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A GEOMETRIC GENERALIZATION OF BUSEMANN-PETTY PROBLEM

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ABSTRACT. The norm defined by Busemann's inequality establishes a class of star body - intersection body. This class of star body plays a key role in the solution of Busemann-Petty problem. In 2003, Giannapoulos [1] defined a norm for a new class of half-section. Based on this norm, we give a geometric generalization of Busemann-Petty problem, and get its answer as a result.

Key words and phrases: Busemann-Petty problem; star body; i-intersection body; Radon transform.

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1. INTRODUCTION

Let $\text{vol}_i(L)$ denote the i-dimensional Lebesgue measure of set $L \subset \mathbb{R}^n$ in its affine hull, and let $G(n,i)$ denote the Grassmann manifold of i-dimensional subspace of $\mathbb{R}^n$. Let B_2^n denote the Euclidean unit ball, S^{n-1} the Euclidean unit sphere in $\mathbb{R}^n$, and $|\cdot|$ the Euclidean norm. A compact, convex set in $\mathbb{R}^n$ with non-empty interiors is called a convex body. We will also work with general star bodies L, which are star-shaped bodies, meaning that $tL \subset L$ for all $t \in [0,1]$, with the additional requirement that their radial function $\rho_L(u) = \max\{\lambda \geq 0 : \lambda u \in L\}$ is a continuous function on S^{n-1}.

Let K be a symmetric convex body in $\mathbb{R}^n$, Busemann inequality states that the function

$$x \mapsto \frac{|x|}{\text{vol}_{n-1}(K \cap x^{\perp})} \tag{1.1}$$

is a norm.

Motivated by this norm, H.Busemann and C.M.Petty posed ten problems about convex bodies in 1956 [2]. The first problem, now known as the Busemann-Petty problem (BP-problem), states:

Supposes that K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, such that

$$\text{vol}_{n-1}(K \cap \xi) \leq \text{vol}_{n-1}(L \cap \xi) \tag{1.2}$$

for all $\xi \in G(n, n-1)$. Does it follow that

$$\text{vol}_n(K) \leq \text{vol}_n(L)?$$

The answer is affirmative if $n \leq 4$ and negative if $n \geq 5$, which was established in a series of papers by Larman and Rogers [3] ($n \geq 12$), Ball [4] ($n \geq 10$), Giannapoulos [5] and Bourgain [6] ($n \geq 7$), Papadimitrakis [7] and Gardner [8] ($n \geq 5$) Gardner [9] proved that the answer is affirmative for $n = 3$. Zhang [10] proved that the answer to the Busemann-Petty problem in the four dimensional case is affirmative . Futhermore, Gardner, Koldobsky and Schlumprecht [11] provided a unified solution to the Busemann-Petty problem in all dimensions. There are many other results related to Busemann-Petty problem (see for example [2], [12], [13] [15],[16],[18],[19]). For Fourier analytic approach to the Busemann-Petty problem and its generalization, the reader is referred to an excellent book of Koldobsky [12] .

The class of intersection bodies, introduced by Lutwak in [14], plays a key role in the thorough solution of the Busemann-Petty problem . A star body K in $\mathbb{R}^n$ is called an intersection body if exist a star body L, such that

$$\rho_K(u) = \text{vol}_{n-1}(L \cap u^{\perp}) = \frac{1}{n-1}\int_{S^{n-1}\cap u^{\perp}} \rho_L(v)^{n-1} dv.$$

The connection between intersection bodies and the Busemann-Petty problem which was first found by Lutwak [14], states that if K is an intersection body in (1.2), then the answer to the question of the Busemann-Petty problem is affirmative.

In [17], Rubin and Zhang firstly give the definition of the (i,k) intersection body . Let positive real integers i and k satisfy $i + k \leq n$, we shall say that an origin-symmetric star body K in $\mathbb{R}^n$ is an (i,k) -intersection body (see [17] for more general definition) if there exits a non-negative measure μ on $G(n,i)$ such that

$$\rho_K^k = R_i^t \mu,$$

where R_i^t is the i-dimensional Radon transform(see [17] or section 2 for its definition).

We denote by $\mathcal{I}_{i,k}^n$ the class of (i,k)-intersection body in $\mathbb{R}^n$. An $(i, n-i)$-intersection body is simply called i-intersection body. the case $i = n-1$ is associate with the notion of intersection body due to Lutwak [14]. Such generalization of the intersection body has essential connections with some generalized Busemann-Petty problem (see [17] for example).

