alpha r^n + \beta \left\{ \left(\frac{R+k}{k+r}\right)^n - |\alpha| \right\} r^n \right| \\ - k^n \left| Q(Rz/k^2) - \alpha Q(rz/k^2) + \beta \left\{ \left(\frac{R+k}{k+r}\right)^n - |\alpha| \right\} Q(rz/k^2) \right|.\end{aligned}$$

Equivalently for $|\alpha| \le 1, |\beta| \le 1$ and $|z| \ge 1$,

$$\begin{aligned}\Bigg| P(Rz) - \alpha P(rz) + \beta \left\{ \left(\frac{R+k}{k+r}\right)^n - |\alpha| \right\} P(rz) \Bigg| \\ + k^n \left| Q(Rz/k^2) - \alpha Q(z/k^2) + \beta \left\{ \left(\frac{R+k}{k+r}\right)^n - |\alpha| \right\} Q(rz/k^2) \right| \\ \le |\mu| \Bigg[\frac{|z|^n}{k^n} \left| R^n - \alpha r^n + \beta \left\{ \left(\frac{R+k}{k+r}\right)^n - |\alpha| \right\} r^n \right| \\ + \left| 1 - \alpha + \beta \left\{ \left(\frac{R+k}{k+r}\right)^n - |\alpha| \right\} \right| \Bigg] M.\end{aligned}$$

Letting $|\mu| \to 1$, we get the conclusion of Lemma 2.4 and this completes proof of Lemma 2.4. ∎

3. PROOFS OF THE THEOREMS

Proof of Theorem 1.3. Let $M = Max_{|z|=k}|P(z)|$. Then the polynomial $F(z) = Mz^n/k^n$ has all its zeros in $|z| \le k$ where $k \le 1$ and,

$$|P(z)| \le |F(z)| \text{ for } |z| = k,$$

therefore by Lemma 2.2, for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \le 1$, $|\beta| \le 1$, $R > r \ge 1$ and $|z| \ge 1$,

$$\begin{aligned}\Bigg| P(Rz) - \alpha P(rz) + \beta \left\{ \left(\frac{R+k}{k+r}\right)^n - |\alpha| \right\} P(rz) \Bigg| \\ \le \left| F(Rz) - \alpha F(rz) + \beta \left\{ \left(\frac{R+k}{k+r}\right)^n - |\alpha| \right\} F(rz) \right|.\end{aligned}$$

Replacing $F(z)$ by Mz^n/k^n, we get the conclusion of Theorem 1.3.
∎

Proof of Theorem 1.7. Let $m = Min_{|z|=k}|P(z)|$. If $P(z)$ has zeros on $|z| = k$, $k \le 1$, then the result is trivially true. Assume that all the zeros of $P(z)$ lie in $|z| < k, k \le 1$, so that $m > 0$. An application of Rouche's Theorem shows that the polynomial $f(z) = k^n P(z) - \lambda m z^n$ has all its zeros in lie $|z| < k, k \le 1$, for every λ with $|\lambda| < 1$. Applying Lemma 2.1 to $f(z)$, we have

$$|f(Rz)| \ge \left(\frac{R+k}{k+r}\right)^n |f(rz)| \quad \text{for} \quad |z| = 1, \quad R > r \ge 1,$$

which implies,

$$|f(Rz)| > |f(rz)| \quad \text{for} \quad R > r \geq 1 \quad \text{and} \quad |z| = 1.$$

Thus by Rouche's Theorem for $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, all the zeros of $F(z) = f(Rz) - \alpha f(rz)$ lie in $|z| < 1$ and we have

$$\begin{aligned} |f(Rz) - \alpha f(rz)| &\geq |f(Rz)| - |\alpha f(rz)| \\ &\geq \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} |f(rz)| \end{aligned}$$

for $|z| = 1$ and $R > r \geq 1$. Again by Rouche's Theorem, it follow that all the zeros of the polynomial

$$\begin{aligned} g(z) =& f(Rz) - \alpha f(rz) + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} f(rz) \\ =& k^n \left[P(Rz) - \alpha P(rz) + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} P(rz) \right] \\ &- \lambda m z^n \left[R^n - \alpha r^n + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} \right] \end{aligned}$$

lie in $|z| < 1$ for $\beta \in \mathbb{C}$ with $|\beta| < 1$. This implies for every λ with $|\lambda| < 1$,

$$\begin{aligned} k^n \Bigg| P(Rz) - \alpha P(rz) + &\beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} P(rz) \Bigg| \\ &\geq |z|^n \left| R^n - \alpha r^n + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} r^n \right| m \end{aligned} \tag{3.1}$$

for $|z| \geq 1$. If inequality (3.1) is not true, then there exists a point $z_0 \in \mathbb{C}$ with $|z_0| \geq 1$ such that

$$\begin{aligned} k^n \Bigg| P(Rz_0) - \alpha P(rz_0) + &\beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} P(rz_0) \Bigg| \\ &< |z_0|^n \left| R^n - \alpha r^n + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} r^n \right| m \end{aligned}$$

