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MULTILINEAR FRACTIONAL INTEGRAL OPERATORS ON HERZ SPACES

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ABSTRACT. We prove the boundedness of the multilinear fractional integral operators of Kenig and Stein type on Herz spaces. We also show that our results are optimal.

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1. INTRODUCTION

Since Lacey and Thiele [6] proved L^p boundedness of the bilinear Hilbert transform, considerable attention has been paid to the study of multilinear singular integral operators. Grafakos and Torres [2] proved the boundedness of multilinear singular integrals on L^p spaces. Kenig and Stein [4] proved the boundedness of multilinear fractional integral operators on L^p spaces. Tang [11] showed the boundedness of multilinear fractional integral operators on Morrey spaces.

On the other hand, many studies have been done for Herz spaces (see, for example, [1], [3] and [8]). Li and Yang [7] considered the boundedness of fractional integral operators on Herz spaces.

In this paper we consider multilinear fractional integral operators on Herz spaces. For simplicity of notation we consider bilinear fractional integral operators.

For the ordinary fractional integral operator J_β the critical index is $q = n/\beta$ (see Theorem 2.1). As we shall see in the next section (see Theorem 2.7), when we consider bilinear fractional integrals on Herz spaces, the subcritical cases are easy. Therefore in this paper we mainly consider the supercritical cases.

2. DEFINITIONS AND THEOREMS

This section is organized as follows. First we recall some known results about the ordinary fractional integral operators J_β. One usually denotes the fractional integrals by I_β, but in this paper we denote bilinear fractional integrals by I_β. Therefore we use this notation. Next we define bilinear fractional integrals I_β and state some known results for I_β. To state our result we need to define new function spaces Lip_ε^λ introduced by the author [5].

The following notation is used: For a set $E \subset \mathbb{R}^n$ we denote the Lebesgue measure of E by $|E|$ and the characteristic function of E by χ_E. We denote the ball of radius R centered at x_0 by $B(x_0, R) = \{x; |x - x_0| < R\}$ and write $A_k = \{x \in \mathbb{R}^n; 2^{k-1} < |x| \leq 2^k\}$ where $k \in \mathbb{Z}$.

First we define homogeneous Herz spaces. Let $0 < p < \infty, 1 \leq q < \infty$ and $\alpha \in \mathbb{R}^1$.

Definition 2.1.

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}); \|f\|_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\int_{A_k} |f(x)|^q dx \right)^{p/q} \right\}^{1/p}.$$

Remark 2.1. $K_q^{0,q}(\mathbb{R}^n) = L^q(\mathbb{R}^n)$.

Next we define the ordinary fractional integral operators.

Definition 2.2.

$$J_\beta f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}} dy \quad \text{for} \quad 0 < \beta < n.$$

The following theorem is well-known (see, for example, [12]).

Theorem 2.1. *J_β is bounded from $L^{q_1}(\mathbb{R}^n)$ to $L^{q_2}(\mathbb{R}^n)$ where $q_1 > 1$ and $1/q_2 = 1/q_1 - \beta/n > 0$.*

Li and Yang [7] proved the boundedness of J_β on Herz spaces.

Theorem 2.2 ([7]). *Let $q_1 > 1, 1/q_2 = 1/q_1 - \beta/n > 0$ and $0 < p < \infty$. Assume that*

$$-n/q_1 + \beta < \alpha < n(1 - 1/q_1). \tag{2.1}$$

Then J_β is bounded from $K_{q_1}^{\alpha,p}(\mathbb{R}^n)$ to $K_{q_2}^{\alpha,p}(\mathbb{R}^n)$.

Remark 2.2. When $\alpha < n(1-1/q_1)$, $\dot{K}^{\alpha,p}_{q_1}(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^n)$. Therefore we always assume this condition (see Theorems 2.7 and 2.11).

When $q \geq n/\beta$, we know the following theorem (see [10] and [12]). We define Lipschitz spaces and modified fractional integral. Let $0 \leq \varepsilon < 1$.

Definition 2.3.

$$Lip_\varepsilon(\mathbb{R}^n) = \Big\{ f; \|f\|_{Lip_\varepsilon} = \sup_Q \inf_c \frac{1}{|Q|^{1+\varepsilon/n}} \int_Q |f(x)-c|dx < \infty \Big\},$$

where the supremum is taken over all balls $Q \subset \mathbb{R}^n$.

We denote $Lip_0(\mathbb{R}^n) = BMO(\mathbb{R}^n)$.

Proposition 2.3. *When* $0 < \varepsilon < 1$,

$$\|f\|_{Lip_\varepsilon} \approx \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^\varepsilon}.$$

Definition 2.4.

$$\widetilde{J}_\beta f(x) = \int_{\mathbb{R}^n} \Big\{ \frac{1}{|x-y|^{n-\beta}} - \frac{1}{|y|^{n-\beta}} \chi_{\{|y|\geq 1\}} \Big\} f(y)dy \quad \text{for} \quad 0 < \beta < n.$$

Theorem 2.4 ([10])**.** *$\widetilde{J}_\beta$ is bounded from $L^q(\mathbb{R}^n)$ to $Lip_{\beta-n/q}(\mathbb{R}^n)$ if*

$$0 \leq \beta - n/q < 1. \tag{2.2}$$

In particular $\widetilde{I}_\beta$ is bounded from $L^{n/\beta}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$.

Next we define bilinear fractional integral operators.

