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RENORMALIZED SOLUTIONS FOR NONLINEAR PARABOLIC EQUATION WITH LOWER ORDER TERMS

A. ABERQI[1], J. BENNOUNA[1], M. MEKKOUR[1] AND H. REDWANE[2]

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[1]:UNIVERSITÉ SIDI MOHAMMED BEN ABDELLAH, DÉPARTEMENT DE MATHÉMATIQUES, LABORATOIRE LAMA, FACULTÉ DES SCIENCES DHAR-MAHREZ, B.P 1796 ATLAS FÉS, MOROCCO.

[2]:FACULTÉ DES SCIENCES JURIDIQUES, ECONOMIQUES ET SOCIALES, UNIVERSITÉ HASSAN 1, B.P. 784. SETTAT, MOROCCO.

aberqi_ahmed@yahoo.fr
jbennouna@hotmail.com
mekkour.mounir@yahoo.fr
redwane_hicham@yahoo.fr

ABSTRACT. In this paper, we study the existence of renormalized solutions for the nonlinear parabolic problem: $\frac{\partial b(u)}{\partial t} - div(a(x,t,u,\nabla u)) + div(\phi(x,t,u)) = f$,where the right side belongs to $L^1(\Omega\times(0,T))$ and $b(u)$ is unbounded function of u, the term $-\mathrm{div}(a(x,t,u,\nabla u))$is a Leray–Lions operator and the function ϕ is a nonlinear lower order and satisfy only the growth condition.

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1. INTRODUCTION

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 2),$ T is a positive real number, and $Q_T = \Omega \times (0,T)$. Let b is a strictly increasing C^1-function, the data f and $b(u_0)$ in $L^1(Q)$ and $L^1(\Omega)$ respectively, $-div\Big(a(x,t,u,\nabla u)\Big)$ is a Leray-Lions operator defined on $W_0^{1,p}(\Omega)$ (see assumptions (2.2)-(2.4) of Section 2). The function $\phi(x,t,u)$ is a Carathéodory assumed to be continuous on u (see assumptions (2.5)-(2.7)). We considere the following nonlinear parabolic problem:

$$
(1.1) \qquad \begin{cases} \dfrac{\partial b(u)}{\partial t} - div(a(x,t,u,\nabla u)) + div(\phi(x,t,u)) = f \quad in \quad Q_T \\ u(x,t) = 0 \quad on \quad \partial\Omega \times (0,T) \\ b(u(x,0)) = b(u_0(x)) \quad in \quad \Omega. \end{cases}
$$

Under our assumptions, problem (1.1) does not admit, in general, a weak solution since the term $\phi(x,t,u)$ may not belong $(L^1_{loc}(Q))^N$. In order to overcome this difficulty, we work with the framework of of renormalized solutions (see definition (3.1)). The notion of renormalized solutions was introduced by R.-J. DiPerna and P.-L. Lions [14] for the study of the Boltzmann equation. It was then used by L. Boccardo and al (see [10]) when the right hand side is in $W^{-1,p'}(\Omega)$ and by J.-M. Rakotoson (see [19]) when the right hand side is in $L^1(\Omega)$.

The existence and uniqueness of a renormalized solution has been proved by D. Blanchard and F. Murat [6] in the case where $a(x,t,s,\xi)$ is independent of s, and with $\phi = 0$ and by D. Blanchard, F. Murat and H. Redwane [5] with the large monotonicity on a.

For the degenerated parabolic equations the existence of weak solutions have been proved by L. Aharouch and al [1] in the case where a is strictly monotone, $\phi = 0$ and $f \in L^{p'}(0,T,W^{-1,p'}(\Omega,\omega^*))$. See also the existence of renormalized solution proved by Y. Akdim and al [3] in the case where $a(x,t,s,\xi)$ is independent of s and $\phi = 0$.

In the case where $b(u) = u$, the existence of renormalized solutions for (1.1) has been established by R.-Di Nardo (see [12]). For the degenerated parabolic equation with $b(u) = u,$ $div(\phi(x,t,u)) = H(x,t,u,\nabla u)$ and $f \in L^1(Q)$, the existence of renormalized solution has been proved by Y. Akdim and al (see [4]).

The case where $b(u) = b(x,u),$ $div(\phi(x,t,u)) = H(x,t,u,\nabla u)$ and $f \in L^1(Q)$, the existence of renormalized solutions has been established by H. Redwane (see [17]) in the classical Sobolev space and by Y. Akdim and al (see [2]) in the degenerate Sobolev space.

It is our purpose, in this paper to generalize the result of ([3], [4], [12]) and we prove the existence of a renormalized solution of (1.1).

The plan of the paper is as follows: In Section 2 we give some preliminaries and basic assumptions. In Section 3 we give the definition of a renormalized solution of (1.1), and we establish (Theorem 3.1) the existence of such a solution.

