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PROPERTIES OF CERTAIN MULTIVALENT FUNCTIONS INVOLVING RUSCHEWEYH DERIVATIVES

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ABSTRACT. Let $A_p(p \in N)$ be the class of functions $f(z) = z^p + \sum_{m=1}^{\infty} a_{p+m} z^{p+m}$ which are analytic in the unit disk. By virtue of the Ruscheweyh derivatives we introduce the new subclasses $C_p(n, \alpha, \beta, \lambda, \mu)$ of A_p. Subordination relations, inclusion relations, convolution properties and a sharp coefficient estimate are obtained. We also give a sufficient condition for a function to be in $C_p(n, \alpha, \beta, \lambda, \mu)$.

Key words and phrases: Analytic function, p-valently close-to-convex function, Convolution, Ruscheweyh derivative, Subordination.

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1. INTRODUCTION AND PRELIMINARIES

Let $A_p(p \in N = \{1, 2, 3, \cdots\})$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{m=1}^{\infty} a_{p+m} z^{p+m}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. Let

$$C_p(\rho) = \left\{ f(z) \in A_p : \text{Re}\frac{f'(z)}{pz^{p-1}} > \rho \quad (z \in U) \right\}$$

for some $\rho(0 \leq \rho < 1)$. It is well known that each function in the class $C_p(\rho)(0 \leq \rho < 1)$ is p-valently close-to-convex of order ρ in U. Denote by K the usual subclass of A_1 consisting of convex univalent functions in U. Further let $UK(\subset K)$ denote the class of functions called uniformly convex (univalent) in U and introduced by Goodman [4]. It was shown in Ronning [10] and Ma and Minda [8] that a function $f(z) \in A_1$ is in UK if and only if

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in U).$$

The uniformly convex and related functions have been studied by several authors (see, e.g., [4, 5, 6, 7, 8, 10, 11, 12, 14, 15, 16, 17]).

If $f(z) = \sum_{m=0}^{\infty} a_m z^m$ and $g(z) = \sum_{m=0}^{\infty} b_m z^m$ are analytic in U, then the Hadamard product or convolution $f(z) * g(z)$ of $f(z)$ and $g(z)$ is defined by

$$f(z) * g(z) = \sum_{m=0}^{\infty} a_m b_m z^m.$$

Let

$$D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z),$$

where $f(z) \in A_p$ and n is any integer greater then $-p$. Then

$$D^{n+p-1} f(z) = \frac{z^p (z^{n-1} f(z))^{(n+p-1)}}{(n+p-1)!}.$$

This symbol $D^{n+p-1} f(z)$ is called the Ruscheweyh derivative of $f(z)$ and was introduced by Ruscheweyh [13] (when $p = 1$) and Goel and Sohi [3]. In this paper we introduce and investigate the following new subclass of A_p:

Definition. A function $f(z) \in A_p$ is said to be in $C_p(n, \alpha, \beta, \lambda, \mu)$ if it satisfies

$$(\text{Re } J_p(n, \beta, f(z)))^2 + \mu > \alpha^2 \left|J_p(n, \beta, f(z) - \lambda\right|^2 \quad (z \in U), \tag{1.1}$$

where

$$J_p(n, \beta, f(z)) = (1-\beta)\frac{D^{n+p-1} f(z)}{z^p} + \beta \frac{(D^{n+p-1} f(z))'}{pz^{p-1}}, \tag{1.2}$$

$$0 < \alpha < 1, \quad \beta \geq 0, \quad \lambda > 0, \quad \alpha^2(\lambda - 1)^2 - 1 < \mu \leq (\alpha\lambda)^2. \tag{1.3}$$

Note that $z^p \in C_p(n, \alpha, \beta, \lambda, \mu)$ and for $n = 1-p, 0 < \alpha < 1, \beta \geq 0, \lambda = 1$ and $\mu = 0$,

$$\begin{aligned} & C_p(1-p,\alpha,\beta,1,0) \\ = & \left\{ f(z)\in A_p : \text{Re}\left((1-\beta)\frac{f(z)}{z^p}+\beta\frac{f'(z)}{pz^{p-1}}\right)\right. \\ & \left. > \alpha\left|(1-\beta)\frac{f(z)}{z^p}+\beta\frac{f'(z)}{pz^{p-1}}-1\right| \quad (z\in U)\right\}. \end{aligned}$$

For $\beta \geq 0$ and $\rho < 1$, Ding, Ling and Bao in [1] have considered the class $Q_\beta(\rho)$ defined by

$$Q_\beta(\rho) = \left\{ f(z)\in A_1 : \text{Re}\left((1-\beta)\frac{f(z)}{z}+\beta f'(z)\right) > \rho \quad (z\in U)\right\}.$$

Let $f(z)$ and $g(z)$ be analytic in U. Then we say that the function $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in U such that $|w(z)| \leq |z|$ and $f(z) = g(w(z))$ for $z \in U$. If $g(z)$ is univalent in U, that $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$. We shall need the following Lemmas.