Let K be a symmetric convex body in $\mathbb{R}^n$ and $E \in G(n, n-k)$, where $2 \leq k \leq n-1$. For every $z \in E^{\perp}$, define

$$E(z) = \{x + tz : x \in E, t > 0\}. \tag{1.3}$$

It is easy to check that $E(z)$ is a $(n-k+1)$-dimensional half-space.

Later in 2003, Giannapoulos gave a generalization of the Busemann-inequality . He proved that

$$z \mapsto \frac{|x|}{\text{vol}_{n-k+1}(K \cap E(z))} \tag{1.4}$$

was a norm on $E^{\perp}$.

Inspired by the above norm , we raise up a generalized Busemann-Petty problem in the following:

Let K, L be two symmetric star bodies and $E \in G(n, n-k)$. Is it true that the inequality

$$\text{vol}_{n-k+1}(K \cap E(z)) \leq \text{vol}_{n-k+1}(L \cap E(z)), \ \forall z \in E^{\perp},$$

implies that

$$\text{vol}_n(K) \leq \text{vol}_n(L)?$$

Taking $k = 2$, the problem is just the Busemann-Petty problem.

In this article,we shall mainly study the specific affirmative answer to the above generalized Busemann-Petty problem when K belongs to the class of $(n-k+1)$-intersection bodies .

Theorem 1.1. *Let K be an $(n-k+1)$-intersection body, L be a symmetric star body in $\mathbb{R}^n$, and $E \in G(n, n-k)$. If*

$$\textit{vol}_{n-k+1}(K \cap E(z)) \leq \textit{vol}_{n-k+1}(L \cap E(z)), \ \forall z \in E^{\perp},$$

then

$$\textit{vol}_n(K) \leq \textit{vol}_n(L).$$

Theorem 1.2. *Let L be an origin-symmetric convex body with C^2 boundary and positive curvature. If L is not an $(n-k+1)$-intersection body in $\mathbb{R}^n$, then there exits an origin-symmetric star body K so that*

$$\textit{vol}_{n-k+1}(K \cap E(z)) \leq \textit{vol}_{n-k+1}(L \cap E(z)), \ \forall z \in E^{\perp}$$

where $E \in G(n, n-k)$, but

$$\textit{vol}_n(K) > \textit{vol}_n(L).$$

Theorem 1.1 and Theorem 1.2 together imply that the answer to the above generalized Busemann-Petty problem is equivalent to ask wether the symmetric star body in $\mathbb{R}^n$ is an $(n-k+1)$-intersection body or not.

In the end, we get the half-section $E(z)$'s Funk theorem as an application .

Theorem 1.3. *Let K, L be symmetric star bodies in $\mathbb{R}^n$ and $E \in G(n, n-k)$. If*

$$\textit{vol}_{n-k+1}(K \cap E(z)) = \textit{vol}_{n-k+1}(L \cap E(z)), \ \forall z \in E^{\perp},$$

then $K = L$.

For $k = 2$, it is just the Funk theorem.

2. NOTATION AND PRELIMINARY WORKS

Let $C(S^{n-1})$ be the space of continuous functions on the unit sphere S^{n-1}, and $C_e(S^{n-1})$ the subspace of $C(S^{n-1})$ that contains the even continuous functions on S^{n-1}. And the subset of $C_e(S^{n-1})$ that contains the infinity differentiable functions will denoted by $C_e^{\infty}(S^{n-1})$. Denoted by $C(G(n,i))$ the space of continuous functions on $G(n,i)$. For $f \in C(S^{n-1}),\ g \in C(G(n,i))$, $1 \le i \le n-1$, the i-dimensional spherical Radon transform $R_i f$ and its dual transform $R_i^t g$ are defined by

$$(R_i f)(\xi) = \int_{S^{n-1}\cap\xi} f(u)d\sigma_i(u),\ (R_i^t g)(u) = \int_{\xi\in G(n,i)} g(\xi)d\upsilon_i(\xi), \tag{2.1}$$

where σ_i is the Haar probability measure on S^{i-1}(and we have identified S^{i-1} with $S^{n-1}\cap\xi$), and υ_i is the Haar probability measure on the homogeneous $\{\xi \in G(n,i) : u \in \xi\}$.

