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**SOME IDENTITIES FOR RAMANUJAN - GÖLLNITZ - GORDON  
CONTINUED FRACTION**

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**ABSTRACT.** In this paper, we obtain certain  $P$ - $Q$  eta-function identities, using which we establish identities providing modular relations between Ramanujan-Göllnitz-Gordon continued fraction  $H(q)$  and  $H(q^n)$  for  $n = 2, 3, 4, 5, 7, 8, 9, 11, 13, 15, 17, 19, 23, 25, 29$  and 55.

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## 1. INTRODUCTION

In Chapter 16, of his Second notebook [15] Ramanujan has defined his theta-function as

$$(1.1) \quad f(a, b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1.$$

Following Ramanujan, we define

$$(1.2) \quad \varphi(q) = f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; -q)_{\infty}}{(q; -q)_{\infty}},$$

$$(1.3) \quad \psi(q) = f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}},$$

$$(1.4) \quad f(-q) = \sum_{n=-\infty}^{\infty} q^{\frac{n(3n-1)}{2}} = (q; q)_{\infty},$$

$$(1.5) \quad \chi(q) := (-q; q^2)_{\infty},$$

where

$$(a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1.$$

On page 229 of his second notebook, Ramanujan recorded the continued fraction which is known as Ramanujan-Göllnitz-Gordon continued fraction along with the two identities as follows:

$$(1.6) \quad H(q) := \frac{q^{\frac{1}{2}}}{1+q} + \frac{q^2}{1+q^3} + \frac{q^4}{1+q^5} + \frac{q^6}{1+q^7} + \cdots, \quad |q| < 1,$$

$$(1.7) \quad \frac{1}{H(q)} - H(q) = \frac{\varphi(q^2)}{q^{\frac{1}{2}}\psi(q^4)},$$

and

$$(1.8) \quad \frac{1}{H(q)} + H(q) = \frac{\varphi(q)}{q^{\frac{1}{2}}\psi(q^4)}.$$

Proofs of the above identities (1.6), (1.7) and (1.8) can be found in [15, p. 221] and for more details one can see, [8] and [18].

Now we define modular equation in brief. Let  $K, K', L$  and  $L'$  denote the complete elliptic integrals of the first kind associated with the moduli  $k, k' := \sqrt{1-k^2}, l$  and  $l' := \sqrt{1-l^2}$  respectively, where  $0 < k, l < 1$ . For a fixed positive integer  $n$ , suppose that

$$(1.9) \quad n \frac{K'}{K} = \frac{L'}{L}.$$

Then a modular equation of degree  $n$  is a relation between  $k$  and  $l$  induced by (1.9). Following Ramanujan, we set  $\alpha = k^2$  and  $\beta = l^2$ . Then we say  $\beta$  is of degree  $n$  over  $\alpha$ .

Recently, in [10] M. S. Mahadeva Naika, S. Chandankumar and M. Manjunatha have established some new modular relations for Ramanujan-Göllnitz-Gordon continued fraction  $H(q)$  with  $H(q^n)$  for  $n = 6, 10, 14$  and  $16$  and also established their explicit evaluations.

In this paper, we prove the relations between  $H(q)$  and  $H(q^n)$  for  $n = 2, 3, 4, 5, 7, 8, 9, 11, 13, 15, 17, 19, 23, 25, 29$  and 55.

## 2. PRELIMINARY SECTION

**Lemma 2.1.**

$$(2.1) \quad f(a, b) = f(a^3b, ab^3) + af\left(\frac{b}{a}, \frac{a}{b}a^4b^4\right).$$

For a proof of (2.1), see [5, Entry 30(ii),(iii), p.46].

**Lemma 2.2.**

$$(2.2) \quad \varphi(q) = \sqrt{z}$$

and

$$(2.3) \quad \varphi(-q) = \sqrt{z}(1-t)^{1/4}.$$

For the proofs of (2.2) and (2.3), see [5, Entry 10(i),(ii), p.122].

**Lemma 2.3.** *If  $\beta$  is of degree 2 over  $\alpha$ , then*

$$(2.4) \quad (1 - \sqrt{1 - \alpha})(1 - \sqrt{\beta}) = 2\sqrt{\beta(1 - \alpha)}.$$

For a proof of (2.4), see [4, Entry 17.3.1, p.385].

**Lemma 2.4.** *If  $\beta$  has degree 3 over  $\alpha$ , then*

$$(2.5) \quad (\alpha\beta)^{1/4} + \{(1 - \alpha)(1 - \beta)\}^{1/4} = 1.$$

For a proof of (2.5), see [5, Entry 5(ii), p.230].

**Lemma 2.5.** *If  $\beta$  has degree 4 over  $\alpha$ , then*

$$(2.6) \quad (1 - \sqrt[4]{1 - \alpha})(1 - \sqrt[4]{\beta}) = 2\sqrt[4]{\beta(1 - \alpha)}.$$

For a proof of (2.6), see [4, Entry 17.3.2, p.385].

**Lemma 2.6.** *If  $\beta$  has degree 5 over  $\alpha$ , then*

$$(2.7) \quad (\alpha\beta)^{1/2} + \{(1 - \alpha)(1 - \beta)\}^{1/2} + 2\{16\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/6} = 1.$$

For a proof of (2.7), see [5, Entry 13(i), p.280].

**Lemma 2.7.** *If  $\beta$  has degree 7 over  $\alpha$ , then*

$$(2.8) \quad (\alpha\beta)^{1/8} + \{(1 - \alpha)(1 - \beta)\}^{1/8} = 1.$$

For a proof of (2.8), see [5, Entry 19(i), p.314].

**Lemma 2.8.** *If  $\beta$  has degree 8 over  $\alpha$ , then*

$$(2.9) \quad (1 - (1 - \alpha)^{1/4})(1 - \beta^{1/4}) = 2\sqrt{2}(\beta(1 - \alpha))^{1/8}.$$

For a proof of (2.9), see [5].

**Lemma 2.9.** *If  $\beta$  has degree 9 over  $\alpha$ , then*

$$(2.10) \quad \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1 - \beta}{1 - \alpha}\right)^{1/8} - \left(\frac{\beta(1 - \beta)}{\alpha(1 - \alpha)}\right)^{1/8} = \sqrt{m}.$$

$$(2.11) \quad \left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1 - \alpha}{1 - \beta}\right)^{1/8} - \left(\frac{\alpha(1 - \alpha)}{\beta(1 - \beta)}\right)^{1/8} = \frac{3}{\sqrt{m}}.$$

For the proofs of (2.10) and (2.11), see [5, Entry 3(x),(xi), p.352].

**Lemma 2.10.** *If  $\beta$  has degree 11 over  $\alpha$ , then*

$$(2.12) \quad (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} = 1.$$

For a proof of (2.12), see [5, Entry 7, p.363].

Let

$$(2.13) \quad U = 1 - \sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)},$$

$$(2.14) \quad V = 64 \left( \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} - \sqrt{\alpha\beta(1-\alpha)(1-\beta)} \right),$$

and

$$(2.15) \quad W = 32\sqrt{\alpha\beta(1-\alpha)(1-\beta)}.$$

**Lemma 2.11.** *Let  $U$ ,  $V$  and  $W$  be defined as in (2.13)–(2.15). If  $\beta$  has degree 13 over  $\alpha$ , then*

$$(2.16) \quad \sqrt{U}(U^3 + 8W) = \sqrt{W}(11U^2 + V).$$

For a proof of (2.16), see [4, Entry 63, p.387].

**Lemma 2.12.** *Let  $\alpha$  and  $\beta$  has degrees 3, 5 or 1, 15 respectively, then*

$$(2.17) \quad \begin{aligned} & (\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} \pm \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \\ & = \left\{ \frac{1}{2} \left( 1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} \right) \right\}^{1/2}, \end{aligned}$$

where the minus sign is chosen in the first case and the plus sign is selected in the second case.

For a proof of (2.17), see [5, Entry 21, p.435].

**Lemma 2.13.** *If  $\beta$  has degree 17 over  $\alpha$ , then*

$$(2.18) \quad \begin{aligned} m &= \left( \frac{\beta}{\alpha} \right)^{1/4} + \left( \frac{(1-\beta)}{(1-\alpha)} \right)^{1/4} + \left( \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \right)^{1/4} \\ &\quad - 2 \left( \frac{\beta(1-\beta)}{\alpha(1-\alpha)} \right)^{1/8} \left\{ 1 + \left( \frac{\beta}{\alpha} \right)^{1/8} + \left( \frac{1-\beta}{1-\alpha} \right)^{1/8} \right\}. \end{aligned}$$

$$(2.19) \quad \begin{aligned} \frac{17}{m} &= \left( \frac{\alpha}{\beta} \right)^{1/4} + \left( \frac{1-\alpha}{1-\beta} \right)^{1/4} + \left( \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right)^{1/4} \\ &\quad - 2 \left( \frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right)^{1/8} \left\{ 1 + \left( \frac{\alpha}{\beta} \right)^{1/8} + \left( \frac{1-\alpha}{1-\beta} \right)^{1/8} \right\}. \end{aligned}$$

For the proofs of equations (2.18) and (2.19), see [4, Entry 17.3.26, p.392].

Let

$$(2.20) \quad X = 1 - \sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)},$$

$$(2.21) \quad Y = 16 \left( \sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} - \sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)} \right),$$

and

$$(2.22) \quad Z = 16\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}.$$

**Lemma 2.14.** Let  $X, Y$  and  $Z$  be defined as in (2.20)-(2.22). If  $\beta$  has degree 19 over  $\alpha$ , then

$$(2.23) \quad X^5 - 7X^2Z - YZ = 0.$$

For a proof of (2.23), see [7, Entry 58, p.386].

**Lemma 2.15.** If  $\beta$  has degree 23 over  $\alpha$ , then

$$(2.24) \quad (\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1.$$

For a proof of (2.24), see [5, Entry 15, p.411].

**Lemma 2.16.** If  $\beta$  has degree 25 over  $\alpha$ , then

$$(2.25) \quad \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} - 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12} = \sqrt{m}.$$

$$(2.26) \quad \left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} - 2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12} = \frac{5}{\sqrt{m}}.$$

For the proofs of (2.25) and (2.26), see [4, Entry 17.3.27, p.392].

**Lemma 2.17.** Let  $U, V$  and  $W$  be defined as in (2.13)-(2.15). If  $\beta$  has degree 29 over  $\alpha$ , then

$$(2.27) \quad \sqrt{U}(U^2 + 17UW^{1/3} - 9W^{2/3}) - W^{1/6}(9X^2 + Y - 13UW^{1/3} + 15W^{2/3}) = 0.$$

For a proof of (2.27), see [7, Entry 65, p.388].

Let

$$(2.28) \quad L = 1 - \{\alpha\beta\}^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8},$$

$$(2.29) \quad M = \left\{ \frac{1}{2} \left( 1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} \right) \right\}^{1/2} - \{\alpha\beta\}^{1/8} \\ - \{(1-\alpha)(1-\beta)\}^{1/8} + \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8},$$

and

$$(2.30) \quad N = 4\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}.$$

**Lemma 2.18.** Let  $L, M$  and  $N$  be defined as in (2.28)-(2.30). If  $\beta$  has degree 55 over  $\alpha$ , then

$$(2.31) \quad M(L^2 - M)^2 = N(L^3 - N).$$

For a proof of (2.31), see [4, Entry 17.4.4, p.393].

### 3. MAIN RESULTS

In this section, we establish several new theta function identities which are useful in proving certain identities for  $H(q)$ .

**Lemma 3.1.**

$$(3.1) \quad \frac{\varphi(-\sqrt{q})}{\varphi(\sqrt{q})} = \frac{1 - 2H(q) - H^2(q)}{1 + 2H(q) - H^2(q)}.$$

*Proof.* Putting  $a = b = q$  in the equation (2.1), we deduce that

$$(3.2) \quad f(q, q) = f(q^4, q^4) + qf(1, q^8).$$

Replacing  $q$  by  $-q$  in the equation (3.2), we find that

$$(3.3) \quad f(-q, -q) = f(q^4, q^4) - qf(1, q^8).$$

From the equations (1.7), (3.2) and (3.3), we obtain (3.1). □

**Lemma 3.2.** *If  $0 < t < 1$ , then*

$$(3.4) \quad \frac{\varphi(-q)}{\varphi(q)} = (1-t)^{1/4}.$$

*Proof.* From the equations (2.2) and (2.3), we obtain (3.4). □

Set

$$(3.5) \quad P = \{(1-\alpha)(1-\beta)\}^{1/4}$$

and

$$(3.6) \quad Q = \left\{ \frac{1-\alpha}{1-\beta} \right\}^{1/4}.$$

where  $\beta$  is of degree  $n$  over  $\alpha$ .

Using the equations (3.5) and (3.6), we find that

$$(3.7) \quad \alpha = 1 - P^2 Q^2$$

and

$$(3.8) \quad \beta = 1 - \frac{P^2}{Q^2}.$$

Throughout this article we are using the following notations:

$$A = \sqrt[4]{1-\alpha}, \quad B = \sqrt[4]{\beta}, \quad B_2 = \sqrt{\beta}, \quad a_1 = \sqrt[8]{\alpha\beta}, \quad a_2 = \sqrt[4]{\alpha\beta}, \quad a_4 = \sqrt{\alpha\beta},$$

$$b_1 = \sqrt[8]{(1-\alpha)(1-\beta)}, \quad b_2 = \sqrt[8]{\frac{1-\alpha}{1-\beta}}, \quad c_1 = \sqrt[8]{\frac{\alpha}{\beta}}, \quad c_2 = \sqrt[4]{\frac{\alpha}{\beta}}, \quad c_3 = \sqrt{\frac{\alpha}{\beta}}.$$

**Lemma 3.3.** *Let  $P$  and  $Q$  be defined as in (3.5) and (3.6). If  $\beta$  is of degree 2 over  $\alpha$ , then*

$$(3.9) \quad \frac{2Q}{P} = \left( \sqrt{PQ} + \frac{1}{\sqrt{PQ}} \right).$$

*Proof.* The equation (2.4) can be rewritten as

$$(3.10) \quad \beta = \left[ \frac{1 - \sqrt{1-\alpha}}{1 + \sqrt{1-\alpha}} \right]^2.$$

Employing the equations (3.7) and (3.8) in the above equation (3.10), we deduce that

$$(3.11) \quad 1 - \frac{P^2}{Q^2} = \left( \frac{1 - PQ}{1 + PQ} \right)^2.$$

On simplification, we obtain the equation (3.9). □

**Lemma 3.4.** *Let  $P$  and  $Q$  be defined as in (3.5) and (3.6). If  $\beta$  is of degree 3 over  $\alpha$ , then*

$$(3.12) \quad Q^2 + \frac{1}{Q^2} + 6 = 4 \left( P + \frac{1}{P} \right).$$

*Proof.* Employing the equations (3.5), (3.7) and (3.8) in the equation (2.5), we find that

$$(3.13) \quad (1 - P^2 Q^2) \left( 1 - \frac{P^2}{Q^2} \right) = (1 - P)^4.$$

On simplification, we obtain the required equation (3.12). □

**Lemma 3.5.** Let  $P$  and  $Q$  be defined as in (3.5) and (3.6). If  $\beta$  is of degree 4 over  $\alpha$ , then

$$(3.14) \quad 64(1 + PQ)^2 \left(1 - \frac{Q^2}{P^2}\right) + 4\frac{P}{Q} = \frac{P}{Q} \left(PQ + \frac{1}{PQ}\right) \left[4 - \frac{P}{Q} \left(PQ + \frac{1}{PQ}\right)\right].$$

*Proof.* The equation (2.6) can be rewritten as

$$(3.15) \quad \beta = \left[\frac{1 - \sqrt[4]{1 - \alpha}}{1 + \sqrt[4]{1 - \alpha}}\right]^4.$$

Employing the equations (3.7) and (3.8) in the above equation (3.15), we find that

$$(3.16) \quad 1 - \frac{P^2}{Q^2} = \left(\frac{1 - A}{1 + A}\right)^4.$$

Isolating the terms having  $A$  on one side of the equation (3.16) and then squaring both sides, we obtain the equation (3.14). □

**Lemma 3.6.** Let  $P$  and  $Q$  be defined in (3.5) and (3.6), If  $\beta$  is of degree 5 over  $\alpha$ , then

$$(3.17) \quad Q^3 + \frac{1}{Q^3} + 10 \left(Q^2 + \frac{1}{Q^2}\right) + 15 \left(Q + \frac{1}{Q}\right) + 20 = 16 \left(P^2 + \frac{1}{P^2}\right).$$

*Proof.* From the equations (3.5) and (2.7), we find that

$$(3.18) \quad (32a_4P) - (1 - a_4 - P^2)^3 = 0.$$

Isolating the terms having  $a_4$  on one side of the equation (3.18) and then squaring both sides, we find that

$$(3.19) \quad \begin{aligned} & (16P^4Q^3 - P^2 - 10P^2Q - 15P^2Q^2 + 16Q^3 \\ & + 20P^2Q^3 - 15P^2Q^4 - 10P^2Q^5 - Q^6P^2) \\ & (16P^4Q^3 + P^2 - 10P^2Q + 16Q^3 + 20P^2Q^3 \\ & + 15P^2Q^4 - 10P^2Q^5 + Q^6P^2 + 15P^2Q^2) = 0. \end{aligned}$$

By examining the behaviour of first factor near  $q = 0$ , it can be seen that there is a neighbourhood about the origin, where the first factor is zero, whereas the second factor is not zero in this neighbourhood. By the Identity Theorem first factor vanishes identically. This completes the proof. □

**Lemma 3.7.** Let  $P$  and  $Q$  be defined in (3.5) and (3.6), If  $\beta$  is of degree 7 over  $\alpha$ , then

$$(3.20) \quad \begin{aligned} & Q^4 + \frac{1}{Q^4} + 140 \left(Q^2 + \frac{1}{Q^2}\right) + 70 \\ & = 64 \left(P^3 + \frac{1}{P^3}\right) + 112 \left(P^2 + \frac{1}{P^2}\right) + 56 \left(P + \frac{1}{P}\right) \left[2 - \left(Q^2 + \frac{1}{Q^2}\right)\right]. \end{aligned}$$

*Proof.* Employing the equations (3.5) and (3.6) in the equation (2.7), we deduce that

$$(3.21) \quad (1 - P^2Q^2) \left(1 - \frac{P^2}{Q^2}\right) = (1 - b_1)^8.$$

On simplification, we find that

$$(3.22) \quad \begin{aligned} & -P^2 - P^2Q^4 + 8Q^2b_1 - 28PQ^2 + 56Q^2Pb_1 \\ & - 70P^2Q^2 + 56Q^2P^2b_1 - 28P^3Q^2 + 8Q^2P^3b_1 = 0. \end{aligned}$$

Isolating the terms having  $b_1$  on one side of the equation (3.22), and then squaring both sides, and employing the equation (3.5), we arrive at the equation (3.20). □

**Lemma 3.8.** Let  $P$  and  $Q$  be defined in (3.5) and (3.6), If  $\beta$  is of degree 8 over  $\alpha$ , then

$$(3.23) \quad P^{15}Q^8 - 65536P^7Q^{12} - 32768P^4Q^9 - 32768P^6Q^{11} \\ + 131072P^5Q^{10} - 471040P^7Q^8 - 242688P^6Q^7 - 242688P^8Q^9 - 26624P^9Q^{10} \\ + 259072P^8Q^5 + 405504P^9Q^6 + 259072P^{10}Q^7 + 59392P^{11}Q^8 + 16384Q^{13}P^4 \\ - 16384Q^{15}P^6 + 32768Q^{13}P^8 + 32768Q^{14}P^5 - 19456Q^{11}P^{10} + 3072Q^9P^{12} \\ - 65536Q^{12}P^3 - 16384Q^9 + 16384Q^{11}P^2 + 32768Q^{10}P + 70P^{11}Q^4 - 56P^{10}Q^3 \\ - 56P^{12}Q^5 + 28P^9Q^2 + 28P^{13}Q^6 - 8P^8Q - 8P^{14}Q^7 + 59392P^7Q^4 - 65536P^3Q^8 \\ + 32768P^2Q^7 + P^7 - 19456P^4Q^5 - 26624P^5Q^6 + 3072P^6Q^3 = 0.$$

*Proof.* Using the equations (3.5) and (3.6) in the equation (2.9), we deduce that

$$(3.24) \quad 1 - 2B + B_2 - 2A - 4AB - 2AB_2 + PQ - 2PQB + PQB_2 = 0.$$

Isolating the terms having  $A$  on one side of the equation (3.24), and then squaring both sides, we find that

$$(3.25) \quad 4PQ^3 + 24PQ^3B + 12PQ^3B_2 + 24PQ^3BB_2 - 2P^3Q - 2Q^2 + 4P^2Q^4BB_2 + P^2 \\ - 6Q^2B_2 + 4Q^2BB_2 - 2P^2Q^4 + 4Q^2B - 6P^2Q^4B_2 + 4P^2Q^4B + P^4Q^2 = 0.$$