Lemma 1.1. *Let $F(z)$ be analytic in U and $h(z)$ be analytic and convex univalent in U with $h(0) = F(0)$. If*

$$F(z) + \frac{1}{c}zF'(z) \prec h(z), \tag{1.4}$$

where $c \neq 0$ and Re $c \geq 0$, then

$$F(z) \prec H(z) = cz^{-c}\int_0^z t^{c-1}h(t)dt \prec h(z)$$

and $H(z)$ is the best dominant of (1.4).

Lemma 1.2. *Let $f(z) = \sum_{m=1}^{\infty} a_m z^m$ be analytic in U and $g(z) \in K$. If $f(z) \prec g(z)$, then $|a_m| \leq 1 (m \in N)$.*

Lemma 1.1 is due to Miller and Mocanu [9] and Lemma 1.2 can be found in Duren [2, p.195].

Throughout this paper we assume, unless otherwise stated, that n is any integer greater than $-p (p \in N)$ and $\alpha, \beta, \lambda, \mu$ satisfy (1.3).

2. Subordination relations

Theorem 2.1. *A function $f(z) \in A_p$ is in $C_p(n,\alpha,\beta,\lambda,\mu)$ if and only if*

$$J_p(n,\beta,f(z)) \prec h(z),$$

where $J_p(n,\beta,f(z))$ is defined by (1.2),

$$h(z) = \varphi\left(\frac{z+b}{1+bz}\right),$$

$$\varphi(\zeta) = \lambda_0 + \frac{\delta}{2\alpha(1-\alpha^2)}\left(\left(\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^{2\sigma/\pi} + \left(\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^{-2\sigma/\pi} - 2\right) \quad (|\zeta|<1), \tag{2.1}$$

$$\lambda_0 = \frac{\delta - \alpha^3\lambda}{\alpha(1-\alpha^2)}, \quad \delta = \sqrt{(\alpha\lambda)^2 - \mu(1-\alpha^2)}, \quad \sigma = \arccos\alpha$$

and $b \in (-1, 1)$ *is determined by* $\varphi(b) = 1$.

Proof. It follows from (1.3) and (2.1) that $0 < (\alpha^2\lambda)^2 \le \delta^2 = (\alpha\lambda)^2 - \mu(1-\alpha^2) < (\alpha\lambda)^2 - (1-\alpha^2)(\alpha^2(\lambda-1)^2 - 1) = (\alpha^2\lambda + 1 - \alpha^2)^2$ and so

$$0 \le \lambda_1 = \frac{\delta - \alpha^2\lambda}{1-\alpha^2} < \min\{1, \lambda_0\}. \tag{2.2}$$

Let $F(z) = J_p(n, \beta, f(z)) = u + iv$ for $f(z) \in A_p$. Then (1.1) can be written as $u^2 + \mu > \alpha^2((u-\lambda)^2 + v^2)$, that is,

$$\left(u + \frac{\alpha^2\lambda}{1-\alpha^2}\right)^2 - \frac{\alpha^2}{1-\alpha^2}v^2 > \frac{(\alpha\lambda)^2 - \mu}{1-\alpha^2} + \left(\frac{\alpha^2\lambda}{1-\alpha^2}\right)^2 = \left(\frac{\delta}{1-\alpha^2}\right)^2. \tag{2.3}$$

In view of $F(0) = 1$, we see that

$$F(U) \subset \Omega = \{w = u + iv : u \text{ and } v \text{ satisfy } (2.3), u > \lambda_1\}.$$

Note that $h(0) = \varphi(b) = 1$. In order to prove our theorem, it suffices to show that the function $h(z)$ given by (2.1) maps U conformally onto the hyperbolic region Ω.