The corresponding duality relation reads (see [1]or [12])

$$\int_{G(n,i)} (R_i f)(\xi)g(\xi)d\xi = \int_{S^{n-1}} f(u)(R_i^t g)(u)du. \tag{2.2}$$

This allows us to define $R_i\mu$ and $R_i^t\nu$ for arbitrary finite Borel measures μ on S^{n-1} an ν on $G(n,i)$ as follows:

$$\int_{G(n,i)} (R_i\mu)(\xi)g(\xi)d\xi = \int_{S^{n-1}} (R_i^t g)(u)d\mu(u),\ g \in C(G(n,i)), \tag{2.3}$$

$$\int_{S^{n-1}} (R_i^t\nu)(u)f(u)du = \int_{G(n,i)} (R_i f)(\xi)d\nu(\xi),\ f \in C(S^{n-1}). \tag{2.4}$$

We will also write (2.2), (2.3)and (2.4) briefly as

$$(R_i f, g) = (f, R_i^t g),\ (R_i\mu, g) = (\mu, R_i^t g),\ (R_i^t\nu, f) = (\nu, R_i f).$$

We shall use the notations $R_{m,i}$ and $R_{m,i}^t$ for i-dimensional spherical Radon transform and its dual in a lower dimensional setting $\mathbb{R}^m \subseteq \mathbb{R}^n$ (as referred above, we also have identified $\mathbb{R}^m$ with $\mathbb{R}^n \cap \eta$, $\eta \in G(n,m)$, $m \le n$).

For i star bodies $K_1, \cdots, K_i$, and $\xi \in G(n,i)$, the dual mixed volume, $\widetilde{v}_\xi(K_1\cap\xi, \cdots, K_i\cap\xi)$, is defined by

$$\widetilde{v}_\xi(K_1\cap\xi, \cdots, K_i\cap\xi) = \frac{1}{i}\int_{S^{n-1}\cap\xi} \rho_{K_1}(u)\cdots\rho_{K_i}(u)du. \tag{2.5}$$

If $K_1 = \cdots = K_i$, then we get the i-th section volume function of K:

$$\text{vol}_i(K\cap\xi) = \frac{1}{i}\int_{S^{n-1}\cap\xi} \rho_K^i(u)du. \tag{2.6}$$

Thus, by (2.1), the Radon transform R_i has following close connection with the central section of star bodies

$$(R_i\rho_K^i)(\xi) = \frac{1}{\omega_i}\text{vol}_i(K\cap\xi), \forall\xi \in G(n,i), \tag{2.7}$$

where ω_i is the volume of the unit ball in $\mathbb{R}^n\cap\xi$.

When $i = n$ in (2.5) , the dual mixed volumes of the star bodies $K_1, \cdots, K_n$ is denoted by $\widetilde{V}(K_1, \cdots, K_n)$. And we will denote by $\widetilde{V}(K_1, i, K_2, n-i)$ the dual mixed volume , where there are i copies of K_1 and $n-i$ copies of K_2.

The last fact needed in this article dual Minkowski inequality reads:

Let K_1, K_2 be star bodies in $\mathbb{R}^n$, $0 < i < n$, then

$$\widetilde{V}(K_1, i, K_2, n-i) \leq V^{\frac{i}{n}}(K_1) V^{\frac{n-i}{n}}(K_2). \tag{2.8}$$

3. THE PROOFS OF THEOREMS

In this section, we shall prove Theorem 1.1, Theorem 1.2 and Theorem 1.3 which presented in section 1.

Proof of Theorem 1.1.