Isolating the terms having  $B$  on one side of the equation (3.25), and then squaring both sides, we deduce that

$$(3.26) \quad 8P^2Q^2 - 4P^2Q^2B_2 + 976P^2Q^6 + 544PQ^5 + 544P^3Q^7 - 8P^4Q^8 - 544P^5Q^5 \\ - 976P^4Q^4 - P^4 + 1072P^2Q^6B_2 + 4P^7Q^3 + 8P^6Q^6 - 6P^6Q^2 + 4P^5Q + 8Q^4B_2 \\ + 8P^4Q^8B_2 + 480PQ^5B_2 - 8Q^4 - 240P^5Q^5B_2 - 536P^4Q^4B_2 - 4P^6Q^6B_2 - P^8Q^4 \\ - 240P^3Q^3B_2 - 544P^3Q^3 + 480P^3Q^7B_2 = 0.$$

Isolating the terms having  $B_2$  on one side of the equation (3.26), and then squaring both sides, we arrive at (3.23).  $\square$

**Lemma 3.9.** Let  $P$  and  $Q$  be defined in (3.5) and (3.6), If  $\beta$  is of degree 9 over  $\alpha$ , then

$$(3.27) \quad Q^6 + \frac{1}{Q^6} - 44 \left( Q^5 + \frac{1}{Q^5} \right) + 514 \left( Q^4 + \frac{1}{Q^4} \right) - 1212 \left( Q^3 + \frac{1}{Q^3} \right) \\ + 111 \left( Q^2 + \frac{1}{Q^2} \right) - 2328 \left( Q + \frac{1}{Q} \right) + 1308 = 256 \left( P^4 + \frac{1}{P^4} \right) \left[ \left( Q + \frac{1}{Q} \right) + 1 \right] \\ - 96 \left( P^2 + \frac{1}{P^2} \right) \left[ 3 \left( Q^3 + \frac{1}{Q^3} \right) - 2 \left( Q^2 + \frac{1}{Q^2} \right) + 13 \left( Q + \frac{1}{Q} \right) + 4 \right].$$

*Proof.* From the equations (2.10) and (2.11), we deduce that

$$(3.28) \quad -c_2 + c_1 - Q + b_1 + c_1Q + c_2b_1 = 0.$$

Isolating the terms having  $b_1$  on one side of the equation (3.28) and then squaring both sides, we deduce that

$$(3.29) \quad Q + Qc_3 - c_3 + 2c_2c_1 + 2c_2c_1Q - c_2 + 2c_1Q - 2Qc_2 - Q^2 + 2c_1Q^2 - c_2Q^2 = 0.$$



Again isolating the terms having  $c_1$  on one side of the equation (3.29) and then squaring both sides, we deduce that

$$(3.30) \quad -6Q^2c_3c_2P^2 - 6Qc_3c_2P^2 - 2Q^3c_3c_2P^2 + 2Q^2c_3c_2 + 6Q^3c_3c_2 - Q^2 + P^2Q^2 + 2Q^3c_2 + 6c_2Q^4 - Q^2c_3 + 6Q^3c_3 + 2Q^3 - 2Q^4 + 6c_3Q^4 + c_3P^2 + 2Q^4P^2 + 6Q^5c_2 + 2c_2Q^6 + 6Q^5c_3 - 2Q^3P^2 + Q^6P^2 - c_3Q^6 - 2Q^5P^2 + 2Q^5c_3c_2 - 6Q^3c_2P^2 - 6Qc_3P^2 - 2Qc_2P^2 - 6c_2Q^2P^2 + 6Q^4c_3c_2 - 2c_2Q^4P^2 - 6Q^2c_3P^2 - 6Q^3c_3P^2 - 2c_3c_2P^2 + c_3Q^4P^2 + 2Q^5 - Q^6 = 0$$

Eliminating  $c_2$  and  $c_3$  in the above equation (3.30) , we deduce that

$$(3.31) \quad (1 + Q)^4(192Q^4P^2 - 288Q^3P^2 - 384Q^6P^2 - 1248Q^5P^2 + 256Q^5 + 256Q^6 + 256Q^7 - 288Q^9P^2 - 1308Q^6P^4 - 111Q^4P^4 + 2328Q^5P^4 + 2328Q^7P^4 - 514Q^2P^4 - P^4 - 514Q^{10}P^4 + 1212Q^9P^4 - Q^{12}P^4 + 44Q^{11}P^4 + 44Q^8P^4 - 1248Q^5P^6 + 192Q^8P^6 - 288Q^9P^6 - 384Q^6P^6 + 192Q^4P^6 - 288Q^3P^6 + 256Q^5P^8 + 256Q^6P^8 + 256Q^7P^8 - 1248Q^7P^2 - 111Q^8P^4 - 1248Q^7P^6 + 192Q^8P^2 + 1212Q^3P^4) = 0.$$

Since  $(1 + Q)^4 \neq 0$ , it completes the proof. □

**Lemma 3.10.** *Let  $P$  and  $Q$  be defined in (3.5) and (3.6), If  $\beta$  is of degree 11 over  $\alpha$ , then*

$$(3.32) \quad Q^6 + \frac{1}{Q^6} + 1298 \left( Q^4 + \frac{1}{Q^4} \right) + 1915 \left( Q^2 + \frac{1}{Q^2} \right) - 7876 = 1024 \left( P^5 + \frac{1}{P^5} \right) + 5632 \left( P^4 + \frac{1}{P^4} \right) - 704 \left( P^3 + \frac{1}{P^3} \right) \left[ 2 \left( Q^2 + \frac{1}{Q^2} \right) - 23 \right] - 176 \left( P^2 + \frac{1}{P^2} \right) \left[ 25 \left( Q^2 + \frac{1}{Q^2} \right) - 106 \right] + 44 \left( P + \frac{1}{P} \right) \left[ 9 \left( Q^4 + \frac{1}{Q^4} \right) - 372 \left( Q^2 + \frac{1}{Q^2} \right) + 422 \right] = 0.$$

*Proof.* Employing the equation (3.5) in the equation (2.12), we find that

$$(3.33) \quad 16a_2P = (1 - a_2 - P)^3.$$

Isolating the terms having  $a_2$  on one side of the equation (3.33) and then squaring both sides, we find that

$$(3.34) \quad 38P^3 + 4Q^2 + 18P^2Q^2 + a_4P^2 - 84a_4P^3Q^2 - 82a_4P^2Q^2 - 4a_4P^4Q^2 + a_4P^2Q^4 + 38P^3Q^4 - 84a_4PQ^2 - 3P^4 - 4a_4Q^2 - 44PQ^2 - 3P^2 - 20P^3Q^2 + 18P^4Q^2 - 44P^5Q^2 + 4P^6Q^2 - 3P^2Q^4 - 3P^4Q^4 = 0.$$

Isolating the terms having  $a_4$  on one side of the equation (3.34) and then squaring both sides, we obtain the equation (3.32). □

**Lemma 3.11.** *Let  $P$  and  $Q$  be defined in (3.5) and (3.6), If  $\beta$  is of degree 13 over  $\alpha$ , then*

$$\begin{aligned}
 (3.35) \quad & Q^7 + \frac{1}{Q^7} + 130 \left( Q^6 + \frac{1}{Q^6} \right) + 5083 \left( Q^5 + \frac{1}{Q^5} \right) + 62036 \left( Q^4 + \frac{1}{Q^4} \right) \\
 & + 195689 \left( Q^3 + \frac{1}{Q^3} \right) + 289822 \left( Q^2 + \frac{1}{Q^2} \right) + 289822 \left( Q^2 + \frac{1}{Q^2} \right) \\
 & + 149435 \left( Q + \frac{1}{Q} \right) - 11752 \\
 = & 4096 \left( P^6 + \frac{1}{P^6} \right) - 6656 \left( P^4 + \frac{1}{P^4} \right) \left[ \left( Q^2 + \frac{1}{Q^2} \right) - 4 \left( Q + \frac{1}{Q} \right) - 10 \right] \\
 & + 208 \left( P^2 + \frac{1}{P^2} \right) \left[ 13 \left( Q^4 + \frac{1}{Q^4} \right) - 104 \left( Q^3 + \frac{1}{Q^3} \right) - 20 \left( Q^2 + \frac{1}{Q^2} \right) \right. \\
 & \left. + 128 \left( Q + \frac{1}{Q} \right) + 1422 \right].
 \end{aligned}$$

*Proof.* Employing the equation (3.5) in the equations (2.13), (2.14) and (2.15), we find that

$$(3.36) \quad U = 1 - a_4 - P^2, \quad V = 64(a_4 + P^2 - a_4P^2), \quad \text{and} \quad W = 32a_4P^2.$$

Employing the equation (3.36) in the equation (2.16), we find that

$$\begin{aligned}
 (3.37) \quad & 7P^6 - 7P^8 - a_4P^6 - 67616a_4P^8Q^4 + 11345a_4P^6Q^4 + 24a_4P^4Q^2 \\
 & - 67616a_4P^4Q^4 - 80a_4P^2Q^4 - 20528P^4Q^6 + 64a_4Q^6 + 147300P^6Q^6 \\
 & - 147300P^8Q^6 + 20528P^{10}Q^6 - 63456P^{12}Q^6 + 64P^{14}Q^6 + 63456P^2Q^6 \\
 & - 95184P^4Q^4 + 31742P^6Q^2 - 95184P^4Q^8 - 52137P^8Q^4 + 112Q^4P^2 - 56Q^2P^4 \\
 & + 112Q^8P^2 + 52137Q^4P^6 - 56Q^{10}P^4 + 52137Q^8P^6 - 52137Q^8P^8 + 56P^{10}Q^2 \\
 & - 112P^{12}Q^4 - 64Q^6 - 112P^{12}Q^8 + 95184P^{10}Q^4 + 95184P^{10}Q^8 - 31742Q^2P^8 \\
 & - 31742Q^{10}P^8 + 31742P^6Q^{10} + 7P^6Q^{12} - 67616a_4P^4Q^8 + 67616a_4P^2Q^6 \\
 & + 20432a_4P^4Q^6 + 196212a_4P^6Q^6 + 20432a_4P^8Q^6 + 67616a_4P^{10}Q^6 - 80a_4P^{10}Q^4 \\
 & - 80a_4P^{10}Q^8 - 67616a_4P^8Q^8 + 24a_4P^8Q^2 + 11345a_4P^6Q^8 + 24a_4P^8Q^{10} \\
 & + 56P^{10}Q^{10} - 7P^8Q^{12} + 3318a_4P^6Q^2 + 3318a_4P^6Q^{10} - 80a_4Q^8P^2 + 24a_4Q^{10}P^4 \\
 & - a_4P^6Q^{12} + 64a_4P^{12}Q^6 = 0.
 \end{aligned}$$

Isolating the terms having  $a_4$  on one side of the equation (3.37) and then squaring both sides, we find that

$$\begin{aligned}
 (3.38) \quad & (168064P^4Q^6 - 130P^6Q^{13} - 149435P^6Q^6 + 168064P^8Q^6 + 26624P^{10}Q^6 \\
 & + 26624P^2Q^6 - 21632P^4Q^4 - 5083P^6Q^2 + 168064P^4Q^8 - 21632P^8Q^4 \\
 & + 26624Q^8P^2 - 195689Q^4P^6 - 21632Q^{10}P^4 - 149435Q^8P^6 + 168064Q^8P^8 \\
 & + 26624P^{10}Q^8 - 21632Q^{10}P^8 - 195689P^6Q^{10} - 5083P^6Q^{12} + 2704P^4Q^3 \\
 & - 62036P^6Q^3 - 6656Q^5P^2 - 4160Q^5P^4 - Q^{14}P^6 + 2704P^8Q^3 - 130P^6Q \\
 & - 289822P^6Q^5 + 66560Q^7P^2 + 295776Q^7P^4 + 11752P^6Q^7 - 4160P^8Q^5
 \end{aligned}$$

$$\begin{aligned}
 &+295776P^8Q^7 - 289822P^6Q^9 - 6656P^{10}Q^5 + 66560P^{10}Q^7 - 4160P^8Q^9 - 4160P^4Q^9 \\
 &-62036P^6Q^{11} + 4096P^{12}Q^7 - 6656P^{10}Q^9 + 2704Q^{11}P^8 - 6656Q^9P^2 + 2704Q^{11}P^4 \\
 &-130P^6Q^{13} + 4096Q^7)(P^6 - 168064P^4Q^6 + 149435P^6Q^6 - 168064P^8Q^6 \\
 &-26624P^{10}Q^6 - 26624P^2Q^6 + 21632P^4Q^4 + 5083P^6Q^2 - 168064P^4Q^8 + 21632P^8Q^4 \\
 &-26624Q^8P^2 + 195689Q^4P^6 + 21632Q^{10}P^4 + 149435Q^8P^6 - 168064Q^8P^8 \\
 &-26624P^{10}Q^8 + 21632Q^{10}P^8 + 195689P^6Q^{10} + 5083P^6Q^{12} + 2704P^4Q^3 - 62036P^6Q^3 \\
 &-6656Q^5P^2 - 4160Q^5P^4 - 289822P^6Q^9 + 2704P^8Q^3 - 130P^6Q - 289822P^6Q^5 \\
 &+295776Q^7P^4 + 11752P^6Q^7 - 4160P^8Q^5 + 295776P^8Q^7 - 6656P^{10}Q^5 + 2704Q^{11}P^4 \\
 &+Q^{14}P^6 - 4160P^8Q^9 - 4160P^4Q^9 - 62036P^6Q^{11} + 4096P^{12}Q^7 - 6656P^{10}Q^9 \\
 &+2704Q^{11}P^8 - 6656Q^9P^2 + 66560Q^7P^2 + 66560P^{10}Q^7 + 4096Q^7 - P^6) = 0.
 \end{aligned}$$

By examining the behaviour of second factor near  $q = 0$ , it can be seen that there is a neighbourhood about the origin, where the second factor is not zero, whereas the first factor is zero in this neighbourhood. By the Identity Theorem first factor vanishes identically. This completes the proof.  $\square$

**Lemma 3.12.** *Let  $P$  and  $Q$  be defined in (3.5) and (3.6), If  $\beta$  is of degree 15 over  $\alpha$ , then*

$$\begin{aligned}
 (3.39) \quad &Q^{12} + \frac{1}{Q^{12}} + 8100 \left( Q^{10} + \frac{1}{Q^{10}} \right) + 1630210 \left( Q^8 + \frac{1}{Q^8} \right) \\
 &- 6072300 \left( Q^6 + \frac{1}{Q^6} \right) + 27556590 \left( Q^4 + \frac{1}{Q^4} \right) - 29064120 \left( Q^2 + \frac{1}{Q^2} \right) + 75698716 \\
 = &65536 \left( P^8 + \frac{1}{P^8} \right) - 16384 \left( P^7 + \frac{1}{P^7} \right) \left[ \left( Q^4 + \frac{1}{Q^4} \right) + 45 \right] \\
 &+4096 \left( P^6 + \frac{1}{P^6} \right) \left[ 45 \left( Q^4 + \frac{1}{Q^4} \right) - 40 \left( Q^2 + \frac{1}{Q^2} \right) + 991 \right] \\
 &+10240 \left( P^5 + \frac{1}{P^5} \right) \left[ 3 \left( Q^6 + \frac{1}{Q^6} \right) + 99 \left( Q^4 + \frac{1}{Q^4} \right) + 226 \left( Q^2 + \frac{1}{Q^2} \right) - 1092 \right] \\
 &-256 \left( P^4 + \frac{1}{P^4} \right) \left[ 1764 \left( Q^6 + \frac{1}{Q^6} \right) - 11415 \left( Q^4 + \frac{1}{Q^4} \right) + 45160 \left( Q^2 + \frac{1}{Q^2} \right) - 67210 \right] \\
 &-64 \left( P^3 + \frac{1}{P^3} \right) \left[ 255 \left( Q^8 + \frac{1}{Q^8} \right) + 32740 \left( Q^6 + \frac{1}{Q^6} \right) \right. \\
 &\quad \left. +280170 + 99180 \left( Q^4 + \frac{1}{Q^4} \right) + 434364 \left( Q^2 + \frac{1}{Q^2} \right) \right] \\
 &+80 \left( P^2 + \frac{1}{P^2} \right) \left[ 3243 \left( Q^8 + \frac{1}{Q^8} \right) - 55544 \left( Q^6 + \frac{1}{Q^6} \right) \right. \\
 &\quad \left. +168116 \left( Q^4 + \frac{1}{Q^4} \right) - 455244 \left( Q^2 + \frac{1}{Q^2} \right) + 367554 \right] \\
 &+8 \left( P + \frac{1}{P} \right) \left[ 245 \left( Q^{10} + \frac{1}{Q^{10}} \right) - 135698 \left( Q^8 + \frac{1}{Q^8} \right) \right. \\
 &\quad \left. +726705 \left( Q^6 + \frac{1}{Q^6} \right) - 2845080 \left( Q^4 + \frac{1}{Q^4} \right) + 4373850 \left( Q^2 + \frac{1}{Q^2} \right) - 7244460 \right].
 \end{aligned}$$

**Lemma 3.13.** *Let  $P$  and  $Q$  be defined in (3.5) and (3.6), If  $\beta$  is of degree 17 over  $\alpha$ , then*

$$\begin{aligned}
 (3.40) \quad & Q^9 + \frac{1}{Q^9} - 306 \left( Q^8 + \frac{1}{Q^8} \right) + 28441 \left( Q^7 + \frac{1}{Q^7} \right) - 793968 \left( Q^6 + \frac{1}{Q^6} \right) \\
 & + 2120308 \left( Q^5 + \frac{1}{Q^5} \right) + 12298888 \left( Q^4 + \frac{1}{Q^4} \right) + 33457156 \left( Q^3 + \frac{1}{Q^3} \right) \\
 & + 27917808 \left( Q^2 + \frac{1}{Q^2} \right) + 9441902 \left( Q + \frac{1}{Q} \right) \\
 & = 65536 \left( P^8 + \frac{1}{P^8} \right) - 835584 \left( P^6 + \frac{1}{P^6} \right) \left[ \frac{1}{6} \left( Q^2 + \frac{1}{Q^2} \right) + \left( Q + \frac{1}{Q} \right) + 3 \right] \\
 & - 835584 \left( P^6 + \frac{1}{P^6} \right) \times \left[ \frac{1}{6} \left( Q^2 + \frac{1}{Q^2} \right) + \left( Q + \frac{1}{Q} \right) + 3 \right] \\
 & + 4352 \left( P^4 + \frac{1}{P^4} \right) \left[ 21 \left( Q^4 + \frac{1}{Q^4} \right) + 475 \left( Q^3 + \frac{1}{Q^3} \right) \right. \\
 & \quad \left. + 2298 \left( Q^2 + \frac{1}{Q^2} \right) + 3477 \left( Q + \frac{1}{Q} \right) + 4866 \right] \\
 & - 544 \left( P^2 + \frac{1}{P^2} \right) \left[ 33 \left( Q^6 + \frac{1}{Q^6} \right) + 2212 \left( Q^5 + \frac{1}{Q^5} \right) + 15234 \left( Q^4 + \frac{1}{Q^4} \right) \right. \\
 & \quad \left. + 22388 \left( Q^3 + \frac{1}{Q^3} \right) + 16015 \left( Q^2 + \frac{1}{Q^2} \right) - 35864 \left( Q + \frac{1}{Q} \right) - 4608 \right].
 \end{aligned}$$

**Lemma 3.14.** *Let  $P$  and  $Q$  be defined in (3.5) and (3.6), If  $\beta$  is of degree 19 over  $\alpha$ , then*

$$\begin{aligned}
 (3.41) \quad & Q^{10} + \frac{1}{Q^{10}} + 39710 \left( Q^8 + \frac{1}{Q^8} \right) + 61482057 \left( Q^6 + \frac{1}{Q^6} \right) \\
 & + 442721128 \left( Q^4 + \frac{1}{Q^4} \right) + 832992338 \left( Q^2 + \frac{1}{Q^2} \right) + 75505688 \\
 & = 262144 \left( P^9 + \frac{1}{P^9} \right) - 2490368 \left( P^8 + \frac{1}{P^8} \right) - 311296 \left( P^7 + \frac{1}{P^7} \right) \left[ 2 \left( Q^2 + \frac{1}{Q^2} \right) - 37 \right] \\
 & + 155648 \left( P^6 + \frac{1}{P^6} \right) \left[ 3 \left( Q^2 + \frac{1}{Q^2} \right) - 73 \right] + 19456 \left( P^5 + \frac{1}{P^5} \right) \left[ -1739 + 25 \left( Q^4 + \frac{1}{Q^4} \right) \right. \\
 & \quad \left. - 2046 \left( Q^2 + \frac{1}{Q^2} \right) \right] + 34048 \left( P^4 + \frac{1}{P^4} \right) \left[ 230 \left( Q^4 + \frac{1}{Q^4} \right) + 439 \left( Q^2 + \frac{1}{Q^2} \right) + 2302 \right] \\
 & - 2432 \left( P^3 + \frac{1}{P^3} \right) \times \left[ -246014 + 55 \left( Q^6 + \frac{1}{Q^6} \right) \right. \\
 & \quad \left. - 26313 \left( Q^4 + \frac{1}{Q^4} \right) - 127659 \left( Q^2 + \frac{1}{Q^2} \right) \right] \\
 & - 608 \left( P^2 + \frac{1}{P^2} \right) \times \left[ 9487 \left( Q^6 + \frac{1}{Q^6} \right) - 57842 \left( Q^4 + \frac{1}{Q^4} \right) \right. \\
 & \quad \left. - 607903 \left( Q^2 + \frac{1}{Q^2} \right) - 1287420 \right]
 \end{aligned}$$

$$+76 \left( P + \frac{1}{P} \right) \times \left[ 103 \left( Q^8 + \frac{1}{Q^8} \right) - 469880 \left( Q^6 + \frac{1}{Q^6} \right) \right. \\ \left. - 3161628 \left( Q^4 + \frac{1}{Q^4} \right) - 4184264 \left( Q^2 + \frac{1}{Q^2} \right) - 747030 \right].$$