2. BASIC ASSUMPTIONS AND PRELIMINARIES

2.1. **Preliminaries.**

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 2),$ T is a positive real number, and $Q_T = \Omega \times (0,T)$. We need the Sobolev embeddings result

Theorem 2.1. *(Gagliardo-Nirenberg) Let v be a function in $W_0^{1,q}(\Omega) \cap L^\rho(\Omega)$ with $q \geq 1,$ $\rho \geq 1$. Then there exists a positive constante C, depending on $N,$ q and ρ, such that*

$$\| v \|_{L^\gamma(\Omega)} \leq C \| \nabla v \|^{\theta}_{(L(\Omega))^N} \| v \|^{1-\theta}_{L^\rho(\Omega)}$$

for every θ and γ satisfying

$$0 \leq \theta \leq 1, \quad 1 \leq \gamma \leq +\infty, \quad \frac{1}{\gamma} = \theta(\frac{1}{q} - \frac{1}{N}) + \frac{1-\theta}{\rho}.$$

An immediate consequence of the previous result:

Corollary 2.2. *Let* $v \in L^q((0,T), L^q(\Omega)) \cap L^\infty((0,T), L^\rho(\Omega))$, *with* $q \geq 1,\ \rho \geq 1$. *Then* $v \in L^\sigma(\Omega)$ *with* $\sigma = q(\frac{N+\rho}{N})$*and*

$$\int_{Q_T} |v|^\sigma dxdt \leq C \parallel v \parallel^{\frac{\rho q}{N}}_{L^\infty(0,T,L^\rho(\Omega))} \int_{Q_T} |\nabla v|^q dxdt.$$

Lemma 2.3. *(see* [12]*) Assume that* Ω *is an open set of* $\mathbb{R}^N$ *of finite measure and* $1 < p < +\infty$. *Let* u *be a measurable function satisfying* $T_k(u) \in L^p((0,T), W_0^{1,p}(\Omega)) \cap L^\infty((0,T), L^2(\Omega))$ *for every* k *and such that:*

$$sup_{t\in(0,T)} \int_\Omega |\nabla T_k(u)|^2 + \int_{Q_T} |\nabla T_k(u)|^p \leq Mk, \quad \forall \quad k > 0$$

where M is a positive constant. Then

$$\| |u|^{p-1} \|_{L^{\frac{p(N+1)-N}{N(p-1)},\infty}(Q_T)} \leq CM^{(\frac{p}{N}+1)\frac{N}{N+p'}} |Q_T|^{\frac{1}{p'}\frac{N}{N+p'}}$$

$$\| |\nabla u|^{p-1} \|_{L^{\frac{p(N+1)-N}{(N+1)(p-1)},\infty}(Q_T)} \leq CM^{\frac{(N+2)(p-1)}{p(N+1)-N}}$$

where C *is a constant depend only on* N *and* p.

2.2. **Assumption(H).** Throughout this paper, we assume that the following assumptions hold true:

(2.1) $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is strictly increasing C^1-function, such that $b' > \beta > 0$ and $b(0) = 0$.

and

$$|a(x,t,s,\xi)| \leq \nu[h(x,t) + |\xi|^{p-1}], \text{ with } \nu > 0 \text{ and } h(x,t) \in L^{p'}(Q_T), \tag{2.2}$$

$$a(x,t,s,\xi)\xi \geq \alpha|\xi|^p, \text{ with } \alpha > 0, \tag{2.3}$$

$$(a(x,t,s,\xi) - a(x,t,s,\eta)(\xi - \eta) > 0, \text{ with } \xi \neq \eta, \tag{2.4}$$

$$|\phi(x,t,s)| \leq c(x,t)|s|^\gamma, \tag{2.5}$$

$$c(x,t) \in (L^\tau(Q_T))^N, \quad \tau = \frac{N+p}{p-1}, \tag{2.6}$$

$$\gamma = \frac{N+2}{N+p}(p-1) \tag{2.7}$$

for almost every $(x,t) \in Q_T$, for every $s \in \mathbb{R}$ and every $\xi,\ \eta \in \mathbb{R}^N$.

$$f \in L^1(Q_T), \tag{2.8}$$

$$u_0 \in L^1(\Omega) \text{ such that } b(u_0) \in L^1(\Omega). \tag{2.9}$$

Throughout the paper, T_k denotes the truncation function at height $k \geq 0$:

$$T_k(r) = max(-k, min(k,r))$$

3. MAIN RESULTS

In this section, we study the existence of renormalized solutions to problem (1.1).

Definition 3.1. A measurable function u is a renormalized solution to problem (1.1), if

$$b(u) \in L^{\infty}((0,T), L^1(\Omega)). \tag{3.1}$$

$$T_k(u) \in L^p((0,T), W_0^{1,p}(\Omega)), \text{ for any } k > 0, \tag{3.2}$$

$$\lim_{n\to+\infty} \int_{\{n\le|u|\le n+1\}} a(x,t,u,\nabla u)\nabla u \, dx\, dt = 0, \tag{3.3}$$

and if for every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support

$$\begin{aligned} &\frac{\partial B_S(u)}{\partial t} - div\Big(a(x,t,u,\nabla u)S'(u)\Big) + S^{''}(u)a(x,t,u,\nabla u)\nabla u \\ &+ div\Big(\phi(x,t,u)S'(u)\Big) - S''(u)\phi(x,t,u)\nabla u = fS'(u) \quad in \quad D^{'}(\Omega), \end{aligned} \tag{3.4}$$

and

$$B_S(u)(t=0) = B_S(u_0) \quad in \quad \Omega, \tag{3.5}$$

where $B_S(z) = \displaystyle\int_0^z b^{'}(s)S^{'}(s)ds$.