Definition 2.5.

$$I_\beta(f_1, f_2)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f_1(y_1)f_2(y_2)}{|(x-y_1, x-y_2)|^{2n-\beta}} dy_1 dy_2 \quad \text{for} \quad 0 < \beta < 2n,$$

where $|(x-y_1, x-y_2)| = (|x-y_1|^2 + |x-y_2|^2)^{1/2}$

Kenig and Stein [4] proved the next result.

Theorem 2.5 ([4])**.** *Let $1 < q_1, q_2 < \infty$ and $1/q = 1/q_1 + 1/q_2 - \beta/n > 0$. Then I_β is bounded from $L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$:*

$$\|I_\beta(f_1, f_2)\|_q \leq C\|f_1\|_{q_1}\|f_2\|_{q_2}.$$

Throughout this paper, C is a positive constant which is independent of essential parameters and not necessarily same at each occurrence.

When $1/q_1 + 1/q_2 - \beta/n \leq 0$, Tang [11] proved the following.

Definition 2.6.

$$\widetilde{I}_\beta(f_1, f_2)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big(\frac{1}{|(x-y_1, x-y_2)|^{2n-\beta}} - \frac{1}{|(y_1, y_2)|^{2n-\beta}} \chi_{\{|(y_1,y_2)|>1\}} \Big) \times f_1(y_1)f_2(y_2)dy_1dy_2.$$

Theorem 2.6 ([11])**.** *Assume that $0 \leq \beta - n/q_1 - n/q_2 < 1$. Then $\widetilde{I}_\beta$ is bounded from $L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n)$ to $Lip_{\beta-n/q_1-n/q_2}(\mathbb{R}^n)$.*

Remark 2.3. In [11] it is said that I_β is bounded, but I_β should be replaced to $\widetilde{I}_\beta$.

Tang also obtained the boundedness on Morrey spaces. In this paper we consider the boundedness of I_β on Herz spaces.

We can easily obtain the next result.

Theorem 2.7. *Let* $1 < q_1, q_2 < \infty, 1/q = 1/q_1 + 1/q_2 - \beta/n > 0$ *and* $1/p = 1/p_2 + 1/p_2$. *Assume that*

$$\alpha_1 < n(1 - 1/q_1), \quad \alpha_2 < n(1 - 1/q_2) \quad \text{and} \quad \alpha_1 + \alpha_2 > -n/q. \tag{2.3}$$

Then I_β *is bounded from* $K_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n) \times K_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)$ *to* $K_q^{\alpha,p}(\mathbb{R}^n)$ *where* $\alpha = \alpha_1 + \alpha_2$*:*

$$\|I_\beta(f_1, f_2)\|_{K_q^{\alpha,p}} \le C\|f_1\|_{K_{q_1}^{\alpha_1,p_1}}\|f_2\|_{K_{q_2}^{\alpha_2,p_2}}.$$

Remark 2.4. Since $|I_\beta(f_1, f_2)(x)| \le |J_{\beta_1}f_1(x) \cdot J_{\beta_1}f_1(x)|$ where $\beta = \beta_1 + \beta_2$, we can prove Theorem 2.7 by Theorem 2.1 and the following Hörder-type inequality.

Lemma 2.8.

$$\|f_1 \cdot f_2\|_{\dot{K}_q^{\alpha,p}} \le \|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}}\|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}}, \tag{2.4}$$

where $\alpha = \alpha_1 + \alpha_2, 1/p = 1/p_2 + 1/p_2$ *and* $1/q = 1/q_1 + 1/q_2$.

Therefore main purpose of this paper is to consider the case $\beta/n \ge 1/q_1 + 1/q_2$.

For the ordinary fractional integrals J_β the author [5] proved the following. Let $0 \le \varepsilon < 1$ and $\lambda \in \mathbb{R}^1$.

Definition 2.7.

$$\begin{aligned} &Lip_\varepsilon^\lambda(\mathbb{R}^n) \\ &= \left\{ f; \|f\|_{Lip_\varepsilon^\lambda} = \sup_{\substack{x\in\mathbb{R}^n \\ R>0}} \inf_c \frac{1}{(|x|+R)^\lambda} \frac{1}{|B(x,R)|^{1+\varepsilon/n}} \int_{B(x,R)} |f(y) - c|dy < \infty \right\}. \end{aligned}$$

Remark 2.5. Lip_ε^λ is a special case of generalized Campanato spaces introduced by Nakai [9]. $Lip_\varepsilon^0(\mathbb{R}^n) = Lip_\varepsilon(\mathbb{R}^n)$ and $Lip_0^0(\mathbb{R}^n) = BMO(\mathbb{R}^n)$.

$$\|f\|_{Lip_\varepsilon^\lambda} \approx \sup_{\substack{x\in\mathbb{R}^n \\ R>0}} \frac{1}{(|x|+R)^\lambda} \frac{1}{|B(x,R)|^{1+\varepsilon/n}} \int_{B(x,R)} |f(y) - f_{B(x,R)}|dy < \infty,$$

where $f_{B(x,R)} = \frac{1}{|B(x,R)|}\int_{B(x,R)} f(y)dy$.

Theorem 2.9 **([5]).** *Let* $q \ge n/\beta,\ 0 < p < \infty$ *and* $\beta - n/q - 1 < \alpha \le n - n/q$*. If* $0 \le \beta - n/q < 1 + \min(0, \alpha)$*, then* $\widetilde{J}_\beta$ *is bounded from* $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ *to* $Lip_{\beta-n/q}^{-\alpha}(\mathbb{R}^n)$.