**Lemma 3.15.** *Let  $P$  and  $Q$  be defined in (3.5) and (3.6), If  $\beta$  is of degree 23 over  $\alpha$ , then*

$$(3.42) \quad Q^{12} + \frac{1}{Q^{12}} + 164772 \left( Q^{10} + \frac{1}{Q^{10}} \right) + 1416790018 \left( Q^8 + \frac{1}{Q^8} \right) \\ - 2130154476 \left( Q^6 + \frac{1}{Q^6} \right) - 7724991761 \left( Q^4 + \frac{1}{Q^4} \right) + 249131895880 \left( Q^2 + \frac{1}{Q^2} \right) \\ = 321023061476 + 4194304 \left( P^{11} + \frac{1}{P^{11}} \right) + 96468992 \left( P^{10} + \frac{1}{P^{10}} \right) \\ - 12058624 \left( P^9 + \frac{1}{P^9} \right) \left[ \left( Q^2 + \frac{1}{Q^2} \right) - 93 \right] \\ + 1507328 \left( P^8 + \frac{1}{P^8} \right) \times \left[ 296 \left( Q^2 + \frac{1}{Q^2} \right) + 4397 \right] \\ + 1130496 \left( P^7 + \frac{1}{P^7} \right) \left[ 11 \left( Q^4 + \frac{1}{Q^4} \right) - 3148 \left( Q^2 + \frac{1}{Q^2} \right) + 21389 \right] \\ + 94208 \left( P^6 + \frac{1}{P^6} \right) \left[ 5617 \left( Q^4 + \frac{1}{Q^4} \right) - 121400 \left( Q^2 + \frac{1}{Q^2} \right) + 566351 \right] \\ + 47104 \left( P^5 + \frac{1}{P^5} \right) \times \left[ 101325 \left( Q^4 + \frac{1}{Q^4} \right) - 856348 \left( Q^2 + \frac{1}{Q^2} \right) + 23576361 \right] \\ - 58888 \left( P^4 + \frac{1}{P^4} \right) \left[ 16164 \left( Q^6 + \frac{1}{Q^6} \right) - 725439 \left( Q^4 + \frac{1}{Q^4} \right) \right. \\ \left. + 11202632 \left( Q^2 + \frac{1}{Q^2} \right) - 28284794 \right] \\ + 1472 \left( P^3 + \frac{1}{P^3} \right) \left[ 593 \left( Q^8 + \frac{1}{Q^8} \right) - 2087612 \left( Q^6 + \frac{1}{Q^6} \right) \right. \\ \left. + 14698260 \left( Q^4 + \frac{1}{Q^4} \right) - 93333412 \left( Q^2 + \frac{1}{Q^2} \right) + 189688822 \right] \\ - 16 \left( P^2 + \frac{1}{P^2} \right) \times \left[ 5258007 \left( Q^8 + \frac{1}{Q^8} \right) - 12214034 \left( Q^6 + \frac{1}{Q^6} \right) \right. \\ \left. - 687894396 \left( Q^4 + \frac{1}{Q^4} \right) - 18978170550 + 10299016728 \left( Q^2 + \frac{1}{Q^2} \right) \right] \\ - 184 \left( P + \frac{1}{P} \right) \left[ 147 \left( Q^{10} + \frac{1}{Q^{10}} \right) - 4008318 \left( Q^8 + \frac{1}{Q^8} \right) + 33717559 \left( Q^6 + \frac{1}{Q^6} \right) \right. \\ \left. - 196168488 \left( Q^4 + \frac{1}{Q^4} \right) + 1477813558 \left( Q^2 + \frac{1}{Q^2} \right) - 2080998580 \right].$$

**Lemma 3.16.** *Let  $P$  and  $Q$  be defined in (3.5) and (3.6), If  $\beta$  is of degree 25 over  $\alpha$ , then*

$$\begin{aligned}
(3.43) \quad & Q^{15} + \frac{1}{Q^{15}} - 1374 \left( Q^{14} + \frac{1}{Q^{14}} \right) + 624051 \left( Q^{13} + \frac{1}{Q^{13}} \right) \\
& - 105877724 \left( Q^{12} + \frac{1}{Q^{12}} \right) + 5588665101 \left( Q^{11} + \frac{1}{Q^{11}} \right) - 6149701233 \left( Q^{10} + \frac{1}{Q^{10}} \right) \\
& + 216547952495 \left( Q^9 + \frac{1}{Q^9} \right) - 274685537880 \left( Q^8 + \frac{1}{Q^8} \right) + 316427310845 \left( Q^7 + \frac{1}{Q^7} \right) \\
& - 103388902030 \left( Q^6 + \frac{1}{Q^6} \right) + 1839454577495 \left( Q^5 + \frac{1}{Q^5} \right) - 2319422065380 \left( Q^4 + \frac{1}{Q^4} \right) \\
& + 2016536239145 \left( Q^3 + \frac{1}{Q^3} \right) - 1982303356330 \left( Q^2 + \frac{1}{Q^2} \right) + 3840832608595 \left( Q + \frac{1}{Q} \right) \\
& - 4110726704080 \\
& = 16777216 \left( P^{12} + \frac{1}{P^{12}} \right) \times \left[ \left( Q^2 + \frac{1}{Q^2} \right) + \left( Q + \frac{1}{Q} \right) + 1 \right] \\
& - 10485760 \left( P^{10} + \frac{1}{P^{10}} \right) \left[ 5 \left( Q^4 + \frac{1}{Q^4} \right) - 80 \left( Q^3 + \frac{1}{Q^3} \right) \right. \\
& \quad \left. + 195 \left( Q^2 + \frac{1}{Q^2} \right) + 110 \left( Q + \frac{1}{Q} \right) \right] \\
& + 327680 \left( P^8 + \frac{1}{P^8} \right) \left[ 185 \left( Q^6 + \frac{1}{Q^6} \right) - 10545 \left( Q^5 + \frac{1}{Q^5} \right) \right. \\
& \quad \left. + 87090 \left( Q^4 + \frac{1}{Q^4} \right) - 158098 \left( Q^3 + \frac{1}{Q^3} \right) + 231157 \left( Q^2 + \frac{1}{Q^2} \right) \right. \\
& \quad \left. - 16013 \left( Q + \frac{1}{Q} \right) + 99552 \right] \\
& + 40960 \left( P^6 + \frac{1}{P^6} \right) \left[ -13095078 + 765 \left( Q^8 + \frac{1}{Q^8} \right) - 99460 \left( Q^7 + \frac{1}{Q^7} \right) \right. \\
& \quad \left. + 1478858 \left( Q^6 + \frac{1}{Q^6} \right) - 4593412 \left( Q^5 + \frac{1}{Q^5} \right) + 6465168 \left( Q^4 + \frac{1}{Q^4} \right) \right. \\
& \quad \left. - 4589852 \left( Q^3 + \frac{1}{Q^3} \right) + 7082838 \left( Q^2 + \frac{1}{Q^2} \right) - 7205212 \left( Q + \frac{1}{Q} \right) \right] \\
& + 2560 \left( P^4 + \frac{1}{P^4} \right) \left[ 548799216 + 2655 \left( Q^{10} + \frac{1}{Q^{10}} \right) - 466519 \left( Q^9 + \frac{1}{Q^9} \right) \right. \\
& \quad \left. + 17418696 \left( Q^8 + \frac{1}{Q^8} \right) - 83433599 \left( Q^7 + \frac{1}{Q^7} \right) + 155764931 \left( Q^6 + \frac{1}{Q^6} \right) \right. \\
& \quad \left. - 207597164 \left( Q^5 + \frac{1}{Q^5} \right) + 294547776 \left( Q^4 + \frac{1}{Q^4} \right) - 312051804 \left( Q^3 + \frac{1}{Q^3} \right) \right. \\
& \quad \left. + 453415006 \left( Q^2 + \frac{1}{Q^2} \right) - 516256354 \left( Q + \frac{1}{Q} \right) \right] \\
& + 320 \left( P^2 + \frac{1}{P^2} \right) \left[ 5665640828 + 1353 \left( Q^{12} + \frac{1}{Q^{12}} \right) + 872568 \left( Q^{11} + \frac{1}{Q^{11}} \right) \right]
\end{aligned}$$

$$\begin{aligned}
 &+33229428 \left( Q^{10} + \frac{1}{Q^{10}} \right) - 262420952 \left( Q^9 + \frac{1}{Q^9} \right) + 843710418 \left( Q^8 + \frac{1}{Q^8} \right) \\
 &-1770861112 \left( Q^7 + \frac{1}{Q^7} \right) + 2370486948 \left( Q^6 + \frac{1}{Q^6} \right) - 2314319272 \left( Q^5 + \frac{1}{Q^5} \right) \\
 &+4574672432 \left( Q^4 + \frac{1}{Q^4} \right) - 7350748752 \left( Q^3 + \frac{1}{Q^3} \right) \\
 &+8589856488 \left( Q^2 + \frac{1}{Q^2} \right) - 8207902832 \left( Q + \frac{1}{Q} \right) \Big].
 \end{aligned}$$

**Lemma 3.17.** *Let  $P$  and  $Q$  be defined in (3.5) and (3.6), If  $\beta$  is of degree 29 over  $\alpha$ , then*

$$\begin{aligned}
 (3.44) \quad &Q^{15} + \frac{1}{Q^{15}} + 2610 \left( Q^{14} + \frac{1}{Q^{14}} \right) + 2290739 \left( Q^{13} + \frac{1}{Q^{13}} \right) \\
 &+773868132 \left( Q^{12} + \frac{1}{Q^{12}} \right) + 85133704333 \left( Q^{11} + \frac{1}{Q^{11}} \right) \\
 &+1722106886726 \left( Q^{10} + \frac{1}{Q^{10}} \right) + 387763344751 \left( Q^9 + \frac{1}{Q^9} \right) \\
 &+20840144814888 \left( Q^8 + \frac{1}{Q^8} \right) + 59747787942141 \left( Q^7 + \frac{1}{Q^7} \right) \\
 &+191351235325442 \left( Q^6 + \frac{1}{Q^6} \right) + 324184390230999 \left( Q^5 + \frac{1}{Q^5} \right) \\
 &+597721984006620 \left( Q^4 + \frac{1}{Q^4} \right) + 1021834090771369 \left( Q^3 + \frac{1}{Q^3} \right) \\
 &+1302069184703878 \left( Q^2 + \frac{1}{Q^2} \right) + 1179308291242323 \left( Q + \frac{1}{Q} \right) \\
 &+914875168239920 \\
 = &268435456 \left( P^{14} + \frac{1}{P^{14}} \right) \\
 &-486539264 \left( P^{12} + \frac{1}{P^{12}} \right) \left[ 2 \left( Q^2 + \frac{1}{Q^2} \right) - 51 \left( Q + \frac{1}{Q} \right) - 180 \right] \\
 &+30408704 \left( P^{10} + \frac{1}{P^{10}} \right) \left[ 237763 + 45 \left( Q^4 + \frac{1}{Q^4} \right) - 1348 \left( Q^3 + \frac{1}{Q^3} \right) \right. \\
 &\quad \left. -28020 \left( Q^2 + \frac{1}{Q^2} \right) + 130092 \left( Q + \frac{1}{Q} \right) \right] \\
 &-1900544 \left( P^8 + \frac{1}{P^8} \right) \left[ 490 \left( Q^6 + \frac{1}{Q^6} \right) 21592 + \left( Q^5 + \frac{1}{Q^5} \right) + 1530060 \left( Q^4 + \frac{1}{Q^4} \right) \right. \\
 &\quad \left. +4143553 \left( Q^3 + \frac{1}{Q^3} \right) + 10062832 \left( Q^2 + \frac{1}{Q^2} \right) + 3649399 \left( Q + \frac{1}{Q} \right) + 4054860 \right] \\
 &+237568 \left( P^6 + \frac{1}{P^6} \right) \times \left[ 811380268 + 1304 \left( Q^8 + \frac{1}{Q^8} \right) + 434044 \left( Q^7 + \frac{1}{Q^7} \right) \right. \\
 &\quad \left. +13788577 \left( Q^6 + \frac{1}{Q^6} \right) + 10411592 \left( Q^5 + \frac{1}{Q^5} \right) + 105951138 \left( Q^4 + \frac{1}{Q^4} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
& +99161472 \left( Q^3 + \frac{1}{Q^3} \right) + 177440463 \left( Q^2 + \frac{1}{Q^2} \right) + 563974332 \left( Q + \frac{1}{Q} \right) \Big] \\
& -3712 \left( P^4 + \frac{1}{P^4} \right) \left[ 14846975376 + 11856 \left( Q^{10} + \frac{1}{Q^{10}} \right) + 11599844 \left( Q^9 + \frac{1}{Q^9} \right) \right. \\
& +448544664 \left( Q^8 + \frac{1}{Q^8} \right) - 763704252 \left( Q^7 + \frac{1}{Q^7} \right) + 5875300048 \left( Q^6 + \frac{1}{Q^6} \right) \\
& +88316609 \left( Q^5 + \frac{1}{Q^5} \right) + 25770706848 \left( Q^4 + \frac{1}{Q^4} \right) + 57741632656 \left( Q^3 + \frac{1}{Q^3} \right) \\
& \left. +80052643040 \left( Q^2 + \frac{1}{Q^2} \right) + 52701143288 \left( Q + \frac{1}{Q} \right) \right] \\
& +464 \left( P^2 + \frac{1}{P^2} \right) \times \left[ 3753 \left( Q^{12} + \frac{1}{Q^{12}} \right) - 6120856 \left( Q^{11} + \frac{1}{Q^{11}} \right) \right. \\
& +772337076 \left( Q^{10} + \frac{1}{Q^{10}} \right) - 2864084968 \left( Q^9 + \frac{1}{Q^9} \right) + 22104272594 \left( Q^8 + \frac{1}{Q^8} \right) \\
& +27146779640 \left( Q^7 + \frac{1}{Q^7} \right) + 141012884964 \left( Q^6 + \frac{1}{Q^6} \right) + 2782862132604 \\
& +319845340808 \left( Q^5 + \frac{1}{Q^5} \right) + 656499614663 \left( Q^4 + \frac{1}{Q^4} \right) \\
& +821445490576 \left( Q^3 + \frac{1}{Q^3} \right) + 1184962932840 \left( Q^2 + \frac{1}{Q^2} \right) \\
& \left. +2247200396144 \left( Q + \frac{1}{Q} \right) \right].
\end{aligned}$$

**Lemma 3.18.** *Let  $P$  and  $Q$  be defined in (3.5) and (3.6), If  $\beta$  is of degree 55 over  $\alpha$ , then*

$$\begin{aligned}
(3.45) \quad & Q^{12} + \frac{1}{Q^{12}} + 164772 \left( Q^{10} + \frac{1}{Q^{10}} \right) + 1416790018 \left( Q^8 + \frac{1}{Q^8} \right) \\
& -2130154476 \left( Q^6 + \frac{1}{Q^6} \right) - 7724991761 \left( Q^4 + \frac{1}{Q^4} \right) + 249131895880 \left( Q^2 + \frac{1}{Q^2} \right) \\
& -321023061476 \\
= & 4194304 \left( P^{11} + \frac{1}{P^{11}} \right) + 96468992 \left( P^{10} + \frac{1}{P^{10}} \right) \\
& -12058624 \left( P^9 + \frac{1}{P^9} \right) \left[ \left( Q^2 + \frac{1}{Q^2} \right) - 93 \right] \\
& -1507328 \left( P^8 + \frac{1}{P^8} \right) \times \left[ 296 \left( Q^2 + \frac{1}{Q^2} \right) - 4397 \right] \\
& +1130496 \left( P^7 + \frac{1}{P^7} \right) \left[ 11 \left( Q^4 + \frac{1}{Q^4} \right) - 3148 \left( Q^2 + \frac{1}{Q^2} \right) - 21389 \right] \\
& +94208 \left( P^6 + \frac{1}{P^6} \right) \left[ 5617 \left( Q^4 + \frac{1}{Q^4} \right) - 121400 \left( Q^2 + \frac{1}{Q^2} \right) - 566351 \right]
\end{aligned}$$



$$\begin{aligned}
& -47104 \left( P^5 + \frac{1}{P^5} \right) \times \left[ 115 \left( Q^6 + \frac{1}{Q^6} \right) - 101325 \left( Q^4 + \frac{1}{Q^4} \right) \right. \\
& \quad \left. + 856348 \left( Q^2 + \frac{1}{Q^2} \right) - 23576361 \right] \\
& -5888 \left( P^4 + \frac{1}{P^4} \right) \times \left[ 16164 \left( Q^6 + \frac{1}{Q^6} \right) - 725439 \left( Q^4 + \frac{1}{Q^4} \right) \right. \\
& \quad \left. + 11202632 \left( Q^2 + \frac{1}{Q^2} \right) - 28284794 \right] \\
& +1472 \left( P^3 + \frac{1}{P^3} \right) \times \left[ 593 \left( Q^8 + \frac{1}{Q^8} \right) - 2087612 \left( Q^6 + \frac{1}{Q^6} \right) \right. \\
& \quad \left. + 14698260 \left( Q^4 + \frac{1}{Q^4} \right) - 93333412 \left( Q^2 + \frac{1}{Q^2} \right) + 189688822 \right] \\
& -368 \left( P^2 + \frac{1}{P^2} \right) \left[ 228609 \left( Q^8 + \frac{1}{Q^8} \right) - 5311208 \left( Q^6 + \frac{1}{Q^6} \right) \right. \\
& \quad \left. - 29908452 \left( Q^4 + \frac{1}{Q^4} \right) + 447783336 \left( Q^2 + \frac{1}{Q^2} \right) - 825137850 \right] \\
& -184 \left( P + \frac{1}{P} \right) \left[ 147 \left( Q^{10} + \frac{1}{Q^{10}} \right) - 4008318 \left( Q^8 + \frac{1}{Q^8} \right) \right. \\
& \quad \left. + 33717559 \left( Q^8 + \frac{1}{Q^8} \right) - 196168488 \left( Q^4 + \frac{1}{Q^4} \right) \right. \\
& \quad \left. + 1477813558 \left( Q^2 + \frac{1}{Q^2} \right) - 2080998580 \right].
\end{aligned}$$

Proofs of the identities (3.39)–(3.45) are similar to the proof of the identity (3.35) given above except that in place of result (2.16), result (2.17) is used for proving (3.39); results (2.18) and (2.19) are used for proving (3.40); result (2.23) is used for proving (3.41); result (2.24) is used for proving (3.42); results (2.25) and (2.26) are used for proving (3.43); result (2.31) is used for proving (3.45).

#### 4. MODULAR RELATIONS BETWEEN $H(q)$ AND $H(q^n)$

In this section, we establish several modular relations between the Ramanujan–Göllnitz–Gordon continued fractions  $H(q)$  and  $H(q^n)$  using the identities established in section 3, for  $n = 2, 3, 4, 5, 7, 8, 9, 11, 13, 15, 17, 19, 23, 25, 29$  and  $55$ .