Equation (3.4) is formally obtained through multiplication of (1.1) by $S'(u)$. However while $a(x,t,u,\nabla u)$ and $\phi(x,t,u)$ does not in general make sense in (1.1), all the terms in (3.4) have a meaning in $D'(Q_T)$. Indeed, if M is such that $suppS^{'} \subset [-M,M]$, the following identifications are made in (3.4):

- $S'(u)a(x,t,u,\nabla u)$ identifies with $S'(u)a(x,t,T_M(u),\nabla T_M(u))$ a.e in Q_T.
- $S''(u)a(x,t,u,\nabla u)\nabla u$ identifies with $S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M(u)$ a.e. in Q_T.
- $S'(u)\phi(x,t,u)$ identifies with $S'(u)\phi(x,t,T_M(u))$ a.e. in Q_T.
- $S''(u)\phi(x,t,u)\nabla u$ identifies with $S''(u)\phi(x,t,T_M(u))\nabla T_M(u))$ a.e. in Q_T.

The above consideration shows that equation (3.4) hold in $D'(\Omega)$, $\frac{\partial B_S(u)}{\partial t}$ belongs to $L^1(Q) + L^{p'}(0,T,W^{-1,p'}(\Omega))$ and $B_S(u) \in L^p(0,T,W_0^{1,p}(\Omega)) \cap L^{\infty}(Q)$. It follows that $B_S(u)$ belongs to $C^0([0,T],L^1(\Omega))$ so the initial condition (3.5) makes sense.

Theorem 3.1. *Assume the assumption (H) hold, then problem (1.1) admits a renormalized solution u in the sense of Definition 3.1.*

Proof. The proof is divided into six steps.

Step 1: Approximate problem and a priori estimates.

For each $\epsilon > 0$, we define the following approximations

$$b_\epsilon(r) = T_{\frac{1}{\epsilon}}(b(r)) + \epsilon r. \quad \forall\, r \in \mathbb{R}, \tag{3.6}$$

$$a_\epsilon(x,t,s,\xi) = a(x,t,T_{\frac{1}{\epsilon}}(s),\xi).a.e \quad in \quad Q \quad \forall\, s\in\mathbb{R},\ \forall\,\xi\in\mathbb{R}^N, \tag{3.7}$$

$$\phi_\epsilon(x,t,r) = \phi(x,t,T_{\frac{1}{\epsilon}}(r)) \text{ a.e. } (x,t)\in Q_T,\ \forall\, r\in\mathbb{R}. \tag{3.8}$$

$$f_\epsilon \in L^{p^{'}}(Q_T) \text{ such that } f_\epsilon \to f \text{ strongly in } L^1(Q_T) \tag{3.9}$$

and

$$u_{0\epsilon} \in D(\Omega) \text{ such that } b_\epsilon(u_{0\epsilon}) \to b(u_0) \text{ strongly in } L^1(\Omega), \tag{3.10}$$

Let us consider the approximate problem :

$$(3.11)\qquad \begin{cases} \dfrac{\partial b_\epsilon(u_\epsilon)}{\partial t} - div(a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)) + div(\phi_\epsilon(x,t,u_\epsilon)) = f_\epsilon \quad in \quad Q_T \\ u_\epsilon(x,t) = 0 \quad on \quad \partial\Omega \times (0,T) \\ b_\epsilon(u_\epsilon(x,0)) = b_\epsilon(u_{0\epsilon}(x)) \quad in \quad \Omega. \end{cases}$$

As a consequence, proving existence of a weak solution $u_\epsilon \in L^p((0,T), W_0^{1,p}(\Omega))$ is an easy task (See [16]).

Step 2: The estimates derived in this step rely on standard techniques for problems of type (3.11).
Let $\tau_1 \in (0,T)$ and t fixed in $(0,\tau_1)$. Using in (3.11), $T_k(u_\epsilon)\chi_{(0,t)}$ as test function, we integrate between $(0,\tau_1)$, and by the condition (3.8) we have

$$\int_\Omega B_k^\epsilon(u_\epsilon(t))dx + \int_{Q_t} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon)\,dx\,ds$$

$$(3.12)\qquad \le \int_{Q_t} c(x,t)|u_\epsilon|^\gamma|\nabla T_k(u_\epsilon)|\,dx\,ds + \int_{Q_t} f_\epsilon T_k(u_\epsilon)\,dx\,ds + \int_\Omega B_k^\epsilon(u_{0\epsilon})dx,$$

where $B_k^\epsilon(r) = \displaystyle\int_0^r T_k(s)b_\epsilon'(s)ds$. Due to definition of B_k^ϵ we have:

$$(3.13)\qquad 0 \le \int_\Omega B_k^\epsilon(u_{0\epsilon})dx \le k\int_\Omega |b_\epsilon(u_{0\epsilon})|dx \le k||b(u_0)||_{L^1(\Omega)} \quad \forall k > 0$$