Corollary 2.10. *If* $0 \le \beta - n/q < 1$*, then* $\widetilde{J}_\beta$ *is bounded from* $\dot{K}_q^{0,p}(\mathbb{R}^n)$ *to* $Lip_{\beta-n/q}(\mathbb{R}^n)$*. In particular* $\widetilde{J}_\beta$ *is bounded from* $\dot{K}_{n/\beta}^{0,p}(\mathbb{R}^n)$ *to* $BMO(\mathbb{R}^n)$.

Remark 2.6. When $p > q$, $L^q(\mathbb{R}^n) \subset K_q^{0,p}(\mathbb{R}^n)$. Therefore this corollary improves Theorem 2.4.

Our main result is the following.

Theorem 2.11. *Let* $0 < \beta < 2n, 0 < p_1, p_2 < \infty$ *and* $0 \le \beta - n/q_1 - n/q_2 < 1$*. Assume that*

$$\alpha_1 < n(1 - 1/q_1), \quad \alpha_2 < n(1 - 1/q_2), \tag{2.5}$$

and

$$\beta - n/q_1 - n/q_2 - 1 < \alpha_1 + \alpha_2 < n + \beta - n/q_1 - n/q_2. \tag{2.6}$$

Then $\widetilde{I}_\beta$ *is bounded from* $\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n) \times \dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)$ *to* $Lip_{\beta-n/q_1-n/q_2}^{-(\alpha_1+\alpha_2)}(\mathbb{R}^n)$.

Corollary 2.12. *If* $\alpha_1 < n(1-1/q_1), \alpha_2 < n(1-1/q_2), \alpha_1+\alpha_2 = 0$ *and* $0 \le \beta = n/q_1+n/q_2 < 1$. *Then* $\widetilde{I}_\beta$ *is bounded from* $\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n) \times \dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)$ *to* $Lip_{\beta-n/q_1-n/q_2}(\mathbb{R}^n)$*:*

$$\|\widetilde{I}_\beta\|_{Lip_{\beta-n/q_1-n/q_2}} \le C\|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}}\|f_2\|_{\dot{K}_{q_1}^{\alpha_1,p_1}}.$$

In particular $\widetilde{I}_\beta$ *is bounded from* $\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{R}^n)\times\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{R}^n)$ *to* $BMO(\mathbb{R}^n)$ *when* $\beta = n/q_1+n/q_2$.

Remark 2.7. If $\beta \ge n$ then the right-hand side of (2.6) is needless, because $\alpha_1 + \alpha_2 < 2n - n/q_1-n/q_2$ by (2.5). Th spaces $Lip^{-(\alpha_1+\alpha_2)}_{\beta-n/q_1-n/q_2}(\mathbb{R}^n)$ do not depend on p_1 and p_2, but the operator norms $\|I_\beta\|_{\dot{K}_{q_1}^{\alpha_1,p_1}\times\dot{K}_{q_2}^{\alpha_2,p_2}\to Lip^{-(\alpha_1+\alpha_2)}_{\beta-n/q_1-n/q_2}}$ depend on p_1 and p_2.

Taking $p_1 = q_1, p_2 = q_2$ and $\alpha_1 = \alpha_2 = \alpha = 0$ in Theore 2.11 we obtain Theorem 2.6.

In Section 4, we shall show that the condition (2.6) is optimal by giving a counterexample.

3. PROOF OF THEOREM 2.11

In order to prove Theorem 2.11 we prepare some lemmas. We use the following notation: $\tilde{A}_k = A_{k-1} \cup A_k \cup A_{k+1}$ and $\chi_k = \chi_{A_k},\ \tilde{\chi}_k = \chi_{\tilde{A}_k}$. The next lemmas are easily obtained from the definition.

Lemma 3.1.

$$\|f\|_{\dot{K}_q^{\alpha,p}} \approx \Big(\sum_k 2^{k\alpha p}\|f\tilde{\chi}_k\|_q^p\Big)^{1/p}.$$

Lemma 3.2.

$$\int_{|x-y|\ge R}\frac{|f(y)|}{|x-y|^s}dy \le CR^{n-n/q-s}\|f\|_q \quad \text{if} \quad s > n-n/q. \tag{3.1}$$

$$\int_{|x-y|\le R}\frac{|f(y)|}{|x-y|^s}dy \le CR^{n-n/q-s}\|f\|_q \quad \text{if} \quad s < n-n/q. \tag{3.2}$$

Lemma 3.3.

$$\int_{|y|\ge 2^k}\frac{|f(y)|}{|y|^s}dy \le C2^{k(n-n/q-\alpha-s)}\|f\|_{\dot{K}_q^{\alpha,p}} \quad \text{if} \quad n-n/q-\alpha < s. \tag{3.3}$$

$$\int_{|y|\le 2^k}|f(y)|dy \le C2^{k(n-n/q-\alpha)}\|f\|_{\dot{K}_q^{\alpha,p}} \quad \text{if} \quad \alpha < n-n/q. \tag{3.4}$$

Lemma 3.4.

$$\int_{\complement\{2^{k-3}\le|y|\le 2^{k+1}\}}\frac{|f(y)|}{|x_0-y|^s}dy \le C2^{k(n-n/q-\alpha-s)}\|f\|_{\dot{K}_q^{\alpha,p}} \tag{3.5}$$

if $2^{k-2} \le |x_0| \le 2^k$ *and* $n-n/q-s < \alpha < n-n/q$.