The linear transformation $w_1 = \frac{\alpha}{\delta}((1-\alpha^2)w + \alpha^2\lambda)$ maps $\Omega^+ = \Omega \cap \{w = u + iv : v > 0\}$ onto

$$\Omega_1^+ = \left\{w_1 = u_1 + iv_1 : \frac{u_1^2}{\cos^2\sigma} - \frac{v_1^2}{\sin^2\sigma} > 1,\ u_1 > \cos\sigma,\ v_1 > 0\right\}$$

so that $w = \lambda_1$ corresponds to $w_1 = \alpha = \cos\sigma$ and $w = \lambda_0$ to $w_1 = 1$. It is clear that $w_2 = w_1 + \sqrt{w_1^2 - 1}$ maps Ω_1^+ conformally onto

$$\Omega_2^+ = \{w_2 : 0 < \arg w_2 < \sigma,\ 1 < |w_2| < +\infty\}$$

so that $w_1 = \cos\sigma$ corresponds to $w_2 = e^{i\sigma}$ and $w_1 = 1$ to $w_2 = 1$. Also the function $t = \frac{1}{2}(w_2^{\pi/\sigma} + w_2^{-\pi/\sigma})$ maps Ω_2^+ conformally onto the upper half plane $\mathrm{Im}(t) > 0$ so that $w_2 = e^{i\sigma}$ corresponds to $t = -1$ and $w_2 = 1$ to $t = 1$. Thus the composite function $t = g(w)$ (say) maps Ω^+ conformally onto $\mathrm{Im}(t) > 0$ so that $w \in [\lambda_1, +\infty)$ corresponds to $t \in [-1, +\infty)$. With the help of the symmetry principle, the function $t = g(w)$ maps Ω conformally onto $G = \{t : |\arg(t+1)| < \pi\}$. Since $t = 2(\frac{1+\zeta}{1-\zeta})^2 - 1$ maps the unit disk $|\zeta| < 1$ onto G, we deduce that

$$\begin{aligned} w = g^{-1}(t) &= \frac{\delta - \alpha^3\lambda}{\alpha(1-\alpha^2)} + \frac{\delta}{2\alpha(1-\alpha^2)}\left(\left(t + \sqrt{t^2-1}\right)^{\sigma/\pi} + \left(t + \sqrt{t^2-1}\right)^{-\sigma/\pi} - 2\right) \\ &= \lambda_0 + \frac{\delta}{2\alpha(1-\alpha^2)}\left(\left(\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^{2\sigma/\pi} + \left(\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^{-2\sigma/\pi} - 2\right) \\ &= \varphi(\zeta) \end{aligned}$$

maps $|\zeta| < 1$ conformally onto Ω so that $\zeta = b \in (-1, 1)$ corresponds to $w = 1$. Now we easily know that $w = h(z) = \varphi(\frac{z+b}{1+bz})$ maps U conformally onto Ω. Therefore the proof of the theorem is complete.

∎

Corollary 2.2. *Let* $f(z) \in C_p(n, \alpha, \beta, \lambda, \mu)$ *and* $h(z)$ *be given by (2.1). Then for* $z \in U$,

$$|\arg J_p(n, \beta, f(z))| < \begin{cases} \arctan\sqrt{\frac{\mu}{(\alpha\lambda)^2 - \mu} + \frac{1}{\alpha^2}}, & \mu < (\alpha\lambda)^2, \\ \frac{\pi}{2}, & \mu = (\alpha\lambda)^2. \end{cases} \tag{2.4}$$

The result is sharp with the extremal function $f_\beta(z)$ defined by $f_\beta(z) \in A_p$ and

$$D^{n+p-1} f_\beta(z) = \begin{cases} \frac{p}{\beta} z^{p(1-1/\beta)} \int_0^z t^{(p/\beta)-1} h(t)dt, & \beta > 0, \\ z^p h(z), & \beta = 0. \end{cases} \tag{2.5}$$

Proof. From the proof of Theorem 2.1 we see that

$$\partial h(U) = \left\{ w = u + iv : u^2 + \mu = \alpha^2 \left((u-\lambda)^2 + v^2 \right),\ u \geq \lambda_1 \right\}. \tag{2.6}$$

If $\mu < (\alpha\lambda)^2$, then $\lambda_1 > 0$. Consider the equations

$$u^2 + \mu = \alpha^2 \left((u-\lambda)^2 + v^2 \right) \quad \text{and} \quad v = ku,$$

where u, v and k are real with $u \geq \lambda_1 > 0$. Elimination of v yields

$$\left(\alpha^2 (k^2+1) - 1 \right) u^2 - 2\alpha^2 \lambda u + (\alpha\lambda)^2 - \mu = 0.$$