Using (3.12) and (2.3) we obtain:

$$\int_\Omega B_k^\epsilon(u_\epsilon(t))dx + \alpha\int_{Q_t} |\nabla T_k(u_\epsilon)|^p\,dx\,ds$$

$$(3.14)\qquad \le \int_{Q_t} c(x,t)|u_\epsilon|^\gamma|\nabla T_k(u_\epsilon)|\,ds\,dx + k(||b(u_0)||_{L^1(\Omega)} + ||f_\epsilon||_{L^1(Q)})$$

If we take the supremum for $t \in (0,\tau_1)$ and we define $M = sup(||f_\epsilon||_{L^1(Q)}) + ||b(u_0)||_{L^1(\Omega)}$, we deduce from that above inequality (3.12) and (3.13)

$$(3.15)\quad \frac{\beta}{2}\int_\Omega |T_k(u_\epsilon)|^2\,dx + \alpha\int_{Q_t} |\nabla T_k(u_\epsilon)|^p\,dx\,ds \le Mk + \int_{Q_t} c(x,t)|u_\epsilon|^\gamma|\nabla T_k(u_\epsilon)|\,dx\,ds.$$

By Gagliardo-Niremberg and Young inequalities we have:

$$\int_{Q_t} c(x,t)|u_\epsilon|^\gamma|\nabla T_k(u_\epsilon)|\,dx\,ds \le C\frac{\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})}sup_{t\in(0,\tau_1)}\int_\Omega |T_k(u_\epsilon)|^2\,dx$$

$$(3.16)\qquad + C\,\frac{N+2+\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})}\Big(\int_{Q_{\tau_1}} |\nabla T_k(u_\epsilon)|^p\,dx\,ds\Big)^{(\frac{1}{p}+\frac{N\gamma}{(N+2)p})\frac{N+2}{N+2-\gamma}}.$$

Since $\gamma = \frac{(N+2)}{N+p}(p-1)$ and by using (3.15) and (3.16), we obtain

$$\frac{\beta}{2}\int_\Omega |T_k(u_\epsilon)||^2\,dx + \alpha\int_{Q_t} |\nabla T_k(u_\epsilon)|^p\,dx\,ds$$

$$\le Mk + C\frac{\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})}sup_{t\in(0,\tau_1)}\int_\Omega |T_k(u_\epsilon)|^2\,dx$$

$$+C\frac{N+2+\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})}\int_{Q_{\tau_1}} |\nabla T_k(u_\epsilon)|^p\,dx\,ds$$

Which is equivalent to

$$\Big(\frac{\beta}{2}-C\frac{\gamma}{N+2}||c(x,t)||_{L^{\tau}(Q_{\tau_1})}\Big)sup_{t\in(0,\tau_1)}\int_{\Omega}|T_k(u_\epsilon)|^2\,dx+\alpha\int_{Q_{\tau_1}}|\nabla T_k(u_\epsilon)|^p\,dx\,ds$$
$$-C\frac{N+2+\gamma}{N+2}||c(x,t)||_{L^{\tau}(Q_{\tau_1})}\int_{Q_{\tau_1}}|\nabla T_k(u_\epsilon)|^p\,dx\,ds\le Mk$$

If we choose τ_1 such that

(3.17) $$\Big(\frac{\beta}{2}-C\frac{\gamma}{N+2}||c(x,t)||_{L^{\tau}(Q_{\tau_1})}\Big)\ge 0,$$

and

(3.18) $$\Big(\alpha-C\frac{N+2+\gamma}{N+2}||c(x,t)||_{L^{\tau}(Q_{\tau_1})}\Big)\ge 0,$$

then, let us denote by C the minimum between (3.17) and (3.18), we obtain

(3.19) $$sup_{t\in(0,\tau_1)}\int_{\Omega}|T_k(u_\epsilon)|^2\,dx+\int_{Q_{\tau_1}}|\nabla T_k(u_\epsilon)|^p\,dx\,dt\le CMk$$

Then, by (3.19) and lemma 3.1, we conclude that $T_k(u_\epsilon)$ is bounded in $L^p(0,T,W_0^{1,p}(\Omega))$ independently of ϵ and for any $k\ge 0$, so there exists a subsequence still denoted by u_ϵ such that

(3.20) $$T_k(u_\epsilon)\rightharpoonup\sigma_k\quad in\quad L^p(0,T,W_0^{1,p}(\Omega))$$

We turn now to prove the almost every convergence of u_ϵ and $b_\epsilon(u_\epsilon)$. Let $g_k\in C^2(\mathbb{R})$ such that $g_k(s)=s$ for $|s|\le\frac{k}{2}$ and $g_k(s)=k$ for $|s|\ge k$. Pointwise multiplication of the approximate equation (3.11) by $g_k'(b_\epsilon(u_\epsilon))$ leads to