**Theorem 4.1.** *If  $x = H(q)$  and  $y = H(q^2)$ , then*

$$(4.1) \quad y^2 - y + x^2y + x^2 = 0.$$

*Proof.* Employing the equation (3.4) in the equations (3.5) and (3.6), we find that

$$(4.2) \quad P = \frac{\varphi(-q)\varphi(-q^2)}{\varphi(q)\varphi(q^2)} \quad \text{and} \quad Q = \frac{\varphi(-q)\varphi(q^2)}{\varphi(q)\varphi(-q^2)}.$$

Replacing  $q$  by  $\sqrt{q}$  in the equation (4.2) and from the equation (3.1), we deduce that

$$(4.3) \quad P = \frac{(1 - 2x - x^2)(1 - 2y - y^2)}{(1 + 2x - x^2)(1 + 2y - y^2)} \quad \text{and} \quad Q = \frac{(1 - 2x - x^2)(1 + 2y - y^2)}{(1 + 2x - x^2)(1 - 2y - y^2)}.$$

Employing the equation (4.3) in the equation (3.9), we deduce that

$$(4.4) \quad (y^2 - y + x^2y + x^2)(-1 + 2x + x^2)(-2xy^2 + y^2 + x^2y^2 + 1 + 4xy + 2x + x^2) \\ (x^2y^2 + y + 1 - x^2y)(x^2y^2 + 2xy^2 + y^2 + 1 - 4xy - 2x + x^2) = 0.$$

From the definitions of  $x$  and  $y$ , we have  $x = o(q^{1/2})$  and  $y = o(q)$  as  $q \rightarrow 0$ , it can be seen that the first factor vanishes for  $q$  sufficiently small whereas the other factors does not vanish. Thus by the identity theorem, first factor vanishes identically. Hence the proof.  $\square$

**Remark:** For a different proof of the equation (4.1), see [8].

**Theorem 4.2.** If  $x = H(q)$  and  $y = H(q^3)$ , then

$$(4.5) \quad y^3x^4 + x^3 + 3yx^2 - y - 3y^2x^3 + 3y^2x - 3y^3x^2 - y^4x = 0.$$

*Proof.* Employing the equation (3.4) in the equations (3.5) and (3.6), we deduce that

$$(4.6) \quad P = \frac{\varphi(-q)\varphi(-q^3)}{\varphi(q)\varphi(q^3)} \quad \text{and} \quad Q = \frac{\varphi(-q)\varphi(q^3)}{\varphi(q)\varphi(-q^3)}.$$

Replacing  $q$  by  $\sqrt{q}$  in the equation (4.6) and from the equation (3.1), we deduce that

$$(4.7) \quad P = \frac{(1 - 2x - x^2)(1 - 2y - y^2)}{(1 + 2x - x^2)(1 + 2y - y^2)} \quad \text{and} \quad Q = \frac{(1 - 2x - x^2)(1 + 2y - y^2)}{(1 + 2x - x^2)(1 - 2y - y^2)}.$$

Employing the equation (4.7) in the equation (3.12), we deduce that

$$(4.8) \quad (y^3x^4 + x^3 + 3yx^2 - y - 3y^2x^3 + 3y^2x - 3y^3x^2 - y^4x)(-1 + 2x + x^2) \\ (yx^4 + x - 3yx^2 + 3y^2x^3 - 3y^2x + 3y^3x^2 - y^3 - y^4x^3) = 0.$$

From the definitions of  $x$  and  $y$ , we have  $x = o(q^{1/2})$  and  $y = o(q^{3/2})$  as  $q \rightarrow 0$ , it can be seen that the first factor vanishes for  $q$  sufficiently small whereas other factors does not vanish. Thus by the identity theorem, first factor vanishes identically. Hence the proof.  $\square$

**Remark:** For a different proofs of the equation (4.5), see [8] and [18].

**Theorem 4.3.** If  $x = H(q)$  and  $y = H(q^4)$ , then

$$(4.9) \quad -y + 4x^2y - 4x^2y^3 + x^4y + x^4y^2 + x^4y^3 + y^2 + x^4 - y^3 + y^4 = 0.$$

*Proof.* Employing the equation (3.4) in the equations (3.5) and (3.6), we find that

$$(4.10) \quad P = \frac{\varphi(-q)\varphi(-q^4)}{\varphi(q)\varphi(q^4)} \quad \text{and} \quad Q = \frac{\varphi(-q)\varphi(q^4)}{\varphi(q)\varphi(-q^4)}.$$

Replacing  $q$  by  $\sqrt{q}$  in the equation (4.10) and from the equation (3.1), we deduce that

$$(4.11) \quad P = \frac{(1 - 2x - x^2)(1 - 2y - y^2)}{(1 + 2x - x^2)(1 + 2y - y^2)} \quad \text{and} \quad Q = \frac{(1 - 2x - x^2)(1 + 2y - y^2)}{(1 + 2x - x^2)(1 - 2y - y^2)}.$$

Employing the equation (4.11) in the equation (3.14), we deduce that

$$(4.12) \quad (-y + 4x^2y - 4x^2y^3 + x^4y + x^4y^2 + x^4y^3 + y^2 + x^4 - y^3 + y^4) \\ (-1 + 2x + x^2)^3(1 + 4x + 4y + 8xy + 8xy^3 - 4xy^4 - 8x^2y + 12x^2y^2 + 8x^2y^3 \\ + 6x^2y^4 + 8x^3y + 8x^3y^3 - 4x^3y^4 + 4x^4y + 2x^4y^2 - 4x^4y^3 + x^4y^4 + 6x^2 + 2y^2 \\ + 4x^3 + x^4 - 4y^3 + y^4)(1 + y + 4x^2y - 4x^2y^3 - x^4y + x^4y^2 - x^4y^3 + x^4y^4 + y^2 + y^3) \\ (1 - 4x + 4y - 8xy - 8xy^3 + 4xy^4 - 8x^2y + 12x^2y^2 + 8x^2y^3 + 6x^2y^4 - 8x^3y - 8x^3y^3 \\ + 4x^3y^4 + 4x^4y + 2x^4y^2 - 4x^4y^3 + x^4y^4 + 6x^2 + 2y^2 - 4x^3 + x^4 - 4y^3 + y^4) = 0.$$

From the definitions of  $x$  and  $y$ , we have  $x = o(q^{1/2})$  and  $y = o(q^2)$  as  $q \rightarrow 0$ , it can be seen that the first factor of the equation (4.12) vanishes for  $q$  sufficiently small whereas as other factors does not vanish. Thus by the identity theorem, first factor vanishes identically. Hence the proof.  $\square$

**Remark:** For a different proof of the equation (4.9), see [8].

**Theorem 4.4.** *If  $x = H(q)$  and  $y = H(q^5)$ , then*

$$(4.13) \quad y - 5xy^4 - 5x^2y + 10x^2y^3 - 10x^3y^2 + 10x^3y^4 + 5x^4y^5 \\ + 5x^5y^2 - x^6y^5 - 10x^4y^3 - x^5 + xy^6 = 0.$$

*Proof.* Employing the equation (3.4) in the equations (3.5) and (3.6), we find that

$$(4.14) \quad P = \frac{\varphi(-q)\varphi(-q^5)}{\varphi(q)\varphi(q^5)} \quad \text{and} \quad Q = \frac{\varphi(-q)\varphi(q^5)}{\varphi(q)\varphi(-q^5)}.$$

Replacing  $q$  by  $\sqrt{q}$  in the equation (4.14) and then from equation (3.1), we deduce that

$$(4.15) \quad P = \frac{(1 - 2x - x^2)(1 - 2y - y^2)}{(1 + 2x - x^2)(1 + 2y - y^2)} \quad \text{and} \quad Q = \frac{(1 - 2x - x^2)(1 + 2y - y^2)}{(1 + 2x - x^2)(1 - 2y - y^2)}.$$

Employing the equation (4.15) in the equation (3.17), we deduce that

$$(4.16) \quad (-y + 5xy^4 + 5x^2y - 10x^2y^3 + 10x^3y^2 - 10x^3y^4 + 10x^4y^3 - 5x^4y^5 \\ - 5x^5y^2 + x^6y^5 + x^5 - xy^6) \times (x - 5xy^2 - 10x^2y^3 + 5x^2y^5 + 10x^3y^2 \\ - 10x^3y^4 - 5x^4y + 10x^4y^3 + 5x^5y^4 + x^6y - y^5 - x^5y^6)(-1 + 2x + x^2)^2 = 0.$$

From the definitions of  $x$  and  $y$ , we have  $x = o(q^{1/2})$  and  $y = o(q^{5/2})$  as  $q \rightarrow 0$ , it can be seen that the first factor vanishes for  $q$  sufficiently small whereas the other factors does not vanish. Thus by the identity theorem, first factor vanishes identically. Hence the proof.  $\square$

**Remark:** For a different proof of the equation (4.13), see [18].

**Theorem 4.5.** *If  $x = H(q)$  and  $y = H(q^7)$ , then*

$$(4.17) \quad x^8 + 7x^7y^5 - 7yx^7 - 7x^7y^3 - y^7x^7 + 28x^6y^2 - 7yx^5 \\ + 7y^7x^5 - 49x^5y^5 - 7x^5y^3 + 70x^4y^4 + 7yx^3 - 7x^3y^5 \\ - 49x^3y^3 - 7y^7x^3 + 28x^2y^6 - xy - 7xy^5 + 7xy^3 - 7xy^7 + y^8 = 0.$$

*Proof.* Employing the equation (3.4) in the equations (3.5) and (3.6), we find that

$$(4.18) \quad P = \frac{\varphi(-q)\varphi(-q^7)}{\varphi(q)\varphi(q^7)} \quad \text{and} \quad Q = \frac{\varphi(-q)\varphi(q^7)}{\varphi(q)\varphi(-q^7)}.$$

Replacing  $q$  by  $\sqrt{q}$  in the equation (4.18) and then from equation (3.1), we deduce that

$$(4.19) \quad P = \frac{(1 - 2x - x^2)(1 - 2y - y^2)}{(1 + 2x - x^2)(1 + 2y - y^2)} \quad \text{and} \quad Q = \frac{(1 - 2x - x^2)(1 + 2y - y^2)}{(1 + 2x - x^2)(1 - 2y - y^2)}.$$

From the equations (4.19) and (3.20), we deduce that

$$(4.20) \quad \begin{aligned} & (1 - 7xy^5 + 7xy + 7xy^3 + xy^7 + 28x^2y^2 + 7yx^3 - 7yx^5 + yx^7 + 70x^4y^4 \\ & + 49x^3y^5 + 7x^5y^5 + 7x^7y^5 + 28x^6y^6 + 7x^3y^3 + 49x^5y^3 - 7x^7y^3 - 7y^7x^3 \\ & + 7y^7x^5 + 7y^7x^7 + y^8x^8)(x^8 + 7x^7y^5 - 7yx^7 - 7x^7y^3 - y^7x^7 + 28x^6y^2 \\ & - 7yx^5 + 7y^7x^5 - 49x^5y^5 - 7x^5y^3 + 70x^4y^4 + 7yx^3 - 7x^3y^5 - 49x^3y^3 \\ & - 7y^7x^3 + 28x^2y^6 - xy - 7xy^5 + 7xy^3 - 7xy^7 + y^8)(-1 + 2x + x^2)^3 = 0. \end{aligned}$$

From the definitions of  $x$  and  $y$ , we have  $x = o(q^{1/2})$  and  $y = o(q^{7/2})$  as  $q \rightarrow 0$ , it can be seen that the second factor vanishes for  $q$  sufficiently small whereas the other factors does not vanish. Thus by the identity theorem, second factor vanishes identically. Hence the proof.  $\square$

**Remark:** For a different proof of (4.17), see [18].

**Theorem 4.6.** If  $x = H(q)$  and  $y = H(q^8)$ , then

$$\begin{aligned} & -3y^2 + 12x^4y^5 + 16x^4y^6 + 12x^4y^7 - 12x^4y - 16x^6y^2 \\ & + 8x^6y^3 + 8x^6y^5 + 16x^6y^6 - 8x^6y^7 - 8x^6y + 7x^8y^2 + 3x^8y^3 \\ & + 3x^8y^4 - x^8y^5 - 3x^8y^6 + x^8y^7 + 5x^8y + 16x^2y^2 - 8x^2y^3 - \\ & 8x^2y^5 - 16x^2y^6 + 8x^2y^7 + 8x^2y + 16x^4y^2 - 12x^4y^3 - 32x^4y^4 \\ & + x^8 + y^3 + 3y^4 - 3y^5 + 7y^6 - 5y^7 + y^8 - y = 0. \end{aligned}$$

*Proof.* Employing the equation (3.4) in the equations (3.5) and (3.6), we deduce that

$$(4.21) \quad P = \frac{\varphi(-q)\varphi(-q^8)}{\varphi(q)\varphi(q^8)} \quad \text{and} \quad Q = \frac{\varphi(-q)\varphi(q^8)}{\varphi(q)\varphi(-q^8)}.$$

Replacing  $q$  by  $\sqrt{q}$  in the equation (4.21) and then from equation (3.1), we deduce that

$$(4.22) \quad P = \frac{(1 - 2x - x^2)(1 - 2y - y^2)}{(1 + 2x - x^2)(1 + 2y - y^2)} \quad \text{and} \quad Q = \frac{(1 - 2x - x^2)(1 + 2y - y^2)}{(1 + 2x - x^2)(1 - 2y - y^2)}.$$

Employing the equation (4.22) in the equation (3.23), we deduce that

$$(4.23) \quad \begin{aligned} & (-3y^2 + 12x^4y^5 + 16x^4y^6 + 12x^4y^7 - 12x^4y - 16x^6y^2 + 8x^6y^3 + 8x^6y^5 \\ & + 16x^6y^6 - 8x^6y^7 - 8x^6y + 7x^8y^2 + 3x^8y^3 + 3x^8y^4 - x^8y^5 - 3x^8y^6 + x^8y^7 \\ & + 5x^8y + 16x^2y^2 - 8x^2y^3 - 8x^2y^5 - 16x^2y^6 + 8x^2y^7 + 8x^2y - 5y^7 + 16x^4y^2 \\ & - 12x^4y^3 - 32x^4y^4 + x^8 + y^3 + 3y^4 - 3y^5 + 7y^6 + y^8 - y)(8x - 8y + 48x^2y^2 \\ & + 400x^3y + 48x^3y^3 + 48x^3y^5 - 688x^3y^6 + 400x^3y^7 - 56x^3y^8 + 688x^3y^2 + 208x^4y \\ & + 208x^4y^3 + 228x^4y^4 - 208x^4y^5 + 376x^4y^6 - 208x^4y^7 + 70x^4y^8 + 376x^4y^2 + 400yx^5 \\ & + 160yx^6 - 16yx^7 - 8yx^8 + 688x^5y^2 + 48x^6y^2 - 48x^7y^2 + 20x^8y^2 + 56x^3 + 70x^4 \\ & - 8y^3 - 26y^4 + 8y^5 + 20y^6 + 8y^7 + y^8 - 16xy + 80xy^3 + 80xy^5 + 48xy^6 - 16xy^7 \end{aligned}$$

$$\begin{aligned}
 & -8xy^8 + 1 - 48xy^2 + 160x^2y + 160x^2y^3 + 296x^2y^4 - 160x^2y^5 + 48x^2y^6 - 160x^2y^7 \\
 & + 28x^2y^8 + 28x^2 + 20y^2 + 48x^5y^5 + 28x^6 + 8x^7 + x^8 + 160x^6y^3 + 80x^7y^3 + 48x^5y^3 \\
 & - 8x^8y^3 + 296x^6y^4 - 26x^8y^4 - 160x^6y^5 + 80x^7y^5 + 56x^5 + 8x^8y^5 + 48x^6y^6 \\
 & + 48x^7y^6 - 688x^5y^6 + 20x^8y^6 - 160x^6y^7 - 16x^7y^7 + 400x^5y^7 + 8x^8y^7 + 28x^6y^8 \\
 & - 8x^7y^8 - 56x^5y^8 + x^8y^8)(70x^4y^8 - 8x - 8y + 48x^2y^2 - 400x^3y - 48x^3y^3 \\
 & - 48x^3y^5 + 688x^3y^6 - 400x^3y^7 + 56x^3y^8 - 688x^3y^2 + 208x^4y + 208x^4y^3 \\
 & + 228x^4y^4 - 208x^4y^5 + 376x^4y^6 - 208x^4y^7 + 376x^4y^2 - 400yx^5 - 160x^2y^5 + 16yx^7 \\
 & - 8yx^8 - 688x^5y^2 + 48x^6y^2 + y^8 + 20x^8y^2 - 56x^3 + 70x^4 - 8y^3 - 26y^4 + 8y^5 + 20y^6 \\
 & + 160yx^6 + 48x^7y^2 + 16xy - 80xy^3 - 80xy^5 - 48xy^6 + 16xy^7 + 8xy^8 + 48xy^2 \\
 & + 160x^2y + 160x^2y^3 + 296x^2y^4 + 8y^7 + 48x^2y^6 - 160x^2y^7 + 28x^2y^8 + 28x^2 + 20y^2 \\
 & - 56x^5 + 28x^6 + 20x^8y^6 + x^8 + 160x^6y^3 - 80x^7y^3 - 48x^5y^3 - 8x^7 - 8x^8y^3 \\
 & + 296x^6y^4 - 26x^8y^4 - 160x^6y^5 - 80x^7y^5 - 48x^5y^5 + 8x^8y^5 + 48x^6y^6 - 48x^7y^6 \\
 & + 688x^5y^6 - 160x^6y^7 + 16x^7y^7 - 400x^5y^7 + 8x^8y^7 + 28x^6y^8 + 8x^7y^8 + 56x^5y^8 \\
 & + x^8y^8)(1 + 5y - 16x^2y^2 - 12x^4y - 12x^4y^3 - 32x^4y^4 + 12x^4y^5 + 16x^4y^6 + 12x^4y^7 \\
 & + 16x^4y^2 + 8yx^6 - yx^8 + 16x^6y^2 - 3x^8y^2 + 3y^3 + 3y^4 - y^5 - 3y^6 + y^7 - 8x^2y \\
 & + 8x^2y^3 + 8x^2y^5 + 16x^2y^6 - 8x^2y^7 + 7y^2 - 8x^6y^3 + x^8y^3 + 3x^8y^4 - 8x^6y^5 \\
 & - 3x^8y^5 - 16x^6y^6 + 7x^8y^6 + 8x^6y^7 - 5x^8y^7 + x^8y^8)(-1 + 2x + x^2)^7 = 0.
 \end{aligned}$$

From the definitions of  $x$  and  $y$ , we have  $x = o(q^{1/2})$  and  $y = o(q^4)$  as  $q \rightarrow 0$ , it can be seen that the first factor vanishes for  $q$  sufficiently small whereas other factors does not vanish. Thus by the identity theorem, first factor vanishes identically. Hence the proof.  $\square$

**Theorem 4.7.** *If  $x = H(q)$  and  $y = H(q^9)$ , then*

$$\begin{aligned}
 (4.24) \quad & y^{10} - yx + 9yx^3 - 18yx^5 - 6yx^7 + 9yx^9 - 8y^2x^2 + 48y^2x^4 - 3y^2x^8 - 12y^2x^{10} \\
 & + y^2x^{12} + 9y^3x + x^{10} - 73y^3x^3 + 114y^3x^5 - 66y^3x^7 - 33y^3x^9 + 9y^3x^{11} \\
 & + 48y^4x^2 - 288y^4x^4 + 210y^4x^6 + 48y^4x^8 - y^{11}x^{11} - 18y^5x + 114y^5x^3 - 288y^5x^5 \\
 & + 312y^5x^7 - 66y^5x^9 - 6y^5x^{11} + 210y^6x^4 - 504y^6x^6 + 210y^6x^8 - 66y^7x^3 \\
 & + 312y^7x^5 - 288y^7x^7 + 114y^7x^9 - 18y^7x^{11} - 3y^8x^2 + 48y^8x^4 + 210y^8x^6 - 288y^8x^8 \\
 & + 9y^9x - 33y^9x^3 - 66y^9x^5 + 114y^9x^7 - 73y^9x^9 + 9y^9x^{11} - 12y^{10}x^2 - 3y^{10}x^4 \\
 & + 48y^{10}x^8 + y^{12}x^2 + 48y^8x^{10} - y^{11}x - 3y^4x^{10} + 9y^{11}x^3 - 6y^7x - 8y^{10}x^{10} \\
 & - 6y^{11}x^5 - yx^{11} - 18y^{11}x^7 + 9y^{11}x^9 = 0.
 \end{aligned}$$

*Proof.* Employing the equation (3.4) in the equations (3.5) and (3.6), we deduce that

$$(4.25) \quad P = \frac{\varphi(-q)\varphi(-q^9)}{\varphi(q)\varphi(q^9)} \quad \text{and} \quad Q = \frac{\varphi(-q)\varphi(q^9)}{\varphi(q)\varphi(-q^9)}.$$

Replacing  $q$  by  $\sqrt{q}$  in the equation (4.25) and then from equation (3.1), we deduce that

$$(4.26) \quad P = \frac{(1 - 2x - x^2)(1 - 2y - y^2)}{(1 + 2x - x^2)(1 + 2y - y^2)} \quad \text{and} \quad Q = \frac{(1 - 2x - x^2)(1 + 2y - y^2)}{(1 + 2x - x^2)(1 - 2y - y^2)}.$$