Suppose $(\alpha^2\lambda)^2 - (\alpha^2(k^2+1) - 1)((\alpha\lambda)^2 - \mu) = 0$, then

$$k^2 = \frac{(\alpha\lambda)^2}{(\alpha\lambda)^2 - \mu} + \frac{1}{\alpha^2} - 1 > 0.$$

If $\mu = (\alpha\lambda)^2$, then $\lambda_1 = 0$. Therefore we conclude that

$$\min\{\theta : |\arg h(z)| < \theta\ (z \in U)\} = \begin{cases} \arctan \sqrt{\frac{\mu}{(\alpha\lambda)^2 - \mu} + \frac{1}{\alpha^2}}, & \mu < (\alpha\lambda)^2, \\ \frac{\pi}{2}, & \mu = (\alpha\lambda)^2. \end{cases} \tag{2.7}$$

Now (2.7) follows immediately from Theorem 2.1.

For the function $f_\beta(z)$ defined by (2.5), it is easy to verify that

$$J_p(n, \beta, f_\beta(z)) = h(z) \quad \text{and} \quad f_\beta(z) \in C_p(n, \alpha, \beta, \lambda, \mu).$$

Hence the bound in (2.7) is sharp. ∎

Corollary 2.3. *Let $f(z) \in C_p(n, \alpha, \beta, \lambda, \mu), \beta > 0,$ and $h(z)$ be given by (2.1). Then*

$$\frac{D^{n+p-1} f(z)}{z^p} \prec \frac{p}{\beta} \int_0^1 \rho^{\frac{p}{\beta}-1} h(\rho z) d\rho. \tag{2.8}$$

Proof. Let us put

$$F(z) = \frac{D^{n+p-1} f(z)}{z^p} \tag{2.9}$$

for $f(z) \in C_p(n, \alpha, \beta, \lambda, \mu)$ and $\beta > 0$. Then it follows from (2.9) and Theorem 2.1 that

$$F(z) + \frac{\beta}{p} z F'(z) = J_p(n, \beta, f(z)) \prec h(z).$$

Therefore, an application of Lemma 1.1 with $c = \frac{p}{\beta} > 0$ yields

$$F(z) \prec \frac{p}{\beta} z^{-\frac{p}{\beta}} \int_0^z t^{\frac{p}{\beta}-1} h(t) dt = \frac{p}{\beta} \int_0^1 \rho^{\frac{p}{\beta}-1} h(\rho z) d\rho \prec h(z), \tag{2.10}$$

which proves (2.8).

∎

Corollary 2.4. *Let $f(z) \in C_p(n, \alpha, \beta, \lambda, \mu)$. Then for $z \in U$, we have*

$$h(-|z|) \leq \mathit{Re}\, J_p(n, \beta, f(z)) \leq h(|z|), \tag{2.11}$$

$$\mathit{Re}\, J_p(n, \beta, f(z)) > \lambda_1, \tag{2.12}$$

$$\frac{p}{\beta}\int_0^1 \rho^{\frac{p}{\beta}-1}h(-\rho|z|)d\rho \le Re\frac{D^{n+p-1}f(z)}{z^p} \le \frac{p}{\beta}\int_0^1 \rho^{\frac{p}{\beta}-1}h(\rho|z|)d\rho, \quad \beta>0, \tag{2.13}$$

and

$$Re\,\frac{D^{n+p-1}f(z)}{z^p} > \frac{p}{\beta}\int_0^1 \rho^{\frac{p}{\beta}-1}h(-\rho)d\rho, \quad \beta>0, \tag{2.14}$$

where $J_p(n,\beta,f(z)), h(z)$ *and* λ_1 *are as given in Theorem 2.1. These results are sharp.*

Proof. From the proof of Theorem 2.1 we know that the univalent function $h(z)$ maps the closed disk $|z| \le r(0<r<1)$ onto a closed region which is convex and symmetric with respect to the real axis. Hence we have

$$h(r) \ge \mathrm{Re}\,h(z) \ge h(-r) > h(-1) = \varphi(-1) = \lambda_1 \quad (|z| \le r). \tag{2.15}$$

According to the definition of the subordination, it follows from Theorem 2.1, Corollary 2.3 and (2.15) that the inequalities (2.11)-(2.14) hold true. Further these results are sharp for the function $f_\beta(z)$ defined by (2.5). ∎