(3.21) $$\frac{\partial g_k(b_\epsilon(u_\epsilon))}{\partial t}-div\Big(a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)g_k'(b_\epsilon(u_\epsilon))\Big)$$
$$+a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)g_k''(b_\epsilon(u_\epsilon))b_\epsilon'(u_\epsilon)\nabla u_\epsilon+div\Big(\phi_\epsilon(x,t,u_\epsilon)g_k'(b_\epsilon(u_\epsilon)\Big)$$
$$-g_k''(b_\epsilon(u_\epsilon))b_\epsilon'(u_\epsilon)\phi_\epsilon(x,t,u_\epsilon)\nabla u_\epsilon=f_\epsilon g_k'(b_\epsilon(u_\epsilon))\text{ in }D'(\Omega)$$

Now each term in (3.21) is taking into account because of (2.2), (3.7)and $T_k(u_\epsilon)$ is bounded in $L^p(0,T,w_0^{1,p}(\Omega))$, we deduce that:

$$-div\Big(a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)g_k^{'}(b_\epsilon(u_\epsilon))\Big)+a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)g_k^{''}(b_\epsilon(u_\epsilon))b_\epsilon^{'}(u_\epsilon)\nabla u_\epsilon+f_\epsilon g_k^{'}(b_\epsilon(u_\epsilon))$$

is bounded in $L^1(Q_T)+L^{p^{'}}(0,T,W^{-1,p^{'}}(\Omega))$ independently of ϵ. Due to definition of b and b_ϵ, we have $\{|b_\epsilon(u_\epsilon)|\le k\}\subset\{|u_\epsilon|\le k^*\}$ where k^* is a constant independent of ϵ. As a first consequence we have:

$$Dg_k(b_\epsilon(u_\epsilon))=g_k^{'}(b_\epsilon(u_\epsilon))b_\epsilon^{'}(T_{k^*}(u_\epsilon))DT_{k^*}(u_\epsilon)\quad\text{a.e in }Q$$

as soon as $k^*<\frac{1}{\epsilon}$. Secondly the following estimate hold true:

$$||g_k^{'}(b_\epsilon(u_\epsilon))b_\epsilon^{'}(T_{k^*}(u_\epsilon))||_{L^\infty(Q)}\le||g_k^{'}||_{L^\infty(Q)}(\max_{|r|\le k^*}(b'(r)+1)).$$

As a consequence, we obtain:

(3.22) $$g_k(b_\epsilon(u_\epsilon))\text{ is bounded in }L^p(0,T,W_0^{1,p}(\Omega)).$$

Since $supp(g_k')$ and $supp(g_k'')$ are both included in [-k,k] by (3.8) it follows that for all $\epsilon < \frac{1}{k}$ we have

$$|\int_{Q_T} \phi_\epsilon(x,t,u_\epsilon)^{p'} g_k'(b_\epsilon(u_\epsilon))^{p'}\,dx\,dt| \leq \int_{Q_T} c(x,t)^{p'} |T_{\frac{1}{\epsilon}}(u_\epsilon)|^{p'\gamma} |g_k'(b_\epsilon(u_\epsilon))|^{p'}\,dx\,dt$$

$$= \int_{\{|u_\epsilon| \leq k^*\}} c(x,t)^{p'} |T_{k^*}(u_\epsilon)|^{p'\gamma} |g_k'(b_\epsilon(u_\epsilon))|^{p'}\,dx\,dt$$

Furthermore, by Hölder and Gagliardo-Niremberg inequality, it results

$$\int_{\{|u_\epsilon| \leq k^*\}} c(x,t)^{p'} |T_{k^*}(u_\epsilon)|^{p'\gamma} |g_k'(b_\epsilon(u_\epsilon))|^{p'}\,dx\,dt$$

$$\leq \|g_k'\|_{L^\infty(\mathbb{R})} ||c(x,t)||^{p'}_{L^\tau(Q_T)} [sup_{t\in(0,T)}(\int_\Omega |T_{k^*}(u_\epsilon)|^2\,dx)^{\frac{p}{N}} + \int_{Q_T} |\nabla T_{k^*}(u_\epsilon)|^p\,dx\,dt] \leq c_{k^*}$$

where c_{k^*} is a constant independently of ϵ which will vary from line to line.
In the same by (3.8) we have:

$$(3.23)\qquad |\int_{Q_T} \phi_\epsilon(x,t,u_\epsilon)^{p'} (g_k''(b_\epsilon(u_\epsilon) b_\epsilon'(u_\epsilon)\nabla u_\epsilon)^{p'}\,dx\,dt|$$

$$\leq \int\int_{Q_T} (g_k''(b_\epsilon(u_\epsilon))^{p'} b_\epsilon'(u_\epsilon)^{p'} |c(x,t)|^{p'} |T_{\frac{1}{\epsilon}}(u_\epsilon)|^{p'} |\nabla u_\epsilon|^{p'}\,dx\,dt.$$