From the equations (4.26) and (3.27), we deduce that

$$\begin{aligned}
 (4.27) \quad & (y^{10} - yx + 9yx^3 - 18yx^5 - 6yx^7 + 9yx^9 - yx^{11} - 8y^2x^2 + 48y^2x^4 - 3y^2x^8 \\
 & - 12y^2x^{10} + y^2x^{12} + 9y^3x - 73y^3x^3 + 114y^3x^5 - 66y^3x^7 - 33y^3x^9 \\
 & + 9y^3x^{11} + 48y^4x^2 - 288y^4x^4 + 210y^4x^6 + 48y^4x^8 - 3y^4x^{10} - 18y^5x \\
 & + 114y^5x^3 - 288y^5x^5 + 312y^5x^7 - 66y^5x^9 - 6y^5x^{11} + 210y^6x^4 - 504y^6x^6 \\
 & + 210y^6x^8 - 6y^7x - 66y^7x^3 + 312y^7x^5 - 288y^7x^7 + 114y^7x^9 - 18y^7x^{11} \\
 & - 3y^8x^2 + 48y^8x^4 + 210y^8x^6 - 288y^8x^8 + 9y^9x - 33y^9x^3 - 66y^9x^5 + 114y^9x^7 \\
 & - 73y^9x^9 + 9y^9x^{11} - 12y^{10}x^2 - 3y^{10}x^4 + 48y^{10}x^8 - 8y^{10}x^{10} \\
 & + 9y^{11}x^3 - 6y^{11}x^5 - 18y^{11}x^7 + 9y^{11}x^9 - y^{11}x^{11} + y^{12}x^2 \\
 & + x^{10})(x^2 + y^2 + yx - 9yx^3 + 6yx^5 + 18yx^7 - 9yx^9 + yx^{11} - 12y^2x^2 - 3y^2x^4 \\
 & + 48y^2x^8 - 8y^2x^{10} - 9y^3x + 33y^3x^3 + 66y^3x^5 - 114y^3x^7 + 73y^3x^9 \\
 & - 9y^3x^{11} - 3y^4x^2 + 48y^4x^4 + 210y^4x^6 - 288y^4x^8 + 48y^4x^{10} + 6y^5x + 66y^5x^3 \\
 & - 312y^5x^5 + 288y^5x^7 - 114y^5x^9 + 18y^5x^{11} + 210y^6x^4 - 504y^6x^6 + 210y^6x^8 \\
 & + 18y^7x - 114y^7x^3 + 288y^7x^5 - 312y^7x^7 + 66y^7x^9 + 6y^7x^{11} + 48y^8x^2 - 288y^8x^4 \\
 & + 210y^8x^6 + 48y^8x^8 - 3y^8x^{10} - 9y^9x + 73y^9x^3 - 114y^9x^5 + 66y^9x^7 \\
 & + 33y^9x^9 - 9y^9x^{11} - 8y^{10}x^2 + 48y^{10}x^4 - 3y^{10}x^8 - 12y^{10}x^{10} \\
 & + y^{10}x^{12} + y^{11}x - 9y^{11}x^3 + 18y^{11}x^5 + 6y^{11}x^7 - 9y^{11}x^9 \\
 & + y^{11}x^{11} + y^{12}x^{10} - y^{11}x + 48y^8x^{10}) = 0.
 \end{aligned}$$

From the definitions of  $x$  and  $y$ , we have  $x = o(q^{1/2})$  and  $y = o(q^{9/2})$  as  $q \rightarrow 0$ , it can be seen that the first factor vanishes for  $q$  sufficiently small whereas other factors does not vanish. Thus by the identity theorem, first factor vanishes identically. Hence the proof.  $\square$

**Theorem 4.8.** *If  $x = H(q)$  and  $y = H(q^{11})$ , then*

$$\begin{aligned}
 (4.28) \quad & -y - 11xy^2 + 33xy^4 + 11xy^6 - 33xy^8 + 11xy^{10} - xy^{12} + 11x^2y - 66x^2y^3 \\
 & + 66x^2y^5 - 66x^2y^9 + 11x^2y^{11} + 66x^3y^2 - 231x^3y^4 - 55x^3y^6 + 396x^3y^8 \\
 & - 66x^3y^{10} - 33x^4y + 396x^4y^3 - 528x^4y^5 - 198x^4y^7 + 231x^4y^9 - 33x^4y^{11} \\
 & + 198x^5y^4 + 396x^5y^6 - 528x^5y^8 + 66x^5y^{10} + 11x^6y - 55x^6y^3 + 396x^6y^5 \\
 & - 396x^6y^7 + 55x^6y^9 - 11x^6y^{11} - 66x^7y^2 + 528x^7y^4 - 396x^7y^6 - 198x^7y^8 \\
 & + 33x^8y - 231x^8y^3 + 198x^8y^5 + 528x^8y^7 - 396x^8y^9 + 33x^8y^{11} + 66x^9y^2 \\
 & - 396x^9y^4 + 55x^9y^6 + 231x^9y^8 - 66x^9y^{10} - 11x^{10}y + 66x^{10}y^3 \\
 & - 66x^{10}y^7 + 66x^{10}y^9 - 11x^{10}y^{11} - 11x^{11}y^2 + 33x^{11}y^4 \\
 & - 11x^{11}y^6 - 33x^{11}y^8 + 11x^{11}y^{10} + x^{12}y^{11} + x^{11} = 0.
 \end{aligned}$$

*Proof.* Employing the equation (3.4) in the equations (3.5) and (3.6), we find that

$$(4.29) \quad P = \frac{\varphi(-q)\varphi(-q^{11})}{\varphi(q)\varphi(q^{11})} \quad \text{and} \quad Q = \frac{\varphi(-q)\varphi(q^{11})}{\varphi(q)\varphi(-q^{11})}.$$

Replacing  $q$  by  $\sqrt{q}$  in the equation (4.29) and then from equation (3.1), we deduce that

$$(4.30) \quad P = \frac{(1 - 2x - x^2)(1 - 2y - y^2)}{(1 + 2x - x^2)(1 + 2y - y^2)} \quad \text{and} \quad Q = \frac{(1 - 2x - x^2)(1 + 2y - y^2)}{(1 + 2x - x^2)(1 - 2y - y^2)}.$$

Employing the equation (4.30) in the equation (3.32), we deduce that

$$(4.31) \quad \begin{aligned} & (-y - 11xy^2 + 33xy^4 + 11xy^6 - 33xy^8 + 11xy^{10} - xy^{12} + 11x^2y - 66x^2y^3 \\ & + 66x^2y^5 - 66x^2y^9 + 11x^2y^{11} + 66x^3y^2 - 231x^3y^4 - 55x^3y^6 + 396x^3y^8 \\ & - 66x^3y^{10} - 33x^4y + 396x^4y^3 - 528x^4y^5 - 198x^4y^7 + 231x^4y^9 \\ & - 33x^4y^{11} + 198x^5y^4 + 396x^5y^6 - 528x^5y^8 + 66x^5y^{10} + 11x^6y \\ & - 55x^6y^3 + 396x^6y^5 - 396x^6y^7 + 55x^6y^9 - 11x^6y^{11} - 66x^7y^2 \\ & + 528x^7y^4 - 396x^7y^6 - 198x^7y^8 + 33x^8y - 231x^8y^3 + 198x^8y^5 \\ & + 528x^8y^7 - 396x^8y^9 + 33x^8y^{11} + 66x^9y^2 - 396x^9y^4 + 55x^9y^6 \\ & + 231x^9y^8 - 66x^9y^{10} - 11x^{10}y + 66x^{10}y^3 - 66x^{10}y^7 + 66x^{10}y^9 \\ & - 11x^{10}y^{11} - 11x^{11}y^2 + 33x^{11}y^4 - 11x^{11}y^6 - 33x^{11}y^8 \\ & + 11x^{11}y^{10} + x^{12}y^{11} + x^{11})(x - 11xy^2 + 33xy^4 - 11xy^6 - 33xy^8 \\ & + 11xy^{10} + 11x^2y - 66x^2y^3 + 66x^2y^7 - 66x^2y^9 + 11x^2y^{11} + 66x^3y^2 \\ & - 396x^3y^4 + 55x^3y^6 + 231x^3y^8 - 66x^3y^{10} - 33x^4y + 231x^4y^3 - 198x^4y^5 \\ & - 528x^4y^7 + 396x^4y^9 - 33x^4y^{11} - 66x^5y^2 + 528x^5y^4 - 396x^5y^6 - 198x^5y^8 \\ & - 11x^6y + 55x^6y^3 - 396x^6y^5 + 396x^6y^7 - 55x^6y^9 + 11x^6y^{11} + 198x^7y^4 \\ & + 396x^7y^6 - 528x^7y^8 + 66x^7y^{10} + 33x^8y - 396x^8y^3 + 528x^8y^5 + 198x^8y^7 \\ & - 231x^8y^9 + 33x^8y^{11} + 66x^9y^2 - 231x^9y^4 - 55x^9y^6 + 396x^9y^8 - 66x^9y^{10} \\ & - 11x^{10}y + 66x^{10}y^3 - 66x^{10}y^5 + 66x^{10}y^9 - 11x^{10}y^{11} - 11x^{11}y^2 \\ & + 33x^{11}y^4 + 11x^{11}y^6 - 33x^{11}y^8 + 11x^{11}y^{10} - x^{11}y^{12} + x^{12}y - y^{11}) = 0. \end{aligned}$$

From the definitions of  $x$  and  $y$ , we have  $x = o(q^{1/2})$  and  $y = o(q^{11/2})$  as  $q \rightarrow 0$ , it can be seen that the first factor vanishes for  $q$  sufficiently small whereas the other factors does not vanish. Thus by the identity theorem, first factor vanishes identically. Hence the proof.  $\square$

**Remark:** For a different proof of the equation (4.28), see [18].

**Theorem 4.9.** *If  $x = H(q)$  and  $y = H(q^{13})$ , then*

$$(4.32) \quad \begin{aligned} & -y - 52xy^4 + 52xy^6 + 52xy^8 - 52xy^{10} + 13xy^{12} - xy^{14} + 13x^2y - 130x^2y^3 \\ & + 312x^2y^5 - 312x^2y^9 + 52x^2y^{11} - 52x^3y^2 + 572x^3y^4 - 520x^3y^6 - 520x^3y^8 \\ & + 858x^3y^{10} - 130x^3y^{12} - 52x^4y + 858x^4y^3 - 2327x^4y^5 + 1612x^4y^9 - 572x^4y^{11} \\ & + 52x^4y^{13} + 312x^5y^2 - 1612x^5y^4 + 1300x^5y^6 + 2587x^5y^8 - 2327x^5y^{10} + 312x^5y^{12} \\ & + 52x^6y - 520x^6y^3 + 2587x^6y^5 - 1716x^6y^7 - 1300x^6y^9 + 520x^6y^{11} - 52x^6y^{13} \\ & + 1716x^7y^6 - 1716x^7y^8 + 52x^8y - 520x^8y^3 + 1300x^8y^5 + 1716x^8y^7 - 2587x^8y^9 \end{aligned}$$

$$\begin{aligned}
&+520x^8y^{11} - 52x^8y^{13} - 312x^9y^2 + 2327x^9y^4 - 2587x^9y^6 - 1300x^9y^8 + 1612x^9y^{10} \\
&- 312x^9y^{12} - 52x^{10}y + 572x^{10}y^3 - 1612x^{10}y^5 + 2327x^{10}y^9 - 858x^{10}y^{11} + 52x^{10}y^{13} \\
&+ 130x^{11}y^2 - 858x^{11}y^4 + 520x^{11}y^6 + 520x^{11}y^8 - 572x^{11}y^{10} + 52x^{11}y^{12} - 52x^{12}y^3 \\
&+ 312x^{12}y^5 - 312x^{12}y^9 + 130x^{12}y^{11} - 13x^{12}y^{13} - 13x^{13}y^2 + 52x^{13}y^4 - 52x^{13}y^6 \\
&- 52x^{13}y^8 + 52x^{13}y^{10} + x^{14}y^{13} + x^{13} = 0.
\end{aligned}$$

*Proof.* Employing the equation (3.4) in the equations (3.5) and (3.6), we find that

$$(4.33) \quad P = \frac{\varphi(-q)\varphi(-q^{13})}{\varphi(q)\varphi(q^{13})} \quad \text{and} \quad Q = \frac{\varphi(-q)\varphi(q^{13})}{\varphi(q)\varphi(-q^{13})}.$$

Replacing  $q$  by  $\sqrt{q}$  in the equation (4.33) and then from equation (3.1), we deduce that

$$(4.34) \quad P = \frac{(1-2x-x^2)(1-2y-y^2)}{(1+2x-x^2)(1+2y-y^2)} \quad \text{and} \quad Q = \frac{(1-2x-x^2)(1+2y-y^2)}{(1+2x-x^2)(1-2y-y^2)}.$$

Employing the equation (4.34) in the equation (3.35), we deduce that

$$\begin{aligned}
(4.35) \quad &(-y - 52xy^4 + 52xy^6 + 52xy^8 - 52xy^{10} + 13xy^{12} - xy^{14} + 13x^2y - 130x^2y^3 \\
&+ 312x^2y^5 - 312x^2y^9 + 52x^2y^{11} - 52x^3y^2 + 572x^3y^4 - 520x^3y^6 - 520x^3y^8 \\
&+ 858x^3y^{10} - 130x^3y^{12} - 52x^4y + 858x^4y^3 - 2327x^4y^5 + 1612x^4y^9 - 572x^4y^{11} \\
&+ 52x^4y^{13} + 312x^5y^2 - 1612x^5y^4 + 1300x^5y^6 + 2587x^5y^8 - 2327x^5y^{10} + 312x^5y^{12} \\
&+ 52x^6y - 520x^6y^3 + 2587x^6y^5 - 1716x^6y^7 - 1300x^6y^9 + 520x^6y^{11} - 52x^6y^{13} \\
&+ 1716x^7y^6 - 1716x^7y^8 + 52x^8y - 520x^8y^3 + 1300x^8y^5 + 1716x^8y^7 - 2587x^8y^9 \\
&+ 520x^8y^{11} - 52x^8y^{13} - 312x^9y^2 + 2327x^9y^4 - 2587x^9y^6 - 1300x^9y^8 + 1612x^9y^{10} \\
&- 312x^9y^{12} - 52x^{10}y + 572x^{10}y^3 - 1612x^{10}y^5 + 2327x^{10}y^9 - 858x^{10}y^{11} + 52x^{10}y^{13} \\
&+ 130x^{11}y^2 - 858x^{11}y^4 + 520x^{11}y^6 + 520x^{11}y^8 - 572x^{11}y^{10} + 52x^{11}y^{12} - 52x^{12}y^3 \\
&+ 312x^{12}y^5 - 312x^{12}y^9 + 130x^{12}y^{11} - 13x^{12}y^{13} - 13x^{13}y^2 + 52x^{13}y^4 - 52x^{13}y^6 \\
&- 52x^{13}y^8 + 52x^{13}y^{10} + x^{13} + x^{14}y^{13}) \times (x - 13xy^2 - y^{13} + 52xy^4 - 52xy^6 \\
&- 52xy^8 + 52xy^{10} + 52x^2y^3 - 312x^2y^5 + 312x^2y^9 - 130x^2y^{11} + 13x^2y^{13} + 130x^3y^2 \\
&- 858x^3y^4 + 520x^3y^6 + 520x^3y^8 - 572x^3y^{10} + 52x^3y^{12} + 52x^4y - 572x^4y^3 \\
&+ 1612x^4y^5 - 2327x^4y^9 + 858x^4y^{11} - 52x^4y^{13} - 312x^5y^2 + 2327x^5y^4 + 1612x^5y^{10} \\
&- 312x^5y^{12} - 2587x^5y^6 - 1300x^5y^8 - 52x^6y + 520x^6y^3 - 1300x^6y^5 - 1716x^6y^7 \\
&+ 2587x^6y^9 - 520x^6y^{11} + 52x^6y^{13} - x^13y^{14} + 1716x^7y^6 - 1716x^7y^8 - 52x^8y \\
&+ 520x^8y^3 - 2587x^8y^5 + 1716x^8y^7 + 1300x^8y^9 - 520x^8y^{11} + 52x^8y^{13} + 312x^9y^2 \\
&- 1612x^9y^4 + 1300x^9y^6 + 2587x^9y^8 - 2327x^9y^{10} + 312x^9y^{12} + 52x^{10}y - 858x^{10}y^3 \\
&+ 2327x^{10}y^5 - 1612x^{10}y^9 + 572x^{10}y^{11} - 52x^{10}y^{13} - 52x^{11}y^2 + 572x^{11}y^4 \\
&- 520x^{11}y^6 - 520x^{11}y^8 + 858x^{11}y^{10} - 130x^{11}y^{12} - 13x^{12}y + 130x^{12}y^3 \\
&- 312x^{12}y^5 + 312x^{12}y^9 - 52x^{12}y^{11} - 52x^{13}y^4 + 52x^{13}y^6 + 52x^{13}y^8 - 52x^{13}y^{10} \\
&+ 13x^{13}y^{12} + x^{14}y)(-1 + 2x + x^2)^6 = 0.
\end{aligned}$$

From the definitions of  $x$  and  $y$ , we have  $x = o(q^{1/2})$  and  $y = o(q^{13/2})$  as  $q \rightarrow 0$ , it can be seen that the first factor vanishes for  $q$  sufficiently small whereas other factors does



not vanish. Thus by the identity theorem, first factor vanishes identically. Hence the proof.  $\square$

**Theorem 4.10.** *If  $x = H(q)$  and  $y = H(q^{15})$ , then*

$$\begin{aligned}
(4.36) \quad & 2200x^3y^{13} - 210x^3y^{15} + 15xy^3 + 9380x^4y^{12} - 2850x^4y^{14} - 300x^4y^{16} + 30720x^5y^{11} \\
& - 16120x^5y^{13} + 10x^5y^{15} - 57694x^6y^{12} - xy - 75xy^5 + 125xy^7 + 30xy^9 + 15x^2y^2 \\
& - 210x^2y^4 + 693x^2y^6 + 300x^2y^8 - 221120x^{10}y^8 + 15x^3y - 330x^3y^3 + 1410x^3y^5 \\
& + 485x^3y^7 - 2610x^3y^9 - 210x^4y^2 + 1720x^4y^4 - 1966x^4y^6 - 5580x^4y^8 - 230x^4y^{10} \\
& - 75x^5y + 1410x^5y^3 - 3930x^5y^5 - 11265x^5y^7 + 1350x^5y^9 + 693x^6y^2 - 1966x^6y^4 \\
& - 14655x^6y^6 + 5520x^6y^8 + 68225x^6y^{10} - 162xy^{11} + 30xy^{13} + 30xy^{15} - 210x^2y^{12} \\
& + 435x^2y^{14} - 8x^2y^{16} - 948x^3y^{11} - 347150x^{15}y^{11} + 220290x^{15}y^{13} - 261446x^{15}y^{15} \\
& + 132664x^{16}y^{12} - 162x^{11}y + 300x^8y^2 + 68225x^{10}y^6 - 948x^{11}y^3 + 485x^7y^3 \\
& - 5580x^8y^4 - 230x^{10}y^4 - 995x^{10}y^2 + 1350x^9y^5 + 30720x^{11}y^5 - 261446x^9y^9 \\
& - 995x^2y^{10} - 11265x^7y^5 + 5520x^8y^6 + 113518x^9y^7 - 133770x^{11}y^7 + 8071x^7y^7 \\
& + 135270x^8y^8 - 2610x^9y^3 + 220290x^{11}y^9 + 113518x^7y^9 - 221120x^8y^{10} \\
& + 256960x^{10}y^{10} - 210x^{12}y^2 + 9380x^{12}y^4 - 57694x^{12}y^6 + 132664x^{12}y^8 \\
& - 617680x^{12}y^{10} + 2200x^{13}y^3 - 16120x^{13}y^5 + 48890x^{13}y^7 - 347150x^{13}y^9 \\
& - 750394x^{11}y^{11} + 30x^{13}y - 2850x^{14}y^4 + 10203x^{14}y^6 - 139368x^{14}y^8 \\
& + 645180x^{14}y^{10} + 30x^{15}y + 10203x^6y^{14} - 12940x^6y^{16} + 125yx^7 + 30yx^9 \\
& - 617680x^{10}y^{12} + 1068900x^{12}y^{12} + 132664x^8y^{12} + 945854x^{11}y^{13} - 750394x^{13}y^{13} \\
& + 48890x^7y^{13} - 347150x^9y^{13} + 645180x^{10}y^{14} - 617680x^{12}y^{14} - 139368x^8y^{14} \\
& + 1350x^{15}y^{19} - 347150x^{11}y^{15} + 220290x^{13}y^{15} - 45790x^7y^{15} + 323050x^9y^{15} \\
& - 139368x^{10}y^{16} + 132664x^{12}y^{16} + 105800x^8y^{16} + 435x^{14}y^2 + 945854x^{13}y^{11} \\
& - 133770x^7y^{11} + 220290x^9y^{11} - 617680x^{14}y^{12} + 256960x^{14}y^{14} - 221120x^{14}y^{16} \\
& + x^{16} - 210x^{15}y^3 + 10x^{15}y^5 - 45790x^{15}y^7 + 323050x^{15}y^9 - 8x^{16}y^2 - 300x^{16}y^4 \\
& - 12940x^{16}y^6 + 105800x^{16}y^8 - 139368x^{16}y^{10} - 221120x^{16}y^{14} + 135270x^{16}y^{16} \\
& + 21444x^7y^{17} - 2388x^7y^{19} - 12940x^8y^{18} - 2388x^5y^{17} - 8x^8y^{22} + x^8y^{24} \\
& - 45790x^9y^{17} - 210x^9y^{21} + 10x^9y^{19} + 30x^9y^{23} + 10203x^{10}y^{18} - 2850x^{10}y^{20} \\
& + 435x^{10}y^{22} + 48890x^{11}y^{17} + 2200x^{11}y^{21} - 16120x^{11}y^{19} + 30x^{11}y^{23} + x^3y^{21} \\
& - 5x^3y^{19} - 120x^4y^{18} + 30x^4y^{20} - 300x^8y^{20} - 5x^5y^{21} + 461x^5y^{19} + 2930x^6y^{18} \\
& - 120x^6y^{20} - 57694x^{12}y^{18} + 9380x^{12}y^{20} - 210x^{12}y^{22} - 133770x^{13}y^{17} \\
& - 948x^{13}y^{21} + 30720x^{13}y^{19} - 162x^{13}y^{23} + 68225x^{14}y^{18} - 230x^{14}y^{20} - 995x^{14}y^{22} \\
& + 113518x^{15}y^{17} - x^{23}y^{23} - 2610x^{15}y^{21} + 30x^{15}y^{23} + 5520x^{16}y^{18} - 5580x^{16}y^{20} \\
& + 300x^{16}y^{22} - 2388x^{17}y^5 + 21444x^{17}y^7 - 45790x^{17}y^9 + 48890x^{17}y^{11} - 133770x^{17}y^{13} \\
& + 113518x^{17}y^{15} + 8071x^{17}y^{17} + 485x^{17}y^{21} - 11265x^{17}y^{19} + 125x^{17}y^{23} - 120x^{18}y^4 \\
& + 2930x^{18}y^6 - 12940x^{18}y^8 + 10203x^{18}y^{10} - 57694x^{18}y^{12} + 68225x^{18}y^{14}
\end{aligned}$$