3. PROPERTIES OF $C_p(n,\alpha,\beta,\lambda,\mu)$.

If λ_0 in (2.1) satisfies the restricted condition $\lambda_0 \le 1$, then we can prove the following:

Theorem 3.1. *Let* $f(z) \in C_p(n,\alpha,\beta,\lambda,\mu)$. *If* $\lambda_1 < \rho < \lambda_0 \le 1$, *then Re* $J_p(n,\beta,f(z)) > \rho$ *in* $|z|<r$, *where*

$$r = r(\rho,\alpha,\lambda,\mu) = \frac{b+\tan^2\theta}{1+b\tan^2\theta}, \quad \theta = \frac{\pi}{4\sigma}\arccos\left(\frac{\alpha}{\delta}\left(\rho(1-\alpha^2)+\alpha^2\lambda\right)\right), \tag{3.1}$$

and $J_p(n,\beta,f(z)), \lambda_0, \lambda_1, \delta, \sigma$ *and* b *are as given in Theorem 2.1. The result is sharp.*

Proof. Since $\lambda_0 \le 1$ and $\varphi(b)=1$, it follows from (2.1) in Theorem 2.1 that $0 \le b < 1$. In view of $0 \le \lambda_1 < \rho < \lambda_0 (\le 1)$, (2.1) and (3.1), we have

$$0 < \frac{\alpha}{\delta}\left(\rho(1-\alpha^2)+\alpha^2\lambda\right) < \frac{\alpha}{\delta}\left(\lambda_0(1-\alpha^2)+\alpha^2\lambda\right) = 1,$$

$$0 < \theta < \frac{\pi}{4\sigma}\arccos\left(\frac{\alpha}{\delta}\left(\lambda_1(1-\alpha^2)+\alpha^2\lambda\right)\right) = \frac{\pi}{4}.$$

Hence $0 \le b < r < 1$. Let $h(z)$ be given by (2.1). Then it follows from (2.1) and (3.1) that

$$\begin{aligned} h(-r) &= \varphi\left(\frac{b-r}{1-br}\right) \\ &= \lambda_0 + \frac{\delta}{2\alpha(1-\alpha^2)}\left(\left(\frac{1+i\sqrt{(r-b)/(1-br)}}{1-i\sqrt{(r-b)/(1-br)}}\right)^{2\sigma/\pi}\right. \\ &\quad \left. + \left(\frac{1+i\sqrt{(r-b)/(1-br)}}{1-i\sqrt{(r-b)/(1-br)}}\right)^{-2\sigma/\pi} - 2\right) \\ &= \lambda_0 + \frac{\delta}{\alpha(1-\alpha^2)}\left(\cos\left(\frac{4\sigma}{\pi}\arctan\sqrt{\frac{r-b}{1-br}}\right) - 1\right) \\ &= \lambda_0 + \frac{\delta}{\alpha(1-\alpha^2)}\left(\cos\frac{4\sigma\theta}{\pi} - 1\right) \\ &= \lambda_0 + \frac{\delta}{\alpha(1-\alpha^2)}\left(\frac{\alpha}{\delta}\left(\rho(1-\alpha^2)+\alpha^2\lambda\right) - 1\right) = \rho, \end{aligned}$$

which leads to

$$\inf_{|z|<r} \operatorname{Re} h(z) = h(-r) = \rho. \tag{3.2}$$

If $f(z) \in C_p(n,\alpha,\beta,\lambda,\mu)$ then, by Theorem 2.1 and (3.2), we get

$$\operatorname{Re} J_p(n,\beta,f(z)) > \rho \quad (|z| < r).$$

Obviously the function $f_\beta(z)$ defined by (2.5) shows that the result is sharp. ∎

Theorem 3.2. $C_p(n,\alpha,\beta_1,\lambda,\mu) \subset C_p(n,\alpha,\beta_2,\lambda,\mu)$ for $0 \le \beta_2 < \beta_1$.