Furthermore, by Hölder and Gagliardo-Niremberg inequality,we obtain for $\epsilon < \frac{1}{k^*}$:

$$\int_{Q_T} (g_k''(b_\epsilon(u_\epsilon))^{p'} b_\epsilon'(u_\epsilon)^{p'} |c(x,t)|^{p'} |T_{\frac{1}{\epsilon}}(u_\epsilon)|^{p'} |\nabla u_\epsilon|^{p'}\,dx\,dt$$

$$= \int_{Q_T} (g_k''(b_\epsilon(u_\epsilon))^{p'} b_\epsilon'(u_\epsilon)^{p'} |c(x,t)|^{p'} |T_k(u_\epsilon)|^{p'} |\nabla T_k(u_\epsilon)|^{p'}\,dx\,dt$$

$$\leq \|g_k''\|_{L^\infty(\mathbb{R})} \times \sup_{|r|\leq k^*} |b'(r)| \int_{Q_T} |c(x,t)|^{p'} |T_{k^*}(u_\epsilon)|^{p'} |\nabla T_k(u_\epsilon)|^{p'}\,dx\,dt \leq c_{k^*}$$

We conclude by (3.21) that

(3.24) $$\frac{\partial g_k(b_\epsilon(u_\epsilon))}{\partial t} \text{ is bounded in } L^1(Q) + L^{p'}(0,T,W^{-1,p'}(\Omega)).$$

Arguing again as in [8], estimates (3.22) and (3.24) imply that, for a subsequence, still indexed by ϵ,

(3.25) $$u_\epsilon \longrightarrow u \text{ a.e. } Q_T,$$

where u is a measurable function defined on Q_T.

Let us prove that $b(u)$ belongs to $L^\infty((0,T),L^1(\Omega))$. We take $T_k(b_\epsilon(u_\epsilon))$ as test function in (3.11), by (3.8) we have

$$(3.26)\qquad \int_\Omega B_k^\epsilon(u_\epsilon)dx + \int_{Q_T} a_\epsilon(x,t,u,\nabla u_\epsilon)\nabla T_k(b_\epsilon(u_\epsilon))\,dx\,dt$$

$$\leq \int_{Q_T} |c(x,t)||T_{\frac{1}{\epsilon}}(u_\epsilon)|^\gamma |\nabla T_k(b_\epsilon(u_\epsilon))|\,dx\,dt + k\,(||f_\epsilon||_{L^1(Q_T)}) + ||b(u_0)||_{L^1(\Omega)}).$$

with $B_k(r) = \int_0^{b(r)} T_k(s)ds$. On the other hand, we have

$$(3.27)\qquad \int_{Q_T} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(b_\epsilon(u_\epsilon))\,dx\,ds$$

$$= \int_{\{|b_\epsilon(u_\epsilon)|\le k\}} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k'(b_\epsilon(u_\epsilon))b_\epsilon'(u_\epsilon)\nabla u_\epsilon\,dx\,ds \ge 0.$$

Since $b'(s)\ge\beta$, then for $0<\epsilon<\frac{1}{k}$ and for almost $t\in(0,T)$, we have

$$\int_{Q_T} |c(x,t)||T_{\frac{1}{\epsilon}}(u_\epsilon)|^\gamma|\nabla T_k(b_\epsilon(u_\epsilon))|\,dx\,dt \tag{3.28}$$

$$\le \max_{|s|\le\frac{k}{\beta}}(b'(s))||c(x,t)||_{L^\tau(Q_T)}\Big(\sup_{t\in(0,T)}(\int_\Omega |T_{\frac{k}{\beta}}(u_\epsilon)|^2\,dx)^{\frac{p-1}{N+p}}\times||\nabla T_{\frac{k}{\beta}}(u_\epsilon)||_{L^p(Q_T)}^{\frac{p(N+1)}{N+p}}\Big)\le c_k.$$

Using (3.13), (3.28) and (3.26) in (3.27), we have

$$\int_\Omega B_k^\epsilon(u_\epsilon(t)) \le c_k + k\Big(||f||_{L^1(Q_T)}+||b(u_0)||_{L^1(\Omega)}\Big)$$

Passing to limit-inf as $\epsilon\to0$, we obtain that:

$$\int_\Omega B_k(u(t))\,dx \le c_k + k\Big(||f_\epsilon||_{L^1(Q_T)})+||b(u_0)||_{L^1(\Omega)}\Big) \text{ for almost } t\in(0,T).$$

Due to definition of B_k, we have

$$k\int_\Omega |b(u(x,t))|dx \le \int_\Omega B_k(u)(t)dx + \frac{3}{2}k^2mes(\Omega)$$

$$\le k(||f||_{L^1(\Omega)}+||b(u_0)||_{L^1(\Omega)}) + c_k + \frac{3}{2}k^2mes(\Omega).$$

then we conclude $b(u)\in L^\infty((0,T),L^1(\Omega))$.