For all $z \in E^{\perp}$, by (1.4) and (2.7), we have

$$R_{n-k+1}\rho_K^{n-k+1}(E(z)) \leq R_{n-k+1}\rho_L^{n-k+1}(E(z)).$$

Since $K \in \mathcal{I}^n_{n-k+1}$, there exists $\mu \in G(n, n-k+1)$, such that

$$\rho_K^{k-1} = R^t_{n-k+1}\mu.$$

Therefore, we have

$$\begin{aligned}
\text{vol}_n(K) &= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-k+1}(u)\rho_K^{k-1}(u)du \\
&= \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-k+1}(u) R^t_{n-k+1}\mu du \\
&= \frac{1}{n}\int_{G(n,n-k+1)} (R_{n-k+1}\rho_K^{n-k+1})(E(z))d\mu \\
&\leq \frac{1}{n}\int_{G(n,n-k+1)} (R_{n-k+1}\rho_L^{n-k+1})(E(z))d\mu \\
&= \frac{1}{n}\int_{S^{n-1}} \rho_L^{n-k+1}(u) R^t_{n-k+1}\mu du \\
&= \frac{1}{n}\int_{S^{n-1}} \rho_L^{n-k+1}(u)\rho_K^{k-1}(u)du \\
&\leq \Big(\frac{1}{n}\int_{S^{n-1}} (\rho_L^{n-k+1}(u))^{\frac{n}{n-k+1}}du\Big)^{\frac{n-k+1}{n}} \Big(\frac{1}{n}\int_{S^{n-1}} (\rho_K^{k-1}(u))^{\frac{n}{k-1}}du\Big)^{\frac{k-1}{n}} \\
&= (\text{vol}_n(L))^{\frac{n-k+1}{n}} (\text{vol}_n(K))^{\frac{k-1}{n}}.
\end{aligned}$$

This gives

$$\text{vol}_n(K)^n \leq (\text{vol}_n(L))^{n-k+1}(\text{vol}_n(K))^{k-1},$$

$$\text{vol}_n(K) \leq \text{vol}_n(L).$$

This proves the theorem. ∎

Before we prove Theorem 1.2, we require the following notations. We will denotes by $\mathcal{M}$ the spaces of signed Borel measures. In particular, $\mathcal{M}(S^{n-1})$ denotes the spaces of signed measures on S^{n-1}. Let $\mathbb{X} = R_i(C(S^{n-1}))$, denoted by $\mathcal{M}^+(\mathbb{X})$ the set of non-negative linear functionals on $\mathbb{X}$. We will consider the convex cone $\mathcal{N}_i$ defined by $\mathcal{N}_i = \{R^t_i\mu : \mu \in \mathcal{M}^+(\mathbb{X})\}$ in $\mathcal{M}(S^{n-1})$. It can be shown that this convex cone $\mathcal{N}_i$ is closed under w^*-topology of $\mathcal{M}(S^{n-1})$ (see [17]).

Lemma 3.1. *(see* [17]*) Let* $\rho \in \mathcal{M}(S^{n-1})$. *If* $\rho \notin \mathcal{N}_i$, *then there exists* $g \in C^{\infty}(S^{n-1})$ *so that*

$$(\rho, g) > 0, \ R_i g < 0.$$

Remark. If the Radon-Nikodym derivative of the measure ρ with respect to the Lebesgue measure on S^{n-1} is an even continuous function and $\rho \notin \mathcal{N}_i$, then the function g in Lemma 3.1 can be chosen in $C_e^{\infty}(S^{n-1})$ (see [17]).

Proof of Theorem 1.2.

Since $L \notin \mathcal{I}_{k-1}^n$, we have $\rho_L^{k-1} \notin \mathcal{N}_{n-k+1}$. By applying Lemma 3.1 in $\mathbb{R}^n$, there exists a $g \in C_e^\infty(S^{n-1})$ so that

$$(\rho_L^{k-1}, g) > 0,\ R_{n-k+1}g < 0. \tag{3.1}$$

We can define an origin-symmetric convex body L_ε in $\mathbb{R}^n$ by

$$\rho_{L_\varepsilon}^{n-k+1} = \rho_L^{n-k+1} + \varepsilon g, \tag{3.2}$$

for $\varepsilon > 0$ sufficiently small. This is possible for the reason that L has C^2 boundary and positive curvature.