Proof. Let $f(z) \in C_p(n,\alpha,\beta_1,\lambda,\mu)$ and $h(z)$ be given by (2.1). Then, by Theorem 2.1 and (2.10) in the proof of Corollary 2.3, we have

$$J_p(n,\beta_1,f(z)) \prec h(z), \quad J_p(n,0,f(z)) \prec h(z). \tag{3.3}$$

Noting that $0 \le \frac{\beta_2}{\beta_1} < 1$ and $h(z)$ is convex univalent in U, it follows from (3.3) that

$$\begin{aligned} J_p(n,\beta_2,f(z)) &= \left(1-\frac{\beta_2}{\beta_1}\right)\frac{D^{n+p-1}f(z)}{z^p} + \frac{\beta_2}{\beta_1}\left((1-\beta_1)\frac{D^{n+p-1}f(z)}{z^p} + \beta_1\frac{(D^{n+p-1}f(z))'}{pz^{p-1}}\right) \\ &= \left(1-\frac{\beta_2}{\beta_1}\right)J_p(n,0,f(z)) + \frac{\beta_2}{\beta_1}J_p(n,\beta_1,f(z)) \\ &\prec h(z). \end{aligned}$$

Hence, using Theorem 2.1, $f(z) \in C_p(n,\alpha,\beta_2,\lambda,\mu)$ and the proof is completed. ∎

Theorem 3.3. $C_p(n+1,\alpha,\beta,\lambda,\mu) \subset C_p(n,\alpha,\beta,\lambda,\mu)$.

Proof. It is known [3] that

$$z(D^{n+p-1}f(z))' = (p+n)D^{n+p}f(z) - nD^{n+p-1}f(z) \tag{3.4}$$

for $f(z) \in A_p$. Let

$$g(z) = J_p(n,\beta,f(z)). \tag{3.5}$$

Then (3.4) and (3.5) lead to

$$pz^p g(z) = \beta(p+n)D^{n+p}f(z) + (p(1-\beta)-\beta n)D^{n+p-1}f(z). \tag{3.6}$$

Differentiating (3.6) and using (3.4) we obtain

$$\begin{aligned} &pz^p(pg(z)+zg'(z)) \\ = \ &\beta(p+n)z(D^{n+p}f(z))' + (p(1-\beta)-\beta n)((p+n)D^{n+p}f(z) - nD^{n+p-1}f(z)) \end{aligned} \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$pz^p((p+n)g(z)+zg'(z)) = \beta(p+n)z(D^{n+p}f(z))' + p(1-\beta)(p+n)D^{n+p}f(z),$$

that is,

$$J_p(n+1,\beta,f(z)) = g(z) + \frac{zg'(z)}{p+n}. \tag{3.8}$$

If $f(z) \in C_p(n+1,\alpha,\beta,\lambda,\mu)$, then from Theorem 2.1 and (3.8) we get

$$g(z) + \frac{zg'(z)}{p+n} \prec h(z),$$

and an application of Lemma 1.1 with $c = p+n > 0$ yields

$$g(z) = J_p(n,\beta,f(z)) \prec h(z).$$

Hence, by Theorem 2.1, $f(z) \in C_p(n,\alpha,\beta,\lambda,\mu)$. This completes the proof. ∎

Remark 3.1. For $\beta \geq 1$, it follows from Theorems 3.3 and 3.2 and (2.12) in Corollary 2.4 (with $n = 1-p$ and $\beta = 1$) that

$$C_p(n,\alpha,\beta,\lambda,\mu) \subset C_p(1-p,\alpha,\beta,\lambda,\mu) \subset C_p(1-p,\alpha,1,\lambda,\mu) \subset C_p(\lambda_1).$$

Thus each function in the class $C_p(n,\alpha,\beta,\lambda,\mu)$ with $\beta \geq 1$ is p-valently close-to-convex of order λ_1 in U.

Theorem 3.4. *Let $f(z) \in C_p(n,\alpha,\beta,\lambda,\mu)$, $g(z) \in A_p$ and Re$\frac{g(z)}{z^p} > \frac{1}{2}$ $(z \in U)$. Then $f(z) * g(z) \in C_p(n,\alpha,\beta,\lambda,\mu)$.*

Proof. . We easily have

$$\frac{D^{n+p-1}(f*g)(z)}{z^p} = \frac{D^{n+p-1}f(z)}{z^p} * \frac{g(z)}{z^p},$$

$$\frac{(D^{n+p-1}(f*g)(z))'}{pz^{p-1}} = \frac{(D^{n+p-1}f(z))'}{pz^{p-1}} * \frac{g(z)}{z^p},$$

and so

$$J_p(n,\beta,f(z)*g(z)) = F(z)*P(z) \quad (z \in U), \tag{3.9}$$

where $f(z) \in C_p(n,\alpha,\beta,\lambda,\mu), F(z) = J_p(n,\beta,f(z))$ and $P(z) = \frac{g(z)}{z^p}$. Since the function $P(z)$ has the integral representation

$$P(z) = \int_{|x|=1} \frac{d\mu(x)}{1-xz} \quad (z \in U),$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x|=1$ and $\int_{|x|=1} d\mu(x) = 1$, it follows from (3.9) that

$$J_p(n,\beta,f(z)*g(z)) = \int_{|x|=1} F(xz)d\mu(x) \quad (z \in U). \tag{3.10}$$