We take

$$\lambda = \frac{k^n \left[P(Rz_0) - \alpha P(rz_0) + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} P(rz_0) \right]}{m|z_0|^n \left[R^n - \alpha r^n + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} r^n \right]},$$

then $|\lambda| < 1$ and with this choice of λ, we have $g(z_0) = 0$ with $|z_0| \geq 1$, which is a contradiction, since all the zeros of $g(z)$ lie in $|z| < 1$. Hence for $R > r \geq 1$, $|\alpha| \leq 1$, $|\beta| < 1$ and $|z| \geq 1$,

$$\begin{aligned} k^n \Bigg| P(Rz) - \alpha P(rz) + &\beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} P(rz) \Bigg| \\ &\geq |z|^n \left| R^n - \alpha r^n + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} r^n \right| \underset{|z|=k}{Min} |P(z)|. \end{aligned} \tag{3.2}$$

For β with $|\beta| = 1$, (3.2) follows from continuity. Hence inequality (3.2) immediately leads to inequality (1.16) and this completes the proof of Theorem 1.7. ∎

Proof of Theorem 1.10. Since $P(z)$ does not vanish in $|z| < k,\ k \leq 1$, by Lemma 2.3, we have for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,

$$\begin{aligned}\Bigg|P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}P(rz)\Bigg| \\ \leq k^n \Bigg|Q\left(Rz/k^2\right) - \alpha Q\left(rz/k^2\right) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}Q\left(rz/k^2\right)\Bigg| \end{aligned} \tag{3.3}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$. Inequality (3.3) in conjunction with Lemma 2.4 gives for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,

$$\begin{aligned} 2\Bigg|P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}P(rz)\Bigg| \\ \leq \Bigg|P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}P(rz)\Bigg| \\ + k^n\Bigg|Q\left(Rz/k^2\right) - \alpha Q\left(rz/k^2\right) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}Q\left(rz/k^2\right)\Bigg| \\ \leq \Bigg[\frac{|z|^n}{k^n}\Bigg|R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}r^n\Bigg| \\ + \Bigg|1 - \alpha + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}\Bigg|\Bigg] \underset{|z|=k}{Max}|P(z)|, \end{aligned}$$

this completes the proof of Theorem 1.10. ∎

Proof of Theorem 1.11. Let $m = Min_{|z|=k}|P(z)|$. If $P(z)$ has a zero on $|z| = k$, then the result follows from Theorem 1.10. We assume that $P(z)$ has all its zeros in $|z| < k$ where $k \leq 1$ so that $m > 0$. Now for every λ with $|\lambda| < 1$, it follows by Rouche's Theorem $h(z) = P(z) - \lambda m$ does not vanish in $|z| < k$. Applying Lemma 2.2 to the polynomial $h(z)$, we get for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$

$$\begin{aligned}\Bigg|h(Rz) - \alpha h(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}h(rz)\Bigg| \\ \leq k^n\Bigg|Q_1(Rz/k^2) - \alpha Q_1(rz/k^2) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}Q_1(rz/k^2)\Bigg|\end{aligned}$$

where $Q_1(z) = z^n\overline{h(1/\overline{z})} = z^n\overline{P(1/\overline{z})} - \overline{\lambda}mz^n$. Equivalently,

$$\begin{aligned}\Bigg|P(Rz) - \alpha P(rz) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}P(rz) \\ - \lambda\left\{1 - \alpha + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}\right\}m\Bigg| \\ \leq k^n\Bigg|Q\left(Rz/k^2\right) - \alpha Q\left(rz/k^2\right) + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}Q\left(rz/k^2\right) \\ - \overline{\lambda}\left\{R^n - \alpha r^n + \beta\left\{\left(\frac{R+k}{k+r}\right)^n - |\alpha|\right\}r^n\right\}\frac{m}{k^{2n}}\Bigg| \quad \text{for } |z| = 1. \end{aligned} \tag{3.4}$$

where $Q(z)=z^nP(1/\overline{z})$. Since all the zeros of $Q(z/k^2)$ lie in $|z|\leq k$, $k\leq 1$, by Theorem 1.7 applied to $Q(z/k^2)$, we have for $R>r\geq 1$ and $|z|=1$,

$$\begin{aligned}\Bigg|Q\left(Rz/k^2\right)-\alpha Q\left(rz/k^2\right)+&\beta\left\{\left(\frac{R+k}{k+r}\right)^n-|\alpha|\right\}Q(rz/k^2)\Bigg|\\ &\geq\frac{1}{k^n}\left|R^n-\alpha r^n+\beta\left\{\left(\frac{R+k}{k+r}\right)^n-|\alpha|\right\}r^n\right|\underset{|z|=k}{Min}|Q(z/k^2)|\\ &=\frac{1}{k^{2n}}\left|R^n-\alpha r^n+\beta\left\{\left(\frac{R+k}{k+r}\right)^n-|\alpha|\right\}r^n\right|\underset{|z|=k}{Min}|P(z)|\\ &=\frac{1}{k^{2n}}\left|R^n-\alpha r^n+\beta\left\{\left(\frac{R+k}{k+r}\right)^n-|\alpha|\right\}r^n\right|m.\end{aligned}\tag{3.5}$$