Lemma 3.2. *The subsequence of u_ϵ defined in Step 1 satisfies*

$$\lim_{n\to+\infty}\limsup_{\epsilon\to0}\int_{\{n\le|u_\epsilon|\le n+1\}} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon\,dx\,dt = 0.$$

Proof. Using $\psi_n(u_\epsilon)\equiv T_{n+1}(u_\epsilon)-T_n(u_\epsilon)$ as a test function in (3.11), and by (3.8) we get

$$\int_0^T <\frac{\partial b_\epsilon(u_\epsilon)}{\partial t},\psi_n(u_\epsilon)>\,dt + \int_{Q_t} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla\psi_n(u_\epsilon)\,dx\,dt \tag{3.29}$$

$$\le \int_\Omega c(x,t)|T_{\frac{1}{\epsilon}}(u_\epsilon)|^\gamma|\nabla\psi_n(u_\epsilon)|\,dx\,dt + \int_{Q_T} f_\epsilon\psi_n(u_\epsilon)\,dx\,dt$$

hence

$$\int_\Omega B_n(u_\epsilon)(T)dx + \int_{Q_t} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla\psi_n(u_\epsilon)$$

$$\le \int_{Q_T} c(x,t)|T_{\frac{1}{\epsilon}}(u_\epsilon)|^\gamma|\nabla\psi_n(u_\epsilon)|\,dx\,dt + \int_\Omega B_n(u_0)_\epsilon dx + \int_{Q_T} f_\epsilon\psi_n(u_\epsilon)\,dx\,dt,$$

where $B_n(r)=\displaystyle\int_0^r b_\epsilon'(s)\psi_n(s)ds$. Since $\psi_n\ge0$ and $B_n(u_\epsilon)(T)\ge0$, then for every $\epsilon<\frac{1}{n+1}$, we have

$$a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla\psi_n(u_\epsilon) = a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla\psi_n(u_\epsilon) \text{ a.e. in } Q$$

As a consequence

$$\int_{Q_T} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla\psi_n(u_\epsilon)\,dx\,dt \le \int_{Q_T} c(x,t)|T_{\frac{1}{\epsilon}}(u_\epsilon)|^\gamma|\nabla\psi_n(u_\epsilon)|\,dx\,dt \tag{3.30}$$

$$+\int_\Omega B_n(u_{0\epsilon})dx + \int_{Q_T} f_\epsilon\psi_n(u_\epsilon)\,dx\,dt.$$

Proceeding as in ([6], [8]) it can be deduced from (3.30) that

$$\psi_n(u_\epsilon) \rightharpoonup \psi_n(u) \;\; weakly \;\; in \;\; L^p(0,T,W_0^{1,p}(\Omega)). \tag{3.31}$$

We have $\nabla\psi_n(u_\epsilon) = \chi_{\{n\le|u_\epsilon|\le n+1\}}\nabla u_\epsilon$ a.e in Q_T, by Young inequality and (2.3), we obtain

$$\frac{\alpha}{p'}\int_{Q_T}|\nabla\psi_n(u_\epsilon)|^p\,dx\,dt \le \int_{Q_T} f^\epsilon\psi_n(u_\epsilon)\,dx\,dt + \int_\Omega B_n(u_{0\epsilon})\,dx$$

$$+\frac{\alpha^{\frac{-p'}{p}}}{p'}\int_{\{n\le|u_\epsilon|\le n+1\}} c(x,t)|T_{n+1}(u_\epsilon)|^\gamma\,dx\,dt. \tag{3.32}$$

Using the weakly convergence of $\psi_n(u_\epsilon)$, the pointwise convergence of u_ϵ and the strongly convergence in L^1 of f^ϵ and $B_n(u_{0\epsilon})$, it follows that

$$\frac{\alpha}{p'}\int_{Q_T}|\nabla\psi_n(u)|^p\,dx\,dt \le \int_{Q_T} f\psi_n(u)\,dx\,dt + \int_\Omega B_n(u_0)\,dx$$

$$+\frac{\alpha^{\frac{-p'}{p}}}{p'}\int_{\{n\le|u|\le n+1\}} c(x,t)|T_{n+1}(u)|^\gamma\,dx\,dt. \tag{3.33}$$

The last inequality, together with the assumptions (2.8), (2.9), shows that $\psi_n(u)$ is bounded in $L^p(0,T,W_0^{1,p}(\Omega))$ independently of n. Thanks to the pointwise convergence of $\psi_n(u)$ to 0 and weakly in $L^p(0,T,W_0^{1,p}(\Omega))$ as $n\to+\infty$, we deduce that

$$\lim_{n\to+\infty}\int_{Q_T} f\psi_n\,dx\,dt = 0,$$

and

$$\lim_{n\to+\infty}\int_{n\le|u|\le n+1}(c(x,t))^{p'}|u|^{\gamma p'}\,dx\,dt = 0.$$

Moreover

$$|B_n(u_0)| \le c\int_0^{u_0}\psi_n(s)ds \to 0 \quad a.e \quad as \quad n\to+\infty,$$

and $|B_n(u_0)| \le |b(u_0)|$ a.e. in Ω, since $b(u_0)\in L^1(\Omega)$, by Lebesgue's convergence theorem we obtain

$$\lim_{n\to+\infty}\int_\Omega B_n(u_0)\,dx = 0.$$

Therefore

$$\lim_{n\to+\infty}\int_{Q_T}|\nabla\psi_n(u)|^p\,dx\,dt = 0,$$

then

$$\psi_n(u) \to 0 \text{ strongly in } L^p(0,T,W_0^{1,p}(\Omega)). \tag{3.34}$$

Finally, passing to the limit in (3.30) as $n\to+\infty$, we get

(3.35)