According to Theorem 2.1 the function $F(z)$ is subordinate to the convex univalent function $h(z)$ given by (2.1). Therefore, we deduce from (3.10) that

$$J_p(n,\beta,f(z)*g(z)) \prec h(z),$$

which leads to $f(z)*g(z)\in C_p(n,\alpha,\beta,\lambda,\mu)$ by Theorem 2.1. ∎

If λ_1 in (2.2) satisfies the restricted condition $\lambda_1 \geq \frac{1}{2}$, then it follows from Theorem 3.4 and (2.12) in Corollary 2.4 (with $n = 1-p$ and $\beta = 0$) that the following:

Corollary 3.5. *Let $f(z)\in C_p(n,\alpha,\beta,\lambda,\mu)$ and $g(z)\in C_p(1-p,\alpha,0,\lambda,\mu)$. If $\lambda_1 \geq \frac{1}{2}$, where λ_1 is given by (2.2), then $f(z)*g(z)\in C_p(n,\alpha,\beta,\lambda,\mu)$.*

Remark 3.2. Note that $C_p(n,\alpha,\beta,\lambda,\mu)\subset C_p(1-p,\alpha,0,\lambda,\mu)$. We known from Corollary 3.5 that $C_p(n,\alpha,\beta,\lambda,\mu)$ is closed with respect to Hadamard product provided $\lambda_1 \geq \frac{1}{2}$.

Let $S^*(\frac{1}{2})$ denote the class of starlike univalent functions of order $\frac{1}{2}$ in U consisting of functions $g(z)\in A_1$ satisfying

$$\mathrm{Re}\frac{zg'(z)}{g(z)} > \frac{1}{2} \quad (z\in U).$$

It is well known that if $g(z)\in S^*(\frac{1}{2})$ then $\mathrm{Re}\frac{g(z)}{z} > \frac{1}{2}$ for $z\in U$. Thus Theorem 3.4 with $p=1$ and $n>-1$ yields the following:

Corollary 3.6. *Let $f(z)\in C_1(n,\alpha,\beta,\lambda,\mu), n\in N\cup\{0\}$, and $g(z)\in S^*(\frac{1}{2})$. Then $f(z)*g(z)\in C_1(n,\alpha,\beta,\lambda,\mu)$.*

Theorem 3.7. *Let $f(z) = z^p + \sum_{m=1}^{\infty} a_{p+m}z^{p+m} \in C_p(n,\alpha,\beta,\lambda,\mu)$. Then*
(3.11)

$$|a_{p+1}| \leq \frac{p\delta(1-b^2)}{\alpha(1-\alpha^2)(p+\beta)(p+n)} \sum_{m=1}^{\infty}\left(\sum_{k=0}^{2m}(-1)^k \binom{2\sigma/\pi}{k}\binom{-2\sigma/\pi}{2m-k}\right) mb^{m-1},$$

where δ,σ and b are as given in Theorem 2.1. The result is sharp.

Proof. . Since

$$\frac{1}{2}\left((1+t)^{\rho}(1-t)^{-\rho} + (1-t)^{\rho}(1+t)^{-\rho}\right)$$
$$= 1 + \sum_{m=1}^{\infty}\left(\sum_{k=0}^{2m}(-1)^k\binom{\rho}{k}\binom{-\rho}{2m-k}\right)t^{2m} \quad (\rho>0, |t|<1),$$

the function $\varphi(\zeta)$ given by (2.1) has the expansion

$$\varphi(\zeta) = \lambda_0 + \frac{\delta}{\alpha(1-\alpha^2)}\sum_{m=1}^{\infty}\left(\sum_{k=0}^{2m}(-1)^k\binom{2\sigma/\pi}{k}\binom{-2\sigma/\pi}{2m-k}\right)\zeta^m \quad (|\zeta|<1).$$

From this we have

$$h(z) = \varphi\left(\frac{z+b}{1+bz}\right) = 1 + Bz + \cdots \quad (z\in U),$$

where

$$B = \varphi'(b)(1-b^2) = \frac{\delta(1-b^2)}{\alpha(1-\alpha^2)}\sum_{m=1}^{\infty}\left(\sum_{k=0}^{2m}(-1)^k\binom{2\sigma/\pi}{k}\binom{-2\sigma/\pi}{2m-k}\right)mb^{m-1}.$$

Also it follows from the proof of Theorem 2.1 that $B = h'(0) > 0$.