Proof. By (3.3) and (3.4), we have

$$\begin{aligned}
&\int_{\complement\{2^{k-3}\le|y|\le 2^{k+1}\}}\frac{|f(y)|}{|x_0-y|^s}dy \le C\int_{|y|\ge 2^{k+1}}\frac{|f(y)|}{|y|^s}dy + C2^{-ks}\int_{|y|\le 2^{k-3}}|f(y)|dy\\
\le& C2^{k(n-n/q-\alpha-s)}\|f\|_{\dot{K}_q^{\alpha,p}} + C2^{-ks}\cdot 2^{k(n-n/q-\alpha)}\|f\|_{\dot{K}_q^{\alpha,p}}\\
\le& C2^{k(n-n/q-\alpha-s)}\|f\|_{\dot{K}_q^{\alpha,p}}.
\end{aligned}$$

□

The following lemma is essential for our proof.

Lemma 3.5. *Let $Q = B(0, 2^k)$ and $2Q = B(0, 2^{k+1})$. Assume the conditions (2.5) and (2.6) in Theorem 2.11. Then*

$$X := \int_{x\in Q}\int_{y_1\in 2Q}\int_{y_2\in 2Q}\frac{|f_1(y_1)||f_2(y_2)|}{|(x-y_1,x-y_2)|^{2n-\beta}}dxdy_1dy_2$$
$$\le C2^{k(-\alpha_1-\alpha_2+n+\beta-n/q_1-n/q_2)}\|f_1\|_{\dot K^{\alpha_1,p_1}_{q_1}}\|f_2\|_{\dot K^{\alpha_2,p_2}_{q_2}}.$$

Proof. By an elementary calculation, we have for $y_1, y_2 \in 2Q$,

$$\int_Q \frac{dx}{|(x-y_1,x-y_2)|^{2n-\beta}} \le \begin{cases} C2^{k(-n+\beta)} & \text{if} \quad n<\beta<2n, \\ C\log\frac{2^{k+3}}{|y_1-y_2|} & \text{if} \quad \beta=n, \\ \frac{C}{|y_1-y_2|^{n-\beta}} & \text{if} \quad 0<\beta<n. \end{cases}$$

When $n < \beta < 2n$, it follows from (3.4) that

$$X \le C2^{k(-n+\beta)}\cdot 2^{k(n-n/q_1-\alpha_1)}\|f\|_{\dot K^{\alpha_1,p_1}_{q_1}}2^{k(n-n/q_2-\alpha_2)}\|f\|_{\dot K^{\alpha_2,p_2}_{q_2}}$$
$$\le C2^{k(-\alpha_1-\alpha_2+n+\beta-n/q_1-n/q_2)}\|f_1\|_{\dot K^{\alpha_1,p_1}_{q_1}}\|f_2\|_{\dot K^{\alpha_2,p_2}_{q_2}}.$$

When $0 < \beta < n$, we write

$$X \le C\sum_{i=-\infty}^{k+1}\sum_{j=-\infty}^{k+1}\iint\frac{|f_1(y_1)|\chi_i(y_1)|f_2(y_2)|\chi_j(y_2)}{|y_1-y_2|^{n-\beta}}dy_1dy_2$$
$$= C\sum_{i=-\infty}^{k+1}\left(\sum_{j=-\infty}^{i-2}+\sum_{j=i-1}^{i+1}+\sum_{j=i+2}^{k+1}\right) =: I+II+III.$$

When $i > k-1$ we define $III = 0$.

First we estimate I. By (3.4) it follows that

$$\sum_{j=-\infty}^{i-2} \le C2^{i(-n+\beta)}\int|f_1(y_1)|\chi_i(y_1)dy_1\int_{|y_2|\le 2^{i-2}}|f_2(y_2)|dy_2$$
$$\le C2^{i(-n+\beta)}\cdot 2^{i(n-n/q_1)}\|f_1\chi_i\|_{q_1}2^{i(n-n/q_2-\alpha_2)}\|f_2\|_{\dot K^{\alpha_2,p_2}_{q_2}},$$

and we obtain

$$I \le C\|f_2\|_{\dot K^{\alpha_2,p_2}_{q_2}}\sum_{i=-\infty}^{k+1}2^{i(-\alpha_1-\alpha_2+n+\beta-n/q_1-n/q_2)}\cdot 2^{i\alpha_1}\|f_1\chi_i\|_{q_1}$$
$$\le C2^{k(-\alpha_1-\alpha_2+n+\beta-n/q_1-n/q_2)}\|f_1\|_{\dot K^{\alpha_1,p_1}_{q_1}}\|f_2\|_{\dot K^{\alpha_2,p_2}_{q_2}}.$$

Next we estimate III. We have

$$III \le C\sum_{i=-\infty}^{k+1}\sum_{j=i+2}^{k+1}2^{j(-n+\beta)}\|f_1\chi_i\|_1\|f_2\chi_j\|_1$$
$$\le C\sum_{j=-\infty}^{k+1}2^{j(\beta-n/q_2)}\|f_2\chi_j\|_{q_2}\sum_{i=-\infty}^{j-2}2^{i(n-n/q_1-\alpha_1)}\cdot 2^{i\alpha_1}\|f_1\chi_i\|_{q_1}$$
$$\le C\|f_1\|_{\dot K^{\alpha_1,p_1}_{q_1}}\sum_{j=-\infty}^{k+1}2^{j(-\alpha_1-\alpha_2+n+\beta-n/q_1-n/q_2)}\cdot 2^{j\alpha_2}\|f_2\chi_j\|_{q_2}$$
$$\le C2^{k(-\alpha_1-\alpha_2+n+\beta-n/q_1-n/q_2)}\|f_1\|_{\dot K^{\alpha_1,p_1}_{q_1}}\|f_2\|_{\dot K^{\alpha_2,p_2}_{q_2}}.$$