$$\begin{aligned}
& + 5520x^{18}y^{16} - 5x^{19}y^3 - 14655x^{18}y^{18} - 1966x^{18}y^{20} + 693x^{18}y^{22} + 461x^{19}y^5 \\
& - 2388x^{19}y^7 + 10x^{19}y^9 - 16120x^{19}y^{11} + 30720x^{19}y^{13} + 1350x^{19}y^{15} - 11265x^{19}y^{17} \\
& + 1410x^{19}y^{21} - 3930x^{19}y^{19} - 75x^{19}y^{23} + 30x^{20}y^4 - 120x^{20}y^6 - 300x^{20}y^8 \\
& - 2850x^{20}y^{10} + 9380x^{20}y^{12} - 230x^{20}y^{14} - 5580x^{20}y^{16} - 1966x^{20}y^{18} + 1720x^{20}y^{20} \\
& - 210x^{20}y^{22} - 8x^{22}y^8 - 5x^{21}y^5 - 210x^{21}y^9 + 2200x^{21}y^{11} - 948x^{21}y^{13} - 2610x^{21}y^{15} \\
& + 485x^{21}y^{17} - 330x^{21}y^{21} + 1410x^{21}y^{19} + y^{16} + 15x^{21}y^{23} + x^{21}y^3 + 435x^{22}y^{10} \\
& - 210x^{22}y^{12} - 995x^{22}y^{14} + 300x^{22}y^{16} + 693x^{22}y^{18} - 210x^{22}y^{20} + x^{24}y^8 + 30x^{23}y^9 \\
& + 30x^{23}y^{11} - 162x^{23}y^{13} + 30x^{23}y^{15} + 125x^{23}y^{17} + 15x^{23}y^{21} - 75x^{23}y^{19} + 15x^{22}y^{22} = 0.
\end{aligned}$$

**Theorem 4.11.** *If  $x = H(q)$  and  $y = H(q^{17})$ , then*

$$\begin{aligned}
(4.37) \quad & -xy + 17xy^3 - 102xy^5 + 238xy^7 - 85xy^9 - 323xy^{11} + 204xy^{13} + 68xy^{15} - 34xy^{17} \\
& + 34x^2y^2 - 442x^2y^4 + 1564x^2y^6 - 748x^2y^8 - 646x^2y^{10} + 1054x^2y^{12} - 952x^2y^{14} \\
& + 289x^2y^{16} + x^{18} + y^{18} + 17x^3y - 544x^3y^3 + 4029x^3y^5 - 8432x^3y^7 - 1105x^3y^9 \\
& + 9112x^3y^{11} - 3009x^3y^{13} - 952x^3y^{15} + 68x^3y^{17} - 442x^4y^2 + 5406x^4y^4 \\
& - 19448x^4y^6 + 15844x^4y^8 + 11254x^4y^{10} - 20298x^4y^{12} + 11696x^4y^{14} - 952x^4y^{16} \\
& - 102x^5y + 4029x^5y^3 - 29274x^5y^5 + 53805x^5y^7 + 9690x^5y^9 - 54825x^5y^{11} \\
& + 10914x^5y^{13} - 3009x^5y^{15} + 204x^5y^{17} + 1564x^6y^2 - 19448x^6y^4 + 74358x^6y^6 \\
& - 75514x^6y^8 - 55624x^6y^{10} + 112472x^6y^{12} - 20298x^6y^{14} + 1054x^6y^{16} + 238x^7y \\
& - 8432x^7y^3 + 53805x^7y^5 - 76432x^7y^7 + 1105x^7y^9 + 43928x^7y^{11} - 54825x^7y^{13} \\
& + 9112x^7y^{15} - 323x^7y^{17} - 748x^8y^2 + 15844x^8y^4 - 75514x^8y^6 + 40426x^8y^8 \\
& + 108766x^8y^{10} - 55624x^8y^{12} + 11254x^8y^{14} - 646x^8y^{16} - 85x^9y - 1105x^9y^3 \\
& + 9690x^9y^5 + 1105x^9y^7 - 67830x^9y^9 + 1105x^9y^{11} + 9690x^9y^{13} - 1105x^9y^{15} \\
& - 85x^9y^{17} - 646x^{10}y^2 + 11254x^{10}y^4 - 55624x^{10}y^6 + 108766x^{10}y^8 + 204x^{13}y \\
& + 40426x^{10}y^{10} - 75514x^{10}y^{12} + 15844x^{10}y^{14} - 748x^{10}y^{16} - 323x^{11}y + 9112x^{11}y^3 \\
& - 54825x^{11}y^5 + 43928x^{11}y^7 + 1105x^{11}y^9 - 76432x^{11}y^{11} + 53805x^{11}y^{13} \\
& - 8432x^{11}y^{15} + 238x^{11}y^{17} + 1054x^{12}y^2 - 20298x^{12}y^4 + 112472x^{12}y^6 - 55624x^{12}y^8 \\
& - 75514x^{12}y^{10} + 74358x^{12}y^{12} - 19448x^{12}y^{14} + 1564x^{12}y^{16} - 3009x^{13}y^3 + 10914x^{13}y^5 \\
& - 54825x^{13}y^7 + 9690x^{13}y^9 + 53805x^{13}y^{11} - 29274x^{13}y^{13} + 4029x^{13}y^{15} - 102x^{13}y^{17} \\
& - 952x^{14}y^2 + 11696x^{14}y^4 - 20298x^{14}y^6 + 11254x^{14}y^8 + 15844x^{14}y^{10} - 19448x^{14}y^{12} \\
& + 5406x^{14}y^{14} - 442x^{14}y^{16} + 68x^{15}y - 952x^{15}y^3 - 3009x^{15}y^5 + 9112x^{15}y^7 - 1105x^{15}y^9 \\
& - 8432x^{15}y^{11} + 4029x^{15}y^{13} - 544x^{15}y^{15} + 17x^{15}y^{17} + 289x^{16}y^2 - 952x^{16}y^4 \\
& + 1054x^{16}y^6 - 646x^{16}y^8 - 748x^{16}y^{10} + 1564x^{16}y^{12} - 442x^{16}y^{14} + 34x^{16}y^{16} - 34x^{17}y \\
& + 68x^{17}y^3 + 204x^{17}y^5 - 323x^{17}y^7 - 85x^{17}y^9 + 238x^{17}y^{11} - 102x^{17}y^{13} + 17x^{17}y^{15} \\
& - x^{17}y^{17} = 0.
\end{aligned}$$

**Theorem 4.12.** *If  $x = H(q)$  and  $y = H(q^{19})$ , then*

$$\begin{aligned}
(4.38) \quad & -22648x^{12}y^{17} + 9310x^{14}y^{17} - 1957x^{16}y^{17} + 171x^{18}y^{17} - 171x^3y^{18} + 608x^5y^{18} \\
& -4465x^7y^{18} + 4902x^9y^{18} + 3667x^{11}y^{18} - 5624x^{13}y^{18} + 817x^{15}y^{18} + 114x^{17}y^{18} \\
& -19x^{19}y^{18} + 209x^4y^{19} - 19x^6y^{19} - 703x^8y^{19} + 437x^{10}y^{19} + 361x^{12}y^{19} \\
& -399x^{14}y^{19} + 133x^{16}y^{19} - 19x^{18}y^{19} + x^{20}y^{19} + x^{19} - xy^{20} + 19xy^2 - 209xy^4 \\
& +19xy^6 + 703xy^8 - 437xy^{10} - 361xy^{12} + 399xy^{14} - 133xy^{16} + 19xy^{18} + 19x^2y \\
& -171x^2y^3 + 608x^2y^5 - 4465x^2y^7 + 4902x^2y^9 + 3667x^2y^{11} - 5624x^2y^{13} \\
& +817x^2y^{15} + 114x^2y^{17} - 19x^2y^{19} - 252529x^{14}y^{11} + 60705x^{16}y^{11} - 347529x^8y^{11} \\
& -101023x^{10}y^{11} + 420147x^{12}y^{11} - 57741x^4y^{11} + 2280x^{10}y^{17} - y - 32813x^{13}y^{10} \\
& -22325x^{15}y^{10} + 2280x^{17}y^{10} + 437x^{19}y^{10} - 133yx^4 + 399yx^6 - 361yx^8 \\
& -437yx^{10} + 703yx^{12} + 19yx^{14} - 209yx^{16} + 19yx^{18} - 114y^2x^3 - 817y^2x^5 \\
& +5624y^2x^7 - 3667y^2x^9 - 4902y^2x^{11} + 4465y^2x^{13} - 608y^2x^{15} + 171y^2x^{17} \\
& -19y^2x^{19} + 1957x^4y^3 - 9310x^6y^3 + 22648x^8y^3 - 2280x^{10}y^3 - 22990x^{12}y^3 \\
& +11704x^{14}y^3 - 646x^{16}y^3 - 114x^{18}y^3 - 646x^3y^4 + 21413x^5y^4 - 72789x^7y^4 \\
& +57741x^9y^4 + 60705x^{11}y^4 - 77957x^{13}y^4 + 16473x^{15}y^4 - 1957x^{17}y^4 + 133x^{19}y^4 \\
& -16473x^4y^5 + 107521x^6y^5 - 196289x^8y^5 + 22325x^{10}y^5 + 187473x^{12}y^5 \\
& -118009x^{14}y^5 + 21413x^{16}y^5 - 817x^{18}y^5 + 11704x^3y^6 - 118009x^5y^6 \\
& +337117x^7y^6 - 262409x^9y^6 - 252529x^{11}y^6 + 398221x^{13}y^6 - 107521x^{15}y^6 \\
& +9310x^{17}y^6 - 399x^{19}y^6 + 77957x^4y^7 - 398221x^6y^7 + 563445x^8y^7 + 32813x^{10}y^7 \\
& -518225x^{12}y^7 + 337117x^{14}y^7 - 72789x^{16}y^7 + 5624x^{18}y^7 - 22990x^3y^8 + 187473x^5y^8 \\
& -518225x^7y^8 + 347529x^9y^8 + 420147x^{11}y^8 - 563445x^{13}y^8 + 196289x^{15}y^8 \\
& -22648x^{17}y^8 + 361x^{19}y^8 - 60705x^4y^9 + 252529x^6y^9 - 420147x^8y^9 + 101023x^{10}y^9 \\
& +347529x^{12}y^9 - 262409x^{14}y^9 + 57741x^{16}y^9 - 3667x^{18}y^9 - 2280x^3y^{10} + 22325x^5y^{10} \\
& +32813x^7y^{10} + 101023x^9y^{10} - 101023x^{11}y^{10} - 4902x^{18}y^{11} + 22648x^3y^{12} \\
& -196289x^5y^{12} + 563445x^7y^{12} - 420147x^9y^{12} - 347529x^{11}y^{12} + 518225x^{13}y^{12} \\
& -187473x^{15}y^{12} + 22990x^{17}y^{12} - 703x^{19}y^{12} + 72789x^4y^{13} - 337117x^6y^{13} \\
& +518225x^8y^{13} - 32813x^{10}y^{13} - 563445x^{12}y^{13} + 398221x^{14}y^{13} - 77957x^{16}y^{13} \\
& +4465x^{18}y^{13} - 9310x^3y^{14} + 107521x^5y^{14} - 398221x^7y^{14} + 252529x^9y^{14} + 262409x^6y^{11} \\
& -337117x^{13}y^{14} + 22990x^8y^{17} - 11704x^{17}y^{14} - 19x^{19}y^{14} - 21413x^4y^{15} + 118009x^6y^{15} \\
& -187473x^8y^{15} - 22325x^{10}y^{15} + 196289x^{12}y^{15} - 107521x^{14}y^{15} + 16473x^{16}y^{15} \\
& -608x^{18}y^{15} + 1957x^3y^{16} - 16473x^5y^{16} + 77957x^7y^{16} - 60705x^9y^{16} - 57741x^{11}y^{16} \\
& +72789x^{13}y^{16} - 21413x^{15}y^{16} + 646x^{17}y^{16} + 209x^{19}y^{16} + 646x^4y^{17} - 11704x^6y^{17} \\
& +118009x^{15}y^{14} + 262409x^{11}y^{14} = 0.
\end{aligned}$$

**Theorem 4.13.** *If  $x = H(q)$  and  $y = H(q^{23})$ , then*

$$\begin{aligned}
(4.39) \quad & 23x^3y - 1610x^3y^3 + 18078x^3y^5 - 68655x^3y^7 + 90620x^3y^9 - 67022x^3y^{11} \\
& - 50462x^3y^{13} + 133584x^3y^{15} - 57270x^3y^{17} + 1725x^3y^{19} - 989x^3y^{21} \\
& - 46x^3y^{23} + 1012x^4y^2 - 23000x^4y^4 + 170292x^4y^6 - 484748x^4y^8 \\
& + 391046x^4y^{10} + 261096x^4y^{12} - 343022x^4y^{14} + 171764x^4y^{16} - 212106x^4y^{18} + \\
& 85514x^4y^{20} - 7222x^4y^{22} - 207x^5y + 18078x^5y^3 - 234646x^5y^5 + 1041739x^5y^7 \\
& - 1593532x^5y^9 + 817558x^5y^{11} + 661066x^5y^{13} - 1635484x^5y^{15} \\
& + 1123090x^5y^{17} - 243133x^5y^{19} + 1725x^5y^{21} + 1242x^5y^{23} - 7222x^6y^2 \\
& + 170292x^6y^4 - 1297292x^6y^6 + 4046252x^6y^8 - 4913329x^6y^{10} - 672750x^6y^{12} \\
& + 5037253x^6y^{14} - 3331688x^6y^{16} + 1300443x^6y^{18} - 212106x^6y^{20} \\
& + 14743x^6y^{22} + 897x^7y - 68655x^7y^3 + 1041739x^7y^5 - 5600293x^7y^7 \\
& + 10840636x^7y^9 - 5288620x^7y^{11} - 5626444x^7y^{13} + 9489340x^7y^{15} \\
& - 6199558x^7y^{17} + 1123090x^7y^{19} - 57270x^7y^{21} - 966x^7y^{23} - 46x^2y^2 \\
& + 1012x^2y^4 - 7222x^2y^6 + 16928x^2y^8 + 1863x^2y^{10} - 12834x^2y^{12} \\
& - 10143x^2y^{14} + 2116x^2y^{16} + 14743x^2y^{18} - 7222x^2y^{20} + 1081x^2y^{22} - xy \\
& + 23xy^3 - 207xy^5 + 897xy^7 - 1748xy^9 + 460xy^{11} + 2944xy^{13} - 2576xy^{15} \\
& - 966xy^{17} + 1242xy^{19} - 46xy^{21} - 46xy^{23} + x^{24} + y^{24} + 1012x^{22}y^{20} - 46x^{22}y^{22} \\
& - 46x^{23}y - 46x^{23}y^3 + 1242x^{23}y^5 + 1081x^{22}y^2 - 7222x^{22}y^4 + 14743x^{22}y^6 \\
& + 2116x^{22}y^8 - 10143x^{22}y^{10} - 12834x^{22}y^{12} + 1863x^{22}y^{14} + 16928x^{22}y^{16} \\
& - 7222x^{22}y^{18} - 966x^{23}y^7 - 2576x^{23}y^9 + 2944x^{23}y^{11} + 460x^{23}y^{13} - 1748x^{23}y^{15} \\
& + 897x^{23}y^{17} - 207x^{23}y^{19} + 23x^{23}y^{21} - x^{23}y^{23} - 46x^{21}y - 989x^{21}y^3 + 1725x^{21}y^5 \\
& - 57270x^{21}y^7 + 133584x^{21}y^9 - 50462x^{21}y^{11} - 67022x^{21}y^{13} + 23x^{21}y^{23} \\
& + 90620x^{21}y^{15} - 68655x^{21}y^{17} + 18078x^{21}y^{19} - 1610x^{21}y^{21} + 16928x^8y^2 \\
& - 484748x^8y^4 + 4046252x^8y^6 - 13628512x^8y^8 + 19391208x^8y^{10} \\
& - 2667816x^8y^{12} - 20124816x^8y^{14} + 17344783x^8y^{16} - 3331688x^8y^{18} \\
& + 171764x^8y^{20} + 2116x^8y^{22} - 1748x^9y + 90620x^9y^3 - 1593532x^9y^5 \\
& + 10840636x^9y^7 - 28673502x^9y^9 + 18981210x^9y^{11} + 19338354x^9y^{13} \\
& - 28274406x^9y^{15} + 9489340x^9y^{17} - 1635484x^9y^{19} + 133584x^9y^{21} \\
& - 2576x^9y^{23} + 1863x^{10}y^2 + 391046x^{10}y^4 - 4913329x^{10}y^6 + 19391208x^{10}y^8 \\
& - 26483212x^{10}y^{10} + 685584x^{10}y^{12} + 28328824x^{10}y^{14} - 20124816x^{10}y^{16} \\
& + 5037253x^{10}y^{18} - 343022x^{10}y^{20} - 10143x^{10}y^{22} + 460x^{11}y - 67022x^{11}y^3 \\
& + 817558x^{11}y^5 - 5288620x^{11}y^7 + 18981210x^{11}y^9 - 17142866x^{11}y^{11} \\
& - 14122322x^{11}y^{13} + 19338354x^{11}y^{15} - 5626444x^{11}y^{17} + 661066x^{11}y^{19} \\
& - 50462x^{11}y^{21} + 2944x^{11}y^{23} - 5626444x^{17}y^{11} + 261096x^{12}y^4 - 672750x^{12}y^6
\end{aligned}$$

$$\begin{aligned}
& -2667816x^{12}y^8 + 685584x^{12}y^{10} + 7517596x^{12}y^{12} + 685584x^{12}y^{14} \\
& -2667816x^{12}y^{16} - 672750x^{12}y^{18} + 261096x^{12}y^{20} - 12834x^{12}y^{22} + 2944x^{13}y \\
& -50462x^{13}y^3 + 661066x^{13}y^5 - 5626444x^{13}y^7 + 19338354x^{13}y^9 - 14122322x^{13}y^{11} \\
& -17142866x^{13}y^{13} + 18981210x^{13}y^{15} - 5288620x^{13}y^{17} + 817558x^{13}y^{19} \\
& -67022x^{13}y^{21} + 460x^{13}y^{23} - 10143x^{14}y^2 - 343022x^{14}y^4 + 5037253x^{14}y^6 \\
& -20124816x^{14}y^8 + 28328824x^{14}y^{10} + 685584x^{14}y^{12} - 26483212x^{14}y^{14} \\
& +19391208x^{14}y^{16} - 4913329x^{14}y^{18} + 391046x^{14}y^{20} + 1863x^{14}y^{22} - 2576x^{15}y \\
& +133584x^{15}y^3 - 1635484x^{15}y^5 + 9489340x^{15}y^7 - 28274406x^{15}y^9 + 19338354x^{15}y^{11} \\
& +18981210x^{15}y^{13} - 28673502x^{15}y^{15} + 10840636x^{15}y^{17} - 1593532x^{15}y^{19} \\
& +90620x^{15}y^{21} - 1748x^{15}y^{23} + 2116x^{16}y^2 + 171764x^{16}y^4 - 3331688x^{16}y^6 \\
& +17344783x^{16}y^8 - 20124816x^{16}y^{10} - 2667816x^{16}y^{12} + 19391208x^{16}y^{14} \\
& -13628512x^{16}y^{16} + 4046252x^{16}y^{18} - 484748x^{16}y^{20} + 16928x^{16}y^{22} - 966x^{17}y \\
& -57270x^{17}y^3 + 1123090x^{17}y^5 - 6199558x^{17}y^7 + 9489340x^{17}y^9 - 5288620x^{17}y^{13} \\
& +10840636x^{17}y^{15} - 5600293x^{17}y^{17} + 1041739x^{17}y^{19} - 68655x^{17}y^{21} + 897x^{17}y^{23} \\
& +14743x^{18}y^2 - 212106x^{18}y^4 + 1300443x^{18}y^6 - 3331688x^{18}y^8 + 5037253x^{18}y^{10} \\
& -672750x^{18}y^{12} - 4913329x^{18}y^{14} + 4046252x^{18}y^{16} - 1297292x^{18}y^{18} + 170292x^{18}y^{20} \\
& -7222x^{18}y^{22} + 1242x^{19}y + 1725x^{19}y^3 - 243133x^{19}y^5 + 1123090x^{19}y^7 - 1635484x^{19}y^9 \\
& +661066x^{19}y^{11} + 817558x^{19}y^{13} - 1593532x^{19}y^{15} + 1041739x^{19}y^{17} - 234646x^{19}y^{19} \\
& +18078x^{19}y^{21} - 207x^{19}y^{23} - 7222x^{20}y^2 + 85514x^{20}y^4 - 212106x^{20}y^6 + 171764x^{20}y^8 \\
& -343022x^{20}y^{10} + 261096x^{20}y^{12} + 391046x^{20}y^{14} - 484748x^{20}y^{16} + 170292x^{20}y^{18} \\
& -23000x^{20}y^{20} + 1012x^{20}y^{22} - 12834x^{12}y^2 = 0.
\end{aligned}$$