Now, choosing the argument of λ on the right hand side of inequality (3.4) such that

$$\begin{aligned}k^n\Bigg|Q\left(Rz/k^2\right)-\alpha Q\left(rz/k^2\right)+\beta\left\{\left(\frac{R+k}{k+r}\right)^n-|\alpha|\right\}Q\left(rz/k^2\right)&\\ -\overline{\lambda}\left\{R^n-\alpha r^n+\beta\left\{\left(\frac{R+k}{k+r}\right)^n-|\alpha|\right\}r^n\right\}\frac{m}{k^{2n}}\Bigg|&\\ =k^n\Bigg[\left|Q\left(Rz/k^2\right)-\alpha Q\left(rz/k^2\right)+\beta\left\{\left(\frac{R+k}{k+r}\right)^n-|\alpha|\right\}Q(rz/k^2)\right|&\\ -|\overline{\lambda}|\left|\left\{R^n-\alpha r^n+\beta\left\{\left(\frac{R+k}{k+r}\right)^n-|\alpha|\right\}r^n\right\}\frac{m}{k^{2n}}\right|\Bigg]&\end{aligned}$$

for $|z|=1$, which is possible by inequality (3.5). We get for $|z|=1$,

$$\begin{aligned}\Bigg|P(Rz)-\alpha P(rz)+&\beta\left\{\left(\frac{R+k}{k+r}\right)^n-|\alpha|\right\}P(rz)\Bigg|\\ &-|\lambda|\left|\left\{1-\alpha+\beta\left\{\left(\frac{R+k}{k+r}\right)^n-|\alpha|\right\}\right\}m\right|\\ \leq k^n\Bigg|Q\left(Rz/k^2\right)&-\alpha Q\left(rz/k^2\right)+\beta\left\{\left(\frac{R+k}{k+r}\right)^n-|\alpha|\right\}Q(rz/k^2)\Bigg|\\ &-\frac{|\lambda|}{k^n}\left|\left\{R^n-\alpha r^n+\beta\left\{\left(\frac{R+k}{k+r}\right)^n-|\alpha|\right\}r^n\right\}m\right|.\end{aligned}$$

Equivalently for $|z|=1, R>r\geq 1$, we have

$$\begin{aligned}\Bigg|P(Rz)-\alpha P(rz)+\beta&\left\{\left(\frac{R+k}{k+r}\right)^n-|\alpha|\right\}P(rz)\Bigg|\\ &-k^n\left|Q\left(Rz/k^2\right)-\alpha Q\left(rz/k^2\right)+\beta\left\{\left(\frac{R+k}{k+r}\right)^n-|\alpha|\right\}Q(rz/k^2)\right|\\ \leq|\lambda|\Bigg[\Bigg|1-\alpha+\beta&\left\{\left(\frac{R+k}{k+r}\right)^n-|\alpha|\right\}\Bigg|\\ &-\frac{1}{k^n}\left|R^n-\alpha r^n+\beta\left\{\left(\frac{R+k}{k+r}\right)^n-|\alpha|\right\}r^n\right|\Bigg]m.\end{aligned}\tag{3.6}$$

Letting $|\lambda| \to 1$ in inequality (3.6), we obtain for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and $|z| = 1$,

$$\left|P(Rz) - \alpha P(rz) + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} P(rz) \right|$$
$$- k^n \left| Q\left(Rz/k^2\right) - \alpha Q\left(rz/k^2\right) + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} Q(rz/k^2) \right|$$
$$\leq \left[\left| 1 - \alpha + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} \right| \right.$$
$$\left. - \frac{1}{k^n} \left| R^n - \alpha r^n + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} r^n \right| \right] m. \tag{3.7}$$

Inequality (3.7) in conjunction with Lemma 2.3 gives for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1, R > r \geq 1$ and $|z| = 1$,

$$2\left|P(Rz) - \alpha P(rz) + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} P(rz) \right|$$
$$\leq \left[\left\{ \frac{1}{k^n} \left| R^n - \alpha r^n + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} r^n \right| \right. \right.$$
$$\left. + \left| 1 - \alpha + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} \right| \right\} \underset{|z|=k}{Max} |P(z)|$$
$$- \left\{ \frac{1}{k^n} \left| R^n - \alpha r^n + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} r^n \right| \right.$$
$$\left. \left. - \left| 1 - \alpha + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} \right| \right\} \underset{|z|=k}{Min} |P(z)| \right],$$

which is equivalent to inequality (1.19) and thus completes the proof of Theorem 1.11. ∎

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