Finally we estimate II. By (3.2) it follows that

$$\sum_{j=i-1}^{i+1} \le C \int |f_1(y_1)|\chi_i(y_1)dy_1 \int \frac{|f_2(y_2)|\tilde{\chi}_i(y_2)}{|y_1-y_2|^{n-\beta}}dy_2$$
$$\le C2^{i(n-n/q_1)}\|f_1\chi_i\|_{q_1}2^{i(\beta-n/q_2)}\|f_2\tilde{\chi}_i\|_{q_2},$$

and we obtain

$$II \le C\sum_{i=-\infty}^{k+1} 2^{i(-\alpha_1-\alpha_2+n+\beta-n/q_1-n/q_2)} \cdot 2^{i\alpha_1}\|f_1\chi_i\|_{q_1}2^{i\alpha_2}\|f_2\tilde{\chi}_i\|_{q_2}$$
$$\le C2^{k(-\alpha_1-\alpha_2+n+\beta-n/q_1-n/q_2)}\|f_1\|_{\dot{K}^{\alpha_1,p_1}_{q_1}}\|f_2\|_{\dot{K}^{\alpha_2,p_2}_{q_2}}.$$

When $\beta = n$, the proof is similar. Therefore we omit the details. We use the following inequality.

$$\sum_{i=-\infty}^{k+1} \log\left(\frac{2^{k+3}}{2^{i-2}}\right)2^{i(-\alpha_1-\alpha_2+2n-n/q_1-n/q_2)} \le C2^{k(-\alpha_1-\alpha_2+2n-n/q_1-n/q_2)}$$

where $-\alpha_1-\alpha_2+2n-n/q_1-n/q_2>0$. □

Now we are in a position to prove Theorem 2.11. Our proof is a bilinear version of the argument in [5].

Proof of Theorem 2. Let $\alpha=\alpha_1+\alpha_2$ and $\varepsilon=\beta-n/q_1-n/q_2$. Fix a ball $Q=B(x_0,R)$ and we estimate

$$\inf_c \frac{(|x_0|+R)^\alpha}{|Q|^{1+\varepsilon/n}}\int_Q |\widetilde{I}_\beta f(x)-c|dx.$$

Let k be the least integer such that $Q\subset B(0,2^k)$. Note that

$$|x_0|+R\approx 2^k. \tag{3.6}$$

We consider three cases:

(i) $Q\cap B(0,2^{k-2})\ne\emptyset$,
(ii) $Q\cap B(0,2^{k-2})=\emptyset$ and $R\ge 2^{k-4}$,
(iii) $Q\cap B(0,2^{k-2})=\emptyset$ and $R<2^{k-4}$.

The case (i) or (ii). Note that $|Q|\ge C2^{kn}$ in both cases. We write

$$f_i(x)=f_i(x)\chi_{B(0,2^{k+1})}+f_i(x)\chi_{\complement B(0,2^{k+1})}=:f_i^b(x)+f_i^c(x)\quad (i=1,2).$$

The symbols b and c stand for *ball* and *complement* respectively.

By the linearity of $\widetilde{I}_\beta$, we calculate $\widetilde{I}_\beta$ by dividing into four terms. Essentially we need to estimate the following three terms: $\widetilde{I}_\beta(f_1^b,f_2^b)$, $\widetilde{I}_\beta(f_1^b,f_2^c)$ and $\widetilde{I}_\beta(f_1^c,f_2^c)$.

First we estimate $\widetilde{I}_\beta(f_1^b,f_2^b)$. Let

$$c=-\iint_{|(y_1,y_2)|\ge 1}\frac{f_1^b(y_1)f_2^b(y_2)}{|(y_1,y_2)|^{2n-\beta}}dy_1dy_2.$$

Then $\widetilde{I}_\beta(f_1^b,f_2^b)-c=I_\beta(f_1^b,f_2^b)$. By Lemma 3.5, we have

$$\int_Q |I_\beta(f_1^b,f_2^b)(x)|dx \le C2^{k(-\alpha+n+\beta-n/q_1-n/q_2)}\|f_1\|_{\dot{K}^{\alpha_1,p_1}_{q_1}}\|f_2\|_{\dot{K}^{\alpha_2,p_2}_{q_2}},$$

and we obtain

$$\frac{(|x_0|+R)^\alpha}{|Q|^{1+\varepsilon/n}}\int_Q |I_\beta(f_1^b,f_2^b)(x)|dx \le C\|f_1\|_{\dot K_{q_1}^{\alpha_1,p_1}}\|f_2\|_{\dot K_{q_2}^{\alpha_2,p_2}}.$$

Next we estimate $\widetilde{I}_\beta(f_1^b,f_2^c)$. Let $c=\widetilde{I}_\beta(f_1^b,f_2^c)(x_0)$. Note that by the conditions (2.5) and (2.6), it follows that $2n-\beta+1>n-n/q_2-\alpha_2$. By (3.3) and (3.4), we have for $x\in Q$,

$$\begin{aligned}|\widetilde{I}_\beta(f_1^b,f_2^c)(x)-c| &\le CR\int_{|y_1|\le 2^{k+1}}|f_1(y_1)|dy_1\int_{|y_1|\ge 2^{k+1}}\frac{|f_2(y_2)|}{|y_2|^{2n-\beta+1}}dy_2\\ &\le CR2^{k(n-n/q_1-\alpha_1)}\|f_1\|_{\dot K_{q_1}^{\alpha_1,p_1}}2^{k(-n-n/q_2-\alpha_2+\beta-1)}\|f_2\|_{\dot K_{q_2}^{\alpha_2,p_2}}\\ &\le CR2^{k(-\alpha+\beta-n/q_1-n/q_2-1)}\|f_1\|_{\dot K_{q_1}^{\alpha_1,p_1}}\|f_2\|_{\dot K_{q_2}^{\alpha_2,p_2}},\end{aligned}$$

and we obtain

$$\frac{(|x_0|+R)^\alpha}{|Q|^{1+\varepsilon/n}}\int_Q |I_\beta(f_1^b,f_2^c)(x)|dx \le C\|f_1\|_{\dot K_{q_1}^{\alpha_1,p_1}}\|f_2\|_{\dot K_{q_2}^{\alpha_2,p_2}}.$$