**Theorem 4.14.** *If  $x = H(q)$  and  $y = H(q^{25})$ , then*

$$\begin{aligned}
(4.40) \quad & -xy - 1050xy^{21} - 350xy^{23} + 3925xy^{19} + 25xy^3 - 250xy^5 + 1250xy^7 - 3075xy^9 \\
& +2355xy^{11} + 4000xy^{13} - 7200xy^{15} + 275xy^{17} + 75xy^{25} - xy^{27} - 74x^2y^2 \\
& - 9150x^2y^{18} - 12900x^2y^{22} + 29400x^2y^{20} + 1350x^2y^4 - 7500x^2y^6 + 7125x^2y^8 \\
& + 38600x^2y^{10} - 40705x^2y^{12} - 9000x^2y^{14} + 1410x^2y^{16} + 1750x^2y^{24} - 2175x^3y^9 \\
& - 80x^2y^{26} + x^2y^{28} + 25x^3y - 54865x^3y^{21} + 16750x^3y^{23} + 199450x^3y^{19} \\
& - 2026x^3y^3 + 1863100x^{24}y^{20} - 67850x^3y^7 + 95740x^3y^{11} + 166955x^3y^{13} + x^{26} \\
& - 8325525x^{24}y^{18} + 3464580x^{24}y^{22} + 4607920x^{24}y^{14} - 37700x^{24}y^4 + 347840x^{24}y^6 \\
& + 1750x^{24}y^2 - 1060020x^{24}y^8 + 38135x^{24}y^{10} - 1816060x^{24}y^{12} + 22075x^3y^5 \\
& + 2221160x^{24}y^{16} - 1436500x^{24}y^{24} + 185555x^{24}y^{26} + y^{26} - 7500x^{24}y^{28} \\
& + 75x^{25}y + 192522400x^{17}y^{13} + 1083170x^{25}y^{23} - 1609775x^{25}y^{19} + 38400x^{25}y^5 \\
& - 235590x^{25}y^7 - 549000x^{25}y^{21} - 87495925x^{17}y^{11} + 82506520x^{17}y^{15} \\
& - 274121325x^{17}y^{17} - 2475x^{25}y^3 - 151835x^{17}y^{25} + 166955x^{17}y^{27} + 4000x^{17}y^{29} \\
& - 9150x^{18}y^2 - 116535050x^{18}y^{18} + 16209500x^{18}y^{22} + 40667030x^{18}y^{20} \\
& + 232860x^{18}y^4 - 1816060x^{18}y^6 + 6767370x^{18}y^8 - 9048640x^{18}y^{10} \\
& - 29557375x^{18}y^{12} + 66718200x^{18}y^{14} + 33346730x^{18}y^{16} - 8325525x^{18}y^{24}
\end{aligned}$$

$$\begin{aligned}
& +1078360x^{18}y^{26} - 40705x^{18}y^{28} + 3925x^{19}y + 18490695x^{19}y^{21} + 5270050x^{19}y^{23} \\
& - 109573050x^{19}y^{19} + 199450x^{19}y^3 - 2004985x^{19}y^5 + 521210x^{19}y^7 + 30451505x^{19}y^9 \\
& - 44846360x^{19}y^{11} - 87495925x^{19}y^{13} + 121201700x^{19}y^{15} + 69658245x^{19}y^{17} - 1609775x^{19}y^{25} \\
& + 95740x^{19}y^{27} + 2355x^{19}y^{29} + 29400x^{20}y^2 + 40667030x^{20}y^{18} + 16295005x^{20}y^{22} \\
& - 68113850x^{20}y^{20} - 277150x^{20}y^4 + 38135x^{20}y^6 + 5897940x^{20}y^8 - 12759005x^{20}y^{10} \\
& - 9048640x^{20}y^{12} - 28632175x^{20}y^{14} + 54510600x^{20}y^{16} + 1863100x^{20}y^{24} - 722775x^{20}y^{26} \\
& + 38600x^{20}y^{28} - 24925925x^{21}y^{21} + 7444970x^{21}y^{23} + 18490695x^{21}y^{19} - 54865x^{21}y^3 \\
& + 780095x^{21}y^5 - 1050x^{21}y - 2482910x^{21}y^7 - 4219440x^{21}y^9 + 30451505x^{21}y^{11} \\
& - 20239885x^{21}y^{13} - 46677200x^{21}y^{15} + 42021150x^{21}y^{17} - 549000x^{21}y^{25} - 2175x^{21}y^{27} \\
& - 12900x^{22}y^2 + 16209500x^{22}y^{18} - 15835700x^{22}y^{22} + 16295005x^{22}y^{20} + 194365x^{22}y^4 \\
& - 1060020x^{22}y^6 + 926960x^{22}y^8 + 5897940x^{22}y^{10} + 6767370x^{22}y^{12} - 7830840x^{22}y^{14} \\
& - 24696800x^{22}y^{16} + 3464580x^{22}y^{24} - 3075x^{21}y^{29} - 262300x^{22}y^{26} + 7125x^{22}y^{28} - 350x^{23}y \\
& + 7444970x^{23}y^{21} - 5618900x^{23}y^{23} + 5270050x^{23}y^{19} + 16750x^{23}y^3 - 235590x^{23}y^5 \\
& + 1248020x^{23}y^7 - 2482910x^{23}y^9 + 521210x^{23}y^{11} + 8747730x^{23}y^{13} - 2424360x^{23}y^{15} \\
& - 13577550x^{23}y^{17} + 1083170x^{23}y^{25} - 67850x^{23}y^{27} + 1250x^{23}y^{29} - 244900x^3y^{15} \\
& - 128235x^3y^{17} - 2475x^3y^{25} + 80x^3y^{27} - x^3y^{29} + 1350x^4y^2 + 232860x^4y^{18} + 194365x^4y^{22} \\
& - 277150x^4y^{20} - 29274x^4y^4 + 185555x^4y^6 - 262300x^4y^8 - 722775x^4y^{10} + 1078360x^4y^{12} \\
& + 123070x^4y^{14} - 482360x^4y^{16} - 37700x^4y^{24} + 2475x^4y^{26} - 80x^4y^{28} + x^4y^{30} - 250x^5y \\
& + 780095x^5y^{21} - 235590x^5y^{23} - 2004985x^5y^{19} + 22075x^5y^3 - 282001x^5y^5 + 1083170x^5y^7 \\
& - 549000x^5y^9 - 1609775x^5y^{11} - 151835x^5y^{13} + 2808680x^5y^{15} + 83110x^5y^{17} + 38400x^5y^{25} \\
& - 2475x^5y^{27} - 7500x^6y^2 + 1750x^6y^{28} - 1816060x^6y^{18} - 1060020x^6y^{22} + 38135x^6y^{20} \\
& + 185555x^6y^4 - 1436500x^6y^6 + 3464580x^6y^8 + 75x^5y^{29} + 1863100x^6y^{10} - 8325525x^6y^{12} \\
& + 2221160x^6y^{14} + 4607920x^6y^{16} + 347840x^6y^{24} - 37700x^6y^{26} + 1250x^7y - 2482910x^7y^{21} \\
& + 1248020x^7y^{23} + 521210x^7y^{19} - 67850x^7y^3 + 1083170x^7y^5 - 5618900x^7y^7 + 7444970x^7y^9 \\
& + 5270050x^7y^{11} - 13577550x^7y^{13} - 2424360x^7y^{15} + 8747730x^7y^{17} - 235590x^7y^{25} \\
& - 350x^7y^{29} + 7125x^8y^2 + 6767370x^8y^{18} + 926960x^8y^{22} + 5897940x^8y^{20} - 262300x^8y^4 \\
& + 3464580x^8y^6 - 15835700x^8y^8 + 16295005x^8y^{10} + 16209500x^8y^{12} + 16750x^7y^{27} \\
& - 24696800x^8y^{14} - 7830840x^8y^{16} - 1060020x^8y^{24} + 194365x^8y^{26} - 12900x^8y^{28} \\
& - 3075x^9y - 4219440x^9y^{21} - 2482910x^9y^{23} - 1050x^9y^{29} + 30451505x^9y^{19} - 549000x^9y^5 \\
& + 7444970x^9y^7 - 24925925x^9y^9 + 18490695x^9y^{11} + 42021150x^9y^{13} - 46677200x^9y^{15} \\
& - 20239885x^9y^{17} + 780095x^9y^{25} - 54865x^9y^{27} + 38600x^{10}y^2 - 9048640x^{10}y^{18} - 2175x^9y^3 \\
& + 5897940x^{10}y^{22} - 12759005x^{10}y^{20} - 722775x^{10}y^4 + 1863100x^{10}y^6 + 16295005x^{10}y^8 \\
& - 68113850x^{10}y^{10} + 40667030x^{10}y^{12} + 54510600x^{10}y^{14} - 28632175x^{10}y^{16} + 38135x^{10}y^{24} \\
& - 277150x^{10}y^{26} + 521210x^{11}y^{23} + 29400x^{10}y^{28} + 30451505x^{11}y^{21} - 44846360x^{11}y^{19} \\
& + 95740x^{11}y^3 - 1609775x^{11}y^5 - 87495925x^{11}y^{17} + 5270050x^{11}y^7 + 18490695x^{11}y^9 \\
& - 109573050x^{11}y^{11} + 69658245x^{11}y^{13} + 121201700x^{11}y^{15} + 2355x^{11}y + 199450x^{11}y^{27} \\
& + 3925x^{11}y^{29} - 40705x^{12}y^2 - 29557375x^{12}y^{18} + 6767370x^{12}y^{22} - 9048640x^{12}y^{20} \\
& + 1078360x^{12}y^4 - 8325525x^{12}y^6 + 16209500x^{12}y^8 + 40667030x^{12}y^{10} - 116535050x^{12}y^{12} \\
& + 33346730x^{12}y^{14} + 66718200x^{12}y^{16} - 1816060x^{12}y^{24} + 232860x^{12}y^{26} - 9150x^{12}y^{28} \\
& - 20239885x^{13}y^{21} + 8747730x^{13}y^{23} - 87495925x^{13}y^{19} + 166955x^{13}y^3 - 151835x^{13}y^5
\end{aligned}$$

$$\begin{aligned}
 & -13577550x^{13}y^7 + 42021150x^{13}y^9 + 69658245x^{13}y^{11} - 274121325x^{13}y^{13} \\
 & + 82506520x^{13}y^{15} + 4000x^{13}y + 192522400x^{13}y^{17} + 83110x^{13}y^{25} - 128235x^{13}y^{27} \\
 & + 275x^{13}y^{29} - 9000x^{14}y^2 + 66718200x^{14}y^{18} - 7830840x^{14}y^{22} - 28632175x^{14}y^{20} \\
 & + 123070x^{14}y^4 + 25x^{29}y^{27} + 2221160x^{14}y^6 - 24696800x^{14}y^8 + 54510600x^{14}y^{10} \\
 & + 33346730x^{14}y^{12} - 187634800x^{14}y^{14} + 1410x^{16}y^2 + 88165480x^{14}y^{16} + 4607920x^{14}y^{24} \\
 & - 482360x^{14}y^{26} + 1410x^{14}y^{28} - 7200x^{15}y - 46677200x^{15}y^{21} + x^{30}y^4 - 2424360x^{15}y^{23} \\
 & + 121201700x^{15}y^{19} - 244900x^{15}y^3 + 2808680x^{15}y^5 - 2424360x^{15}y^7 - 46677200x^{15}y^9 \\
 & + 121201700x^{15}y^{11} + 82506520x^{15}y^{13} - 314920800x^{15}y^{15} + 82506520x^{15}y^{17} \\
 & + 2808680x^{15}y^{25} - 250x^{29}y^{25} - 244900x^{15}y^{27} + 33346730x^{16}y^{18} - 24696800x^{16}y^{22} \\
 & + 54510600x^{16}y^{20} - 482360x^{16}y^4 + 4607920x^{16}y^6 - 7200x^{29}y^{15} - 7200x^{15}y^{29} \\
 & - 7830840x^{16}y^8 - 28632175x^{16}y^{10} + 66718200x^{16}y^{12} + 88165480x^{16}y^{14} \\
 & - 187634800x^{16}y^{16} + 2221160x^{16}y^{24} - 2004985x^{11}y^{25} + 123070x^{16}y^{26} - 9000x^{16}y^{28} \\
 & + 42021150x^{17}y^{21} - 13577550x^{17}y^{23} + 69658245x^{17}y^{19} - 128235x^{17}y^3 + 83110x^{17}y^5 \\
 & + 8747730x^{17}y^7 - 20239885x^{17}y^9 + 780095x^{25}y^9 - 2004985x^{25}y^{11} + 2808680x^{25}y^{15} \\
 & - 151835x^{25}y^{17} - 282001x^{25}y^{25} + 123070x^{26}y^{16} - 7500x^{28}y^{24} + 22075x^{25}y^{27} \\
 & - 250x^{25}y^{29} - 80x^{26}y^2 + 1078360x^{26}y^{18} - 262300x^{26}y^{22} - 722775x^{26}y^{20} \\
 & + 2475x^{26}y^4 - 37700x^{26}y^6 + 194365x^{26}y^8 - 277150x^{26}y^{10} + 232860x^{26}y^{12} \\
 & - 482360x^{26}y^{14} + 83110x^{25}y^{13} - x^{29}y^{29} + 185555x^{26}y^{24} - 29274x^{26}y^{26} \\
 & + 1350x^{26}y^{28} - x^{27}y - 2175x^{27}y^{21} - 67850x^{27}y^{23} + 95740x^{27}y^{19} + 80x^{27}y^3 \\
 & + 16750x^{27}y^7 - 54865x^{27}y^9 + 199450x^{27}y^{11} - 128235x^{27}y^{13} + 166955x^{27}y^{17} \\
 & - 244900x^{27}y^{15} + 22075x^{27}y^{25} - 2475x^{27}y^5 - 2026x^{27}y^{27} + 25x^{27}y^{29} \\
 & + x^{28}y^2 - 40705x^{28}y^{18} + 7125x^{28}y^{22} + 38600x^{28}y^{20} - 80x^{28}y^4 + 1750x^{28}y^6 \\
 & - 12900x^{28}y^8 + 29400x^{28}y^{10} - 9150x^{28}y^{12} + 1410x^{28}y^{14} - 9000x^{28}y^{16} + 275x^{17}y \\
 & + 1350x^{28}y^{26} - 74x^{28}y^{28} - 3075x^{29}y^{21} + 1250x^{29}y^{23} + 2355x^{29}y^{19} + 4000x^{29}y^{17} \\
 & + 75x^{29}y^5 - 350x^{29}y^7 - 1050x^{29}y^9 + 3925x^{29}y^{11} + 275x^{29}y^{13} - x^{29}y^3 = 0
 \end{aligned}$$

**Theorem 4.15.** *If  $x = H(q)$  and  $y = H(q^{29})$ , then*

$$\begin{aligned}
 (4.41) \quad & 67501676x^6y^{15} + 421618356x^{21}y^{22} + 3084410768x^{21}y^{18} - 1888927296x^{21}y^{20} \\
 & - 1743406021x^{21}y^{10} + 2854862196x^{21}y^{12} - 1697818572x^{21}y^{14} - 1443012566x^{21}y^{16} \\
 & + 6206116x^{21}y^4 - 75880008x^{21}y^6 + 2204x^6y - y + 511110471x^{21}y^8 - 35850496x^{21}y^{24} \\
 & + 1339162x^{21}y^{26} - 96280x^{21}y^{28} + 2204x^{22}y - 74585216x^{22}y^{23} + 6279080x^6y^{23} \\
 & + 131459929x^6y^{19} - 35850496x^6y^{21} - 191327819x^6y^{11} + 171454554x^6y^{13} \\
 & - 278690x^{21}y^2 - 211816000x^6y^{17} - 113245x^6y^3 + 1675011x^6y^5 - 14417872x^6y^7 \\
 & + 75880008x^6y^9 - 1222756x^6y^{25} + 143231x^6y^{27} - 72065x^7y^2 + 74585216x^7y^{22} \\
 & + 531992037x^7y^{18} - 277293911x^7y^{20} - 424132511x^7y^{10} - 3770x^6y^{29} \\
 & + 626545029x^7y^{12} - 176717532x^7y^{14} - 418618828x^7y^{16} + 731467x^7y^4 \\
 & + 1384037992x^8y^{17} + 82733984x^7y^8 - 14417872x^7y^{24} + 2184715x^7y^{26} \\
 & - 154889x^7y^{28} - 82733984x^8y^{23} - 1888927296x^{10}y^{21} - 7656x^8y + 421618356x^8y^{21} \\
 & + 1355970429x^8y^{11} - 1305338024x^8y^{13} - 213382116x^8y^{15} - 6279080x^7y^6
 \end{aligned}$$