Substituting (3.2) into the second inequality (3.1) and using (2.7), we have

$$\begin{aligned} 0 &> \varepsilon R_{n-k+1}g \\ &= \varepsilon R_{n-k+1}\rho_{L_\varepsilon}^{n-k+1} - \varepsilon R_{n-k+1}\rho_L^{n-k+1} \\ &= \tfrac{\varepsilon}{\omega_i}\mathrm{vol}_{n-k+1}(L_\varepsilon \cap E(Z)) - \tfrac{\varepsilon}{\omega_i}\mathrm{vol}_{n-k+1}(L \cap E(Z)), \end{aligned}$$

where $E(z) \in G(n, n-k+1)$. This concludes

$$\mathrm{vol}_{n-k+1}(L_\varepsilon \cap E(z)) < \mathrm{vol}_{n-k+1}(L \cap E(z)). \tag{3.3}$$

Substituting (3.2) into the first inequality (3.1) , we get

$$\begin{aligned} 0 &< (\rho_L^{k-1}, g') \\ &= \varepsilon^{-1}(\rho_L^{k-1}, \rho_{L_\varepsilon}^{n-k+1} - \rho_L^{n-k+1}) \\ &= \varepsilon^{-1}(\rho_L^{k-1}, \rho_{L_\varepsilon}^{n-k+1}) - \varepsilon^{-1}(\rho_L^{k-1}, \rho_L^{n-k+1}) \\ &= \varepsilon^{-1}\widetilde{V}(L, k-1, L_\varepsilon, n-k+1) - \varepsilon^{-1}\mathrm{vol}_n(L). \end{aligned}$$

This implies

$$\widetilde{V}(L, k-1, L_\varepsilon, n-k+1) > \mathrm{vol}_n(L). \tag{3.4}$$

But by dual Minkowski inequality (2.8)

$$\widetilde{V}((L, k-1, L_\varepsilon, n-k+1) \leq (\mathrm{v}ol_{k-1}(L))^{\frac{k-1}{n}} (\mathrm{vol}_{n-k+1}(L_\varepsilon))^{\frac{n-k+1}{n}}. \tag{3.5}$$

Combining the last two inequalites (3.4),(3.5), we arrive at

$$\mathrm{vol}_n(L_\varepsilon) > \mathrm{vol}_n(L). \tag{3.6}$$

Then let $K = L_\varepsilon$ in (3.3) and (3.6) we obtain the result desired immediately. □

In the rest part we shall establish a generalization result of Funk Theorem (see [12]), and the following lemma will be needed:

Lemma 3.2. *(see* [12]*) Let f be an even homogeneous of degree $-n+1$ on $\mathbb{R}^n$, continuous on the sphere S^{n-1}. Then the Fourier transform of f is an even homogeneous of degree -1, continuous on $\mathbb{R}^n \backslash \{o\}$ function such that , for every $\xi \in S^{n-1}$,*

$$Rf(\xi) = \int_{S^{n-1}\cap\xi^\perp} f(u)du = \frac{1}{\pi}\hat{f}(\xi).$$

where the spherical Radon transform R is applied to the restriction of f to the sphere.

Proof of Theorem 1.3. From the definition (2.6) , it is easy to verify that

$$\mathrm{vol}_{n-k+1}(K \cap E(z)) = \mathrm{vol}_{n-k+1}(L \cap E(z))$$

$$\Longleftrightarrow \frac{1}{n-k+1}\int_{S^{n-1}\cap E(z)} \rho_K^{n+k-1}(u)du = \frac{1}{n-k+1}\int_{S^{n-1}\cap E(z)} \rho_L^{n+k-1}(u)du.$$

That is

$$\int_{S^{n-1}\cap E(z)} \rho_K^{n+k-1}(u)du = \int_{S^{n-1}\cap E(z)} \rho_L^{n+k-1}(u)du. \tag{3.7}$$

Fixed any $\xi \in S^{n-1}$, and denote by $G_\xi(n-1, n-k+1)$ the Grassmann of $n-k+1$-dimensional subspace of $\xi^\perp$. Integrating both sides of equation (3.7) over $G_\xi(n-1, n-k+1)$, we get

$$\int_{S^{n-1}\cap\xi^\perp} \rho_K^{n+k-1}(u)du = \int_{S^{n-1}\cap\xi^\perp} \rho_L^{n+k-1}(u)du. \tag{3.8}$$

Extend $\rho_K^{n+k-1}(u)$, $\rho_l^{n+k-1}(u)$ to homogeneous functions of degree $-n+1$ on the whole $\mathbb{R}^n$. By Lemma 3.2, the Fourier transforms of their extensions are equal. By the uniqueness theorem for the Fourier transform (see [12]), $\rho_K^{n+k-1}(u) = \rho_L^{n+k-1}(u)$. This implies $\rho_K(u) = \rho_L(u)$, hence $K = L$. ∎

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