Finally we estimate $\widetilde{I}_\beta(f_1^c,f_2^c)$. Let $c=\widetilde{I}_\beta(f_1^c,f_2^c)(x_0)$. Then for $x\in Q$,

$$|\widetilde{I}_\beta(f_1^c,f_2^c)(x)-c|\le CR\iint\frac{|f_1^c(y_1)||f_2^c(y_2)|}{|(y_1,y_2)|^{2n-\beta+1}}dy_1dy_2.$$

Since $\beta-1<n/q_1+n/q_2+\alpha_1+\alpha_2$ we can take s_1 and s_2 such that

$$s_1<n/q_1+\alpha_1,\ s_2<n/q_2+\alpha_2 \quad\text{and}\quad s_1+s_2=\beta-1.$$

By (3.3) we have

$$\begin{aligned}|\widetilde{I}_\beta(f_1^c,f_2^c)(x)-c| &\le CR\int_{|y_1|\ge 2^{k+1}}\frac{|f_1^c(y_1)|}{|y_1|^{n-s_1}}dy_1\int_{|y_2|\ge 2^{k+1}}\frac{|f_2^c(y_2)|}{|y_2|^{n-s_2}}dy_2\\ &\le CR2^{k(-\alpha+\beta-n/q_1-n/q_2-1)}\|f_1\|_{\dot K_{q_1}^{\alpha_1,p_1}}\|f_2\|_{\dot K_{q_2}^{\alpha_2,p_2}},\end{aligned}$$

and we obtain

$$\frac{(|x_0|+R)^\alpha}{|Q|^{1+\varepsilon/n}}\int_Q |I_\beta(f_1^c,f_2^c)(x)|dx \le C\|f_1\|_{\dot K_{q_1}^{\alpha_1,p_1}}\|f_2\|_{\dot K_{q_2}^{\alpha_2,p_2}}.$$

The case (iii). Let $2Q=B(x_0,2R)$. We write

$$\begin{aligned}f_i(x) &= f_i(x)\chi_{\{2^{k-3}<|x|\le 2^{k+1}\}\setminus 2Q}+f_i(x)\chi_{2Q}+f_i(x)\chi_{\complement\{2^{k-3}<|x|\le 2^{k+1}\}}\\ &=: f_i^a(x)+f_i^b(x)+f_i^c(x)\quad (i=1,2).\end{aligned}$$

The symbol a stands for *annulus*.

By the linearity of I_β, we calculate $I_\beta(f_1,f_2)$ by dividing into nine terms. Essentially we need to estimate the following six terms.

$$\begin{aligned}&I:=\widetilde{I}_\beta(f_1^a,f_2^a), && II:=\widetilde{I}_\beta(f_1^a,f_2^b), && III:=\widetilde{I}_\beta(f_1^a,f_2^c),\\ &IV:=\widetilde{I}_\beta(f_1^b,f_2^b), && V:=\widetilde{I}_\beta(f_1^b,f_2^c), && VI:=\widetilde{I}_\beta(f_1^c,f_2^c).\end{aligned}$$

The estimate of I. Let $c=\widetilde{I}_\beta(f_1^a,f_2^a)(x_0)$. For $x\in Q$,

$$|\widetilde{I}_\beta(f_1^a,f_2^a)(x)-c|\le CR\iint\frac{|f_1^a(y_1)||f_2^a(y_2)|}{|(x_0-y_1,x_0-y_2)|^{2n-\beta+1}}dy_1dy_2.$$

Since $\beta-1<n/q_1+n/q_2$ we can take s_1 and s_2 such that

$$s_1<n/q_1,\ s_2<n/q_2 \quad\text{and}\quad s_1+s_2=\beta-1.$$

By (3.1) we have

$$\begin{aligned}|\widetilde{I}_\beta(f_1^a,f_2^a)(x)-c| &\le CR\int_{|x_0-y_1|\ge 2R}\frac{|f_1^a(y_1)|}{|x_0-y_1|^{n-s_1}}dy_1\int_{|x_0-y_2|\ge 2R}\frac{|f_2^a(y_2)|}{|x_0-y_2|^{n-s_2}}dy_2\\ &\le CR\|f_1\tilde{\chi}_k\|_{q_1}R^{s_1-n/q_1}\|f_2\tilde{\chi}_k\|_{q_2}R^{s_2-n/q_2}\\ &\le CR^{\beta-n/q_1-n/q_2}2^{-k\alpha}\|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}}\|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}},\end{aligned}$$

and we obtain

$$\frac{(|x_0|+R)^\alpha}{|Q|^{1+\varepsilon/n}}\int_Q|\widetilde{I}_\beta(f_1^a,f_2^a)(x)-c|dx\le C\|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}}\|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}}.$$