$$\begin{aligned}
& -23925x^8y^3 - 2926448x^8y^5 + 74585216x^8y^7 - 511110471x^8y^9 + 9436310x^8y^{25} \\
& - 514489x^8y^{27} - 2204x^8y^{29} - 1339162x^9y^4 - 511110471x^9y^{22} - 2854862196x^9y^{18} \\
& + 1743406021x^9y^{20} + 1888927296x^9y^{10} + 12760x^{10}y - 3084410768x^9y^{12} + 1443012566x^9y^{14} \\
& + 96280x^9y^2 + 1697818572x^9y^{16} + 35850496x^9y^6 - 421618356x^9y^8 + 20764x^{10}y^{29} \\
& + 75880008x^9y^{24} - 6206116x^9y^{26} + 278690x^9y^{28} - 1132340351x^8y^{19} + 424132511x^{10}y^{23} \\
& + 4284312419x^{10}y^{19} - 4603021694x^{10}y^{11} + 4751192264x^{10}y^{13} + 149551608x^{10}y^{15} \\
& - 4561890704x^{10}y^{17} + 294669x^{10}y^3 + 13505532x^{10}y^5 - 277293911x^{10}y^7 + 1743406021x^{10}y^9 \\
& - 47304336x^{10}y^{25} + 1994388x^{10}y^{27} + 247660x^{11}y^2 + 1355970429x^{11}y^{22} + 7654749462x^{11}y^{18} \\
& - 4603021694x^{11}y^{20} - 4284312419x^{11}y^{10} - xy^{30} + 7678466677x^{11}y^{12} - 4571021180x^{11}y^{14} \\
& - 4040038140x^{11}y^{16} + 3586836x^{11}y^4 - 131459929x^{11}y^6 + 1132340351x^{11}y^8 \\
& - 191327819x^{11}y^{24} - 7656xy^{22} - 1160xy^{18} + 12760x^{20}y - 20764xy^{10} + 10266xy^{12} \\
& + 26158xy^{14} - 26680xy^{16} - 754xy^4 + 3770xy^6 + 2204xy^8 + 2204xy^{24} - 348xy^{26} \\
& + 29xy^{28} + x^{29} + 29x^2y + 72065x^2y^{23} - 247660x^2y^{19} - 96280x^2y^{21} - 36221x^2y^{11} \\
& - 514576x^2y^{13} + 368184x^2y^{15} + 313200x^2y^{17} - 2610x^2y^3 + 31784x^2y^5 - 154889x^2y^7 \\
& + 278690x^2y^9 - 12818x^2y^{25} + 725x^2y^{27} + 20387x^3y^4 - 23925x^3y^{22} + 914109x^3y^{18} \\
& + 294669x^3y^{20} - 1994388x^3y^{10} + 2242628x^3y^{12} + 1218348x^3y^{14} - 725x^3y^2 - 2956724x^3y^{16} \\
& - 143231x^3y^6 + 514489x^3y^8 - 113245x^3y^{24} + 33466x^3y^{26} - 2610x^3y^{28} - 731467x^4y^{23} \\
& - 149551608x^{15}y^{10} + 1339162x^4y^{21} + 10423267x^4y^{11} - 348x^4y - 5256714x^4y^{13} \\
& - 8299104x^4y^{15} + 10334672x^4y^{17} + 33466x^4y^3 - 424937x^4y^5 + 2184715x^4y^7 - 6206116x^4y^9 \\
& + 189776x^4y^{25} - 20387x^4y^{27} + 12818x^5y^2 - 2926448x^5y^{22} - 43689312x^5y^{18} \\
& + 13505532x^5y^{20} + 47304336x^5y^{10} - 61755732x^5y^{12} - 1522964x^5y^{14} + 56288246x^5y^{16} \\
& - 189776x^5y^4 + 1222756x^5y^6 - 9436310x^5y^8 + 1675011x^5y^{24} + 754x^4y^{29} - 424937x^5y^{26} \\
& + 31784x^5y^{28} + 171454554x^{13}y^{24} - 5256714x^{13}y^{26} - 514576x^{13}y^{28} - 26680x^{14}y \\
& + 176717532x^{14}y^{23} + 4571021180x^{14}y^{19} - 1443012566x^{14}y^{21} - 4040038140x^{14}y^{11} \\
& + 4980578233x^{14}y^{13} - 362465432x^{14}y^{15} - 5225278696x^{14}y^{17} - 2956724x^{14}y^3 \\
& + 56288246x^{14}y^5 - 418618828x^{14}y^7 + 1697818572x^{14}y^9 + 1522964x^{14}y^{25} - 1218348x^{14}y^{27} \\
& - 26158x^{14}y^{29} - 368184x^{15}y^2 - 213382116x^{15}y^{22} + 154196016x^{15}y^{18} + 149551608x^{15}y^{20} \\
& - 3586836x^{14}y^{19} - 154196016x^{15}y^{12} + 362465432x^{15}y^{14} - 362465432x^{15}y^{16} \\
& + 8299104x^{15}y^4 - 67501676x^{15}y^6 + 7656x^{22}y^{29} + 213382116x^{15}y^8 + 67501676x^{15}y^{24} \\
& - 8299104x^{15}y^{26} + 368184x^{15}y^{28} + 26158x^{16}y + 418618828x^{16}y^{23} + 4040038140x^{16}y^{19} \\
& - 1697818572x^{16}y^{21} - 4571021180x^{16}y^{11} + 5225278696x^{16}y^{13} + 362465432x^{16}y^{15} \\
& - 4980578233x^{16}y^{17} + 1218348x^{16}y^3 - 1522964x^{16}y^5 - 176717532x^{16}y^7 + 1443012566x^{16}y^9 \\
& - 2204x^{24}y^{29} - 56288246x^{16}y^{25} + 2956724x^{16}y^{27} + 26680x^{16}y^{29} + 514576x^{17}y^2 \\
& + 1384037992x^{17}y^{22} + 8440473896x^{17}y^{18} - 4561890704x^{17}y^{20} - 4751192264x^{17}y^{10} \\
& + 8739973397x^{17}y^{12} - 4980578233x^{17}y^{14} - 5225278696x^{17}y^{16} + 5256714x^{17}y^4 \\
& - 171454554x^{17}y^6 + 1305338024x^{17}y^8 - 211816000x^{17}y^{24} + 10334672x^{17}y^{26} \\
& + x^{30}y^{29} + 313200x^{17}y^{28} + 10266x^{18}y - 531992037x^{18}y^{23} - 7654749462x^{18}y^{19} \\
& + 2854862196x^{18}y^{21} - 2204x^{29}y^6 + 7678466677x^{18}y^{11} - 8440473896x^{18}y^{13} \\
& - 154196016x^{18}y^{15} + 8739973397x^{18}y^{17} + 2242628x^{18}y^3 - 61755732x^{18}y^5 \\
& + 626545029x^{18}y^7 - 3084410768x^{18}y^9 + 43689312x^{18}y^{25} - 914109x^{18}y^{27} + 1160x^{18}y^{29} \\
& + 277293911x^{20}y^{23} - 1132340351x^{19}y^{22} - 7678466677x^{19}y^{18} + 4284312419x^{19}y^{20}
\end{aligned}$$



$$\begin{aligned}
& +4603021694x^{19}y^{10} - 7654749462x^{19}y^{12} + 4040038140x^{19}y^{14} + 4571021180x^{19}y^{16} \\
& - 10423267x^{19}y^4 + 191327819x^{19}y^6 - 1355970429x^{19}y^8 + 131459929x^{19}y^{24} \\
& - 3586836x^{19}y^{26} - 20764x^{20}y - 247660x^{19}y^{28} - 2184715x^{26}y^{23} + 4603021694x^{20}y^{19} \\
& - 1743406021x^{20}y^{21} - 4284312419x^{20}y^{11} + 4561890704x^{20}y^{13} - 149551608x^{20}y^{15} \\
& - 4751192264x^{20}y^{17} - 1994388x^{20}y^3 + 47304336x^{20}y^5 - 424132511x^{20}y^7 \\
& + 1888927296x^{20}y^9 - 1160x^{12}y - 13505532x^{20}y^{25} - 294669x^{20}y^{27} - 12760x^{20}y^{29} \\
& + 10423267x^{11}y^{26} - 36221x^{11}y^{28} - 626545029x^{12}y^{23} - 7678466677x^{12}y^{19} \\
& + 3084410768x^{12}y^{21} + 7654749462x^{12}y^{11} - 8739973397x^{12}y^{13} + 154196016x^{12}y^{15} \\
& + 8440473896x^{12}y^{17} + 914109x^{12}y^3 - 43689312x^{12}y^5 - 10266x^{12}y^{29} + 531992037x^{12}y^7 \\
& - 2854862196x^{12}y^9 + 61755732x^{12}y^{25} - 2242628x^{12}y^{27} - 313200x^{13}y^2 + 514489x^{22}y^3 \\
& - 1305338024x^{13}y^{22} - 8739973397x^{13}y^{18} + 4751192264x^{13}y^{20} + 4561890704x^{13}y^{10} \\
& - 8440473896x^{13}y^{12} + 5225278696x^{13}y^{14} + 4980578233x^{13}y^{16} - 10334672x^{13}y^4 \\
& + 211816000x^{13}y^6 - 1384037992x^{13}y^8 - 1355970429x^{22}y^{19} + 511110471x^{22}y^{21} \\
& + 1132340351x^{22}y^{11} - 1384037992x^{22}y^{13} + 213382116x^{22}y^{15} + 1305338024x^{22}y^{17} \\
& - 9436310x^{22}y^5 + 82733984x^{22}y^7 - 421618356x^{22}y^9 + 2926448x^{22}y^{25} + 23925x^{22}y^{27} \\
& + 154889x^{23}y^2 - 82733984x^{23}y^{22} - 626545029x^{23}y^{18} + 424132511x^{23}y^{20} \\
& + 277293911x^{23}y^{10} - 531992037x^{23}y^{12} + 418618828x^{23}y^{14} + 176717532x^{23}y^{16} \\
& - 2184715x^{23}y^4 + 14417872x^{23}y^6 - 74585216x^{23}y^8 + 6279080x^{23}y^{24} - 731467x^{23}y^{26} \\
& + 72065x^{23}y^{28} + 3770x^{24}y + 14417872x^{24}y^{23} + 191327819x^{24}y^{19} - 75880008x^{24}y^{21} \\
& - 131459929x^{24}y^{11} + 211816000x^{24}y^{13} - 67501676x^{24}y^{15} - 171454554x^{24}y^{17} \\
& - 143231x^{24}y^3 + 1222756x^{24}y^5 - 6279080x^{24}y^7 + 35850496x^{24}y^9 - 1675011x^{24}y^{25} \\
& + 113245x^{24}y^{27} - 31784x^{25}y^2 + 9436310x^{25}y^{22} + 61755732x^{25}y^{18} - 47304336x^{25}y^{20} \\
& - 13505532x^{25}y^{10} + 43689312x^{25}y^{12} - 56288246x^{25}y^{14} + 1522964x^{25}y^{16} + 424937x^{25}y^{24} \\
& - 1675011x^{25}y^6 + 2926448x^{25}y^8 - 1222756x^{25}y^{24} + 189776x^{25}y^{26} - 12818x^{25}y^{28} \\
& - 754x^{26}y + 36221x^{19}y^2 - 10423267x^{26}y^{19} + 6206116x^{26}y^{21} + 3586836x^{26}y^{11} \\
& - 10334672x^{26}y^{13} + 8299104x^{26}y^{15} + 5256714x^{26}y^{17} + 20387x^{26}y^3 - 189776x^{26}y^5 \\
& + 731467x^{26}y^7 - 1339162x^{26}y^9 + 424937x^{26}y^{25} - 33466x^{26}y^{27} + 348x^{26}y^{29} + 2610x^{27}y^2 \\
& - 514489x^{27}y^{22} - 2242628x^{27}y^{18} + 1994388x^{27}y^{20} - 294669x^{27}y^{10} - 914109x^{27}y^{12} \\
& + 2956724x^{27}y^{14} - 1218348x^{27}y^{16} - 33466x^{27}y^4 + 113245x^{27}y^6 + 23925x^{27}y^8 \\
& + 143231x^{27}y^{24} + 348x^{29}y^4 - 20387x^{27}y^{26} + 725x^{27}y^{28} + 154889x^{28}y^{23} + 36221x^{28}y^{19} \\
& - 278690x^{28}y^{21} + 247660x^{28}y^{11} - 313200x^{28}y^{13} - 368184x^{28}y^{15} + 514576x^{28}y^{17} \\
& - 725x^{28}y^3 + 12818x^{28}y^5 - 72065x^{28}y^7 + 96280x^{28}y^9 - 31784x^{28}y^{25} + 2610x^{28}y^{27} \\
& - 29x^{28}y^{29} - 29x^{29}y^2 - 2204x^{29}y^{22} - 10266x^{29}y^{18} + 20764x^{29}y^{20} - 12760x^{29}y^{10} \\
& + 754x^{29}y^{26} + 26680x^{29}y^{14} - 26158x^{29}y^{16} + 7656x^{29}y^8 - 3770x^{29}y^{24} + 1160x^{29}y^{12} = 0
\end{aligned}$$

**Theorem 4.16.** *If  $x = H(q)$  and  $y = H(q^{55})$ , then*

$$(4.42) \quad 897x^7y - 68655x^7y^3 + 1041739x^7y^5 - 5600293x^7y^7 + 10840636x^7y^9$$

$$\begin{aligned}
& -5288620x^7y^{11} - 5626444x^7y^{13} + 9489340x^7y^{15} + 23x^3y - 1610x^3y^3 + 18078x^3y^5 \\
& - 68655x^3y^7 + 90620x^3y^9 - 67022x^3y^{11} - 50462x^3y^{13} + 133584x^3y^{15} - 57270x^3y^{17} \\
& + 1725x^3y^{19} - 989x^3y^{21} - 46x^3y^{23} + 1012x^4y^2 - 23000x^4y^4 + 170292x^4y^6 - 484748x^4y^8 \\
& + 391046x^4y^{10} + 261096x^4y^{12} - 343022x^4y^{14} + 171764x^4y^{16} - 212106x^4y^{18} + 85514x^4y^{20} \\
& - 7222x^4y^{22} - 207x^5y + 18078x^5y^3 - 234646x^5y^5 + 1041739x^5y^7 - 1593532x^5y^9 \\
& - 3331688x^5y^{11} + 817558x^5y^{13} - 1635484x^5y^{15} + 1123090x^5y^{17} \\
& - 243133x^5y^{19} + 1725x^5y^{21} + 1242x^5y^{23} + 170292x^6y^4 - 1297292x^6y^6 + 4046252x^6y^8 \\
& - 4913329x^6y^{10} - 672750x^6y^{12} + 5037253x^6y^{14} - 7222x^6y^{16} + 1300443x^6y^{18} - 212106x^6y^{20} \\
& + 14743x^6y^{22} - 46x^2y^2 + 1012x^2y^4 - 7222x^2y^6 + 16928x^2y^8 + 1863x^2y^{10} - 12834x^2y^{12} \\
& - 10143x^2y^{14} + 2116x^2y^{16} + 14743x^2y^{18} - 7222x^2y^{20} + 1081x^2y^{22} - xy + 23xy^3 - 207xy^5 \\
& + 897xy^7 - 1748xy^9 + 460xy^{11} + 2944xy^{13} - 2576xy^{15} - 966xy^{17} + 1242xy^{19} - 46xy^{21} \\
& - 46xy^{23} + x^{24} - 46x^{23}y - 46x^{23}y^3 + 1242x^{23}y^5 - 966x^{23}y^7 - 2576x^{23}y^9 + 2944x^{23}y^{11} \\
& + 460x^{23}y^{13} - 1748x^{23}y^{15} + y^{24} + 897x^{23}y^{17} - 6199558x^7y^{17} + 1123090x^7y^{19} - 57270x^7y^{21} \\
& - 966x^7y^{23} + 16928x^8y^2 - 484748x^8y^4 + 4046252x^8y^6 - 13628512x^8y^8 + 19391208x^8y^{10} \\
& - 2667816x^8y^{12} - 20124816x^8y^{14} + 17344783x^8y^{16} - 3331688x^8y^{18} + 171764x^8y^{20} \\
& + 2116x^8y^{22} - 1748x^9y + 90620x^9y^3 - 1593532x^9y^5 + 10840636x^9y^7 - 28673502x^9y^9 \\
& + 18981210x^9y^{11} + 19338354x^9y^{13} - 28274406x^9y^{15} + 9489340x^9y^{17} - 1635484x^9y^{19} \\
& + 133584x^9y^{21} - 2576x^9y^{23} + 1863x^{10}y^2 + 391046x^{10}y^4 - 4913329x^{10}y^6 + 19391208x^{10}y^8 \\
& - 10143x^{10}y^{10} + 685584x^{10}y^{12} + 28328824x^{10}y^{14} - 20124816x^{10}y^{16} + 5037253x^{10}y^{18} \\
& - 343022x^{10}y^{20} - 26483212x^{10}y^{10} - 14122322x^{11}y^{13} - 67022x^{11}y^3 + 817558x^{11}y^5 \\
& - 5288620x^{11}y^7 + 18981210x^{11}y^9 - 17142866x^{11}y^{11} + 460x^{11}y + 19338354x^{11}y^{15} \\
& - 5626444x^{11}y^{17} + 661066x^{11}y^{19} - 50462x^{11}y^{21} + 2944x^{11}y^{23} - 12834x^{12}y^2 + 261096x^{12}y^4 \\
& - 672750x^{12}y^6 - 2667816x^{12}y^8 + 685584x^{12}y^{10} + 7517596x^{12}y^{12} + 685584x^{12}y^{14} \\
& - 2667816x^{12}y^{16} - 672750x^{12}y^{18} + 261096x^{12}y^{20} - 12834x^{12}y^{22} + 2944x^{13}y - 50462x^{13}y^3 \\
& + 661066x^{13}y^5 - 5626444x^{13}y^7 + 19338354x^{13}y^9 - 14122322x^{13}y^{11} - 17142866x^{13}y^{13} \\
& + 18981210x^{13}y^{15} - 5288620x^{13}y^{17} + 817558x^{13}y^{19} - 67022x^{13}y^{21} + 460x^{13}y^{23} - 10143x^{14}y^2 \\
& - 343022x^{14}y^4 + 5037253x^{14}y^6 - 20124816x^{14}y^8 + 28328824x^{14}y^{10} + 685584x^{14}y^{12} \\
& - 26483212x^{14}y^{14} + 19391208x^{14}y^{16} - 4913329x^{14}y^{18} + 391046x^{14}y^{20} + 1863x^{14}y^{22} \\
& - 2576x^{15}y + 133584x^{15}y^3 - 1635484x^{15}y^5 + 9489340x^{15}y^7 - 28274406x^{15}y^9 + 2116x^{16}y^2 \\
& + 18981210x^{15}y^{13} - 28673502x^{15}y^{15} + 10840636x^{15}y^{17} - 1593532x^{15}y^{19} + 90620x^{15}y^{21} \\
& - 1748x^{15}y^{23} + 171764x^{16}y^4 - 3331688x^{16}y^6 + 17344783x^{16}y^8 - 20124816x^{16}y^{10} \\
& + 19338354x^{15}y^{11} - 966x^{17}y - 2667816x^{16}y^{12} + 19391208x^{16}y^{14} - 13628512x^{16}y^{16} \\
& + 4046252x^{16}y^{18} - 484748x^{16}y^{20} + 16928x^{16}y^{22} - 57270x^{17}y^3 + 1123090x^{17}y^5 \\
& - 6199558x^{17}y^7 + 9489340x^{17}y^9 - 5626444x^{17}y^{11} - 5288620x^{17}y^{13} + 10840636x^{17}y^{15} \\
& - 5600293x^{17}y^{17} + 1041739x^{17}y^{19} - 68655x^{17}y^{21} + 897x^{17}y^{23} + 5037253x^{18}y^{10}
\end{aligned}$$

$$\begin{aligned}
&+14743x^{18}y^2 - 212106x^{18}y^4 + 1300443x^{18}y^6 - 3331688x^{18}y^8 - 672750x^{18}y^{12} \\
&- 4913329x^{18}y^{14} - 46x^{21}y + 4046252x^{18}y^{16} - 1297292x^{18}y^{18} + 170292x^{18}y^{20} - 7222x^{18}y^{22} \\
&+ 1242x^{19}y + 1725x^{19}y^3 - 243133x^{19}y^5 + 1123090x^{19}y^7 - 1635484x^{19}y^9 + 661066x^{19}y^{11} \\
&+ 817558x^{19}y^{13} - 1593532x^{19}y^{15} + 1041739x^{19}y^{17} - 234646x^{19}y^{19} + 18078x^{19}y^{21} \\
&- 207x^{19}y^{23} - 7222x^{20}y^2 + 85514x^{20}y^4 - 212106x^{20}y^6 + 171764x^{20}y^8 - 343022x^{20}y^{10} \\
&+ 261096x^{20}y^{12} + 391046x^{20}y^{14} - 484748x^{20}y^{16} + 170292x^{20}y^{18} - 23000x^{20}y^{20} + 1012x^{20}y^{22} \\
&- 989x^{21}y^3 + 1725x^{21}y^5 - 57270x^{21}y^7 + 133584x^{21}y^9 - 50462x^{21}y^{11} - 67022x^{21}y^{13} \\
&+ 90620x^{21}y^{15} - 68655x^{21}y^{17} + 18078x^{21}y^{19} - 1610x^{21}y^{21} + 23x^{21}y^{23} + 1081x^{22}y^2 \\
&- 7222x^{22}y^4 + 14743x^{22}y^6 + 2116x^{22}y^8 - 10143x^{22}y^{10} - 12834x^{22}y^{12} + 1863x^{22}y^{14} \\
&+ 16928x^{22}y^{16} - 7222x^{22}y^{18} + 1012x^{22}y^{20} - 46x^{22}y^{22} - 207x^{23}y^{19} + 23x^{23}y^{21} - x^{23}y^{23} = 0.
\end{aligned}$$

Proofs of the identities (4.36)–(4.42) are similar to the proof of the identity (4.32) given above except that in place of result (3.35), result (3.39) is used for proving (4.36); result (3.40) is used for proving (4.37); result (3.42) is used for proving (4.39); result (3.43) is used for proving (4.40); result (3.44) is used for proving (4.41); result (3.45) is used for proving (4.42).

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