If $f(z) = z^p + \sum_{m=1}^{\infty} a_{p+m} z^{p+m} \in C_p(n, \alpha, \beta, \lambda, \mu)$, then we have from Theorem 2.1 that

$$J_p(n, \beta, f(z)) = (1-\beta)(1 + (p+n)a_{p+1}z + \cdots) + \beta \left(1 + \frac{1}{p}(p+1)(p+n)a_{p+1}z + \cdots\right)$$

$$= 1 + \left(1 + \frac{\beta}{p}\right)(p+n)a_{p+1}z + \cdots \prec h(z),$$

that is,

$$\frac{1}{B}\left(1 + \frac{\beta}{p}\right)(p+n)a_{p+1}z + \cdots \prec \frac{1}{B}(h(z) - 1) \in K.$$

Now an application of Lemma 1.2 gives

$$|a_{p+1}| \leq \frac{pB}{(p+\beta)(p+n)},$$

which proves (3.11).

The result is sharp since the equality in (3.11) is attained for the function $f_\beta(z)$ defined by (2.5). The proof of the theorem is completed. ∎

Theorem 3.8. *Let $f(z) \in A_p$ satisfy*

$$|J_p(n, \beta, f(z)) - a| < d \quad (z \in U), \tag{3.12}$$

where

$$d = d(a, \alpha, \lambda, \mu) = \begin{cases} a - \lambda_1, & \lambda_1 < a \leq \lambda + \frac{\lambda_1}{\alpha^2}, \\ \sqrt{\frac{\mu}{\alpha^2} + a^2 - \lambda^2 - \alpha^2(a-\lambda)^2}, & a \geq \lambda + \frac{\lambda_1}{\alpha^2} \end{cases}$$

and $J_p(n, \beta, f(z))$ and λ_1 are as given in Theorem 2.1. Then $f(z) \in C_p(n, \alpha, \beta, \lambda, \mu)$ and the bound d in (3.12) is sharp for each $a (a > \lambda_1)$.

Proof. . Let $h(z)$ be given by (2.1) and d denote the minimum distance from the point $a (a > \lambda_1)$ to the points on the hyperbola $\partial h(U)$ given by (2.6). Then

$$d^2 = \min_{u \geq \lambda_1} g(u), \quad g(u) = (u-a)^2 + \frac{1}{\alpha^2}(u^2 + \mu) - (u - \lambda)^2.$$

If $\lambda_1 < a \leq \lambda + \frac{\lambda_1}{\alpha^2}$, then

$$g'(u) = \frac{2}{\alpha^2}\left(u - \alpha^2(a - \lambda)\right) \geq 0 \quad (u \geq \lambda_1). \tag{3.13}$$

Note that $\lambda_1 \in \partial h(U)$. It follows from (3.13) that

$$d = \sqrt{g(\lambda_1)} = \sqrt{(\lambda_1 - a)^2 + \frac{1}{\alpha^2}(\lambda_1^2 + \mu) - (\lambda_1 - \lambda)^2} = a - \lambda_1.$$

If $a \geq \lambda + \frac{\lambda_1}{\alpha^2}$, then the function $g(u) (u \geq \lambda_1)$ attains its minimum value at $u = \alpha^2(a - \lambda)$ and so

$$d = \sqrt{g(\alpha^2(a-\lambda))} = \sqrt{\frac{\mu}{\alpha^2} + a^2 - \lambda^2 - \alpha^2(a-\lambda)^2}.$$

Consequently, if $f(z) \in A_p$ satisfies (3.12) then the domain of the values of $J_p(n, \beta, f(z))$ for $z \in U$ is contained in $h(U)$, which implies that $J_p(n, \beta, f(z)) \prec h(z)$. Hence $f(z) \in C_p(n, \alpha, \beta, \lambda, \mu)$ by Theorem 2.1. Furthermore, the bound d in (3.12) is sharp for the function $f_\beta(z)$ defined by (2.5). ∎

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