The estimate of II. Let $c=\widetilde{I}_\beta(f_1^a,f_2^b)(x_0)$. For $x\in Q$,

$$|\widetilde{I}_\beta(f_1^a,f_2^b)(x)-c|\le CR\int_{|x_0-y_1|>2R}\frac{|f_1^a(y_1)|}{|x_0-y_1|^{2n-\beta+1}}dy_1\int|f_2^b(y_2)|dy_2.$$

Since $\beta<n/q_1+n/q_2+1$ we have $2n-\beta+1>n-n/q_1$. By (3.1) we have

$$\begin{aligned}|\widetilde{I}_\beta(f_1^a,f_2^b)(x)-c| &\le CR\cdot R^{\beta-n-n/q_1-1}\|f_1\tilde{\chi}_k\|_{q_1}R^{n-n/q_2}\|f_2\tilde{\chi}_k\|_{q_2}\\ &\le CR^{\beta-n/q_1-n/q_2}2^{-k\alpha}\|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}}\|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}},\end{aligned}$$

and we obtain

$$\frac{(|x_0|+R)^\alpha}{|Q|^{1+\varepsilon/n}}\int_Q|\widetilde{I}_\beta(f_1^a,f_2^b)(x)-c|dx\le C\|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}}\|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}}.$$

The estimate of III. Let $c=\widetilde{I}_\beta(f_1^a,f_2^c)(x_0)$. For $x\in Q$,

$$|\widetilde{I}_\beta(f_1^a,f_2^c)(x)-c|\le CR\int|f_1^a(y_1)|dy_1\int\frac{|f_2^c(y_2)|}{|x_0-y_2|^{2n-\beta+1}}dy_2.$$

Since $2n-\beta+1>n-n/q_2-\alpha_2$, it follows from (3.4) and (3.5) that

$$\begin{aligned}|\widetilde{I}_\beta(f_1^a,f_2^c)(x)-c| &\le CR2^{k(-\alpha_1+n-n/q_1)}\|f_1\tilde{\chi}_k\|_{\dot{K}_{q_1}^{\alpha_1,p_1}}2^{k(-\alpha_2-n-n/q_2+\beta-1)}\|f_2\tilde{\chi}_k\|_{\dot{K}_{q_2}^{\alpha_2,p_2}}\\ &\le CR2^{k(-\alpha+\beta-n/q_1-n/q_2-1)}\|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}}\|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}},\end{aligned}$$

and we obtain

$$\begin{aligned}\frac{(|x_0|+R)^\alpha}{|Q|^{1+\varepsilon/n}}\int_Q|\widetilde{I}_\beta(f_1^a,f_2^c)(x)-c|dx &\le C\Big(\frac{R}{2^k}\Big)^{1-\varepsilon}\|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}}\|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}}\\ &\le C\|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}}\|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}}.\end{aligned}$$

The estimate of IV. Let

$$c=-\iint_{|(y_1,y_2)|\ge 1}\frac{f_1^b(y_1)f_2^b(y_2)}{|(y_1,y_2)|^{2n-\beta}}dy_1dy_2.$$

By the same argument as in the proof of Lemma 3.5, we have for $y_1,y_2\in 2Q$,

$$\int_Q\frac{dx}{|(x-y_1,x-y_2)|^{2n-\beta}}\le\begin{cases}CR^{-n+\beta} & \text{if}\quad n<\beta<2n,\\ C\log\frac{8R}{|y_1-y_2|} & \text{if}\quad \beta=n,\\ \frac{C}{|y_1-y_2|^{n-\beta}} & \text{if}\quad 0<\beta<n.\end{cases}$$

The case $n < \beta < 2n$. Since $\int_{2Q} |f_i^b| dx \le CR^{n(1-1/q_i)} \|f_i \tilde{\chi}_k\|_{q_i}$, we have

$$\int_Q |\widetilde{I}_\beta(f_1^b, f_2^b)(x) - c| dx \le CR^{-n+\beta} R^{n(1-1/q_1)} \|f_1 \tilde{\chi}_k\|_{q_1} R^{n(1-1/q_2)} \|f_2 \tilde{\chi}_k\|_{q_2}$$
$$\le C2^{k\alpha} R^{n+\beta-n/q_1-n/q_2} \|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}} \|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}}.$$

The case $0 < \beta < n$. Since $n/q_1 < \beta$ it follows from Hörder's inequality that

$$\int_Q |\widetilde{I}_\beta(f_1^b, f_2^b)(x) - c| dx \le CR^{\beta - n/q_1} \|f_1^b\|_{q_1} \|f_2^b\|_1$$
$$\le C2^{k\alpha} R^{n+\beta-n/q_1-n/q_2} \|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}} \|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}}.$$

The case $\beta = n$ is similar.

The estimate of V is same as that of III.

The estimate of VI. Let $c = \widetilde{I}_\beta(f_1^c, f_2^c)(x_0)$. For $x \in Q$,

$$|\widetilde{I}_\beta(f_1^a, f_2^c)(x) - c| \le CR \iint \frac{|f_1^c(y_1)||f_1^c(y_1)|}{|(x_0 - y_1, x_0 - y_2)|^{2n-\beta+1}} dy_1 dy_2$$
$$= CR \Big(\iint_{\substack{|y_1| \ge 2^{k+1} \\ |y_2| \ge 2^{k+1}}} + \iint_{\substack{|y_1| \ge 2^{k+1} \\ |y_2| \le 2^{k-3}}} + \iint_{\substack{|y_1| \le 2^{k-3} \\ |y_2| \ge 2^{k+1}}} + \iint_{\substack{|y_1| \le 2^{k-3} \\ |y_2| \le 2^{k-3}}} \Big)$$
$$=: CR(VI_1 + VI_2 + VI_3 + VI_4).$$

It suffices to evaluate VI_1, VI_2 and VI_4.

The estimate of VI_1 is same as that of $\tilde{I}_\beta(f_1^c, f_2^c)$ in the case of (i) or (ii). We have

$$VI_1 \le C2^{k(-\alpha+\beta-n/q_1-n/q_2-1)} \|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}} \|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}}.$$

Next we evaluate VI_2. By (3.3) and (3.4) we have

$$VI_2 \le C \int_{|y_1| \ge 2^{k+1}} \frac{|f_1(y_1)|}{|y_1|^{2n-\beta+1}} dy_1 \int_{|y_2| \le 2^{k-3}} |f_2(y_2)| dy_2$$
$$\le C2^{k(-\alpha_1-n-n/q_1+\beta-1)} \|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}} 2^{k(-\alpha_2+n-n/q_2)} \|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}}$$
$$\le C2^{k(-\alpha+\beta-n/q_1-n/q_2-1)} \|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}} \|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}}.$$

Finally we evaluate VI_4. By (3.4) we have

$$VI_4 \le C2^{k(-2n+\beta-1)} \int_{|y_1| \le 2^{k-3}} |f_1(y_1)| dy_1 \int_{|y_2| \le 2^{k-3}} |f_2(y_2)| dy_2$$
$$\le C2^{k(-2n+\beta-1)} \cdot 2^{k(-\alpha_1+n-n/q_1)} \|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}} 2^{k(-\alpha_2+n-n/q_2)} \|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}}$$
$$\le C2^{k(-\alpha+\beta-n/q_1-n/q_2-1)} \|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}} \|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}}.$$

□

4. COUNTEREXAMPLE

In this section we shall show that the condition (2.6) in Thereom 2.11 is optimal by giving a counterexample. We shall show that if $\alpha_1 + \alpha_2 \le \beta - n/q_1 - n/q_2 - 1$ or $\alpha_1 + \alpha_2 \ge n + \beta - n/q_1 - n/q_2$, then Theorem 2.11 is not true for some p_1 and p_2. Let Take p_1 and p_2 such that $1/p_1 + 1/p_2 < 1$. Then we can take two positive sequences such that $\{c_k\}_{k=-\infty}^{\infty} \in \ell^{p_1}$ and $\{d_k\}_{k=-\infty}^{\infty} \in \ell^{p_2}$, but $\sum_{k=-\infty}^{0} c_k d_k = \sum_{k=0}^{\infty} c_k d_k = \infty$.

Counterexample 4.1. *Let*

$$f_1(x) = \sum_{k=10}^{\infty} c_k 2^{-k(\alpha_1+n/q_1)} \chi_{B_k}(x), \quad f_2(x) = \sum_{k=10}^{\infty} d_k 2^{-k(\alpha_2+n/q_2)} \chi_{B_k}(x),$$

$$g_1(x) = \sum_{k=-\infty}^{-10} c_k 2^{-k(\alpha_1+n/q_1)} \chi_{B_k}(x), \quad g_2(x) = \sum_{k=-\infty}^{-10} d_k 2^{-k(\alpha_2+n/q_2)} \chi_{B_k}(x),$$

where $B_k = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n;\ 2^{k-1} < x_i < 2^k \quad \text{for all} \ \ i\}$, *and let* $B = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n;\ -2 < x_i < -1 \quad \text{for all} \ \ i\}$. *Then*

$$f_1, g_1 \in K_{q_1}^{\alpha_1, p_1}(\mathbb{R}^n) \quad \text{and} \quad f_2, g_2 \in K_{q_2}^{\alpha_2, p_2}(\mathbb{R}^n); \tag{4.1}$$

$$\widetilde{I}_\beta(f_1, f_2)(x) = -\infty \quad \text{when} \quad x \in B \quad \text{if} \quad \alpha_1 + \alpha_2 \le \beta - n/q_1 - n/q_2 - 1; \tag{4.2}$$

$$\widetilde{I}_\beta(g_1, g_2) \notin L^1_{loc}(\mathbb{R}^n) \quad \text{if} \quad \alpha_1 + \alpha_2 \ge n + \beta - n/q_1 - n/q_2. \tag{4.3}$$

Proof. The proof of (4.1) is straightforward. We prove (4.2). If $-2 < x_i < -1$,

$$\begin{aligned} \widetilde{I}_\beta(f_1, f_2)(x) &\le -C \sum_{k=10}^{\infty} \frac{c_k 2^{-k(\alpha_1+n/q_1)} d_k 2^{-k(\alpha_2+n/q_2)} 2^{2kn}}{2^{k(2n-\beta+1)}} \\ &\le -C \sum_{k=10}^{\infty} c_k d_k = -\infty. \end{aligned}$$

Finally we prove (4.3). Let $k \le -10$. If $x \in B_k$ then

$$\widetilde{I}_\beta(g_1, g_2)(x) = I_\beta(g_1, g_2)(x) \ge C \frac{c_k 2^{-k(\alpha_1+n/q_1)} d_k 2^{-k(\alpha_2+n/q_2)} 2^{2kn}}{2^{k(2n-\beta)}} \ge C c_k d_k 2^{-kn}.$$

We have

$$\int_{|x| \le \sqrt{n} 2^{-10}} \widetilde{I}_\beta(f_1, f_2)(x) dx \ge C \sum_{k=-\infty}^{-10} c_k d_k = \infty.$$

□

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