∎

Step 4: In this step we introduce a time regularization of the $T_k(u)$ for $k > 0$ in order to perform the monotonicity method. This kind regularization has been introduced at the first time by R. Landes in [15]. Let v_0^μ be a sequence of function in $L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\|v_0^\mu\|_{L^\infty(\Omega)} \le k$ for all $\mu > 0$ and v_0^μ converges to $T_k(u_0)$ a.e. in Ω and $\frac{1}{\mu}\|v_0^\mu\|_{L^p(\Omega)}$ converges to 0. For $k \ge 0$ and $\mu > 0$, let us consider the unique solution $(T_k(u))_\mu \in L^\infty(Q) \cap L^p(0,T : W_0^{1,p}(\Omega))$ of the monotone problem:

$$\frac{\partial (T_k(u))_\mu}{\partial t} + \mu((T_k(u))_\mu - T_k(u)) = 0 \text{ in } D'(\Omega),$$

$$(T_k(u))_\mu(t=0) = \nu_0^\mu \text{ in } \Omega.$$

Remark that $(T_k(u))_\mu \to T_k(u)$ a.e. in Q_T, weakly-$*$ in $L^\infty(Q)$ and strongly in $L^p((0,T), W_0^p(\Omega))$ as $\mu \to +\infty$

$$||(T_k(u))_\mu||_{L^\infty(Q)} \le max(||(T_k(u))||_{L^\infty(Q)}, ||\nu_0^\mu||_{L^\infty(\Omega)}) \le k, \ \ \forall \ \mu > 0 \ \ ,\forall \ k > 0$$

Lemma 3.3. *(see H. Redwane* [18]*) Let $k \ge 0$ be fixed. Let S be an increasing $C^\infty(\mathbb{R})-$function such that $S(r) = r$ for $|r| \le k$, and $suppS'$ is compact. Then*

$$\liminf_{\mu\to+\infty} \lim_{\epsilon\to 0} \int_0^T \int_0^t < \frac{\partial b_\epsilon(u_\epsilon)}{\partial t}, S'(u_\epsilon)(T_k(u_\epsilon) - (T_k(u))_\mu) > \ge 0.$$

where $< .,. >$ denotes the duality pairing between $L^1(\Omega) + W^{-1,p'}(\Omega)$ and $L^\infty(\Omega) \cap W^{1,p}(\Omega)$.

Step 5: We prove the following lemma which is the critical point in the development of the monotonicity method.

Lemma 3.4. *The subsequence of u_ϵ satisfies for any $k \ge 0$*

$$limsup_{\epsilon\to 0} \int_0^T \int_0^t \int_\Omega a(x,t,u_\epsilon, \nabla T_k(u_\epsilon))\nabla T_k(u_\epsilon) \le \int_0^T \int_0^t \int_\Omega \sigma_k \nabla T_k(u).$$

Proof. Let S_n be a sequence of increasing C^∞-function such that

$$S_n(r) = r \text{ for } |r| \le n, \ supp(S'_n) \subset [-(n+1),(n+1)] \text{ and } \|S''_n\|_{L^\infty(\mathbb{R})} \le 1 \text{ for any } n \ge 1.$$

We use the sequence $(T_k(u))_\mu$ of approximation of $T_k(u)$, and plug the test function $S'_n(u_\epsilon)(T_k(u_\epsilon) - (T_k(u))_\mu)$ for $n > 0$ and $\mu > 0$. For fixed $k \ge 0$, let $W_\mu^\epsilon = T_k(u_\epsilon) - (T_k(u))_\mu$ we obtain upon integration over $(0,t)$ and then over $(0,T)$:

$$\begin{aligned} &\int_0^T \int_0^t < \frac{\partial b_\epsilon(u_\epsilon)}{\partial t}, S'_n(u_\epsilon) W_\mu^\epsilon > \,ds\,dt + \int_0^T \int_0^t \int_\Omega a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon) S'_n(u_\epsilon) \nabla W_\mu^\epsilon \,dx\,ds\,dt \\ &+ \int_0^T \int_0^t \int_\Omega a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon) S''_n(u_\epsilon)\nabla u_\epsilon \nabla W_\mu^\epsilon \,dx\,ds\,dt \\ &- \int_0^T \int_0^t \int_\Omega \phi_\epsilon(x,t,u_\epsilon) S'_n(u_\epsilon)\nabla W_\mu^\epsilon \,dx\,ds\,dt \\ &- \int_0^T \int_0^t \int_\Omega S''_n(u_\epsilon)\phi_\epsilon(x,t,u_\epsilon)\nabla u_\epsilon \nabla W_\mu^\epsilon \,dx\,ds\,dt = \int_0^T \int_0^t \int_\Omega f_\epsilon S'_n(u_\epsilon) W_\mu^\epsilon \,dx\,ds\,dt. \end{aligned} \tag{3.36}$$

Now we pass to the limit in (3.36) as $\epsilon \to 0$, $\mu \to +\infty$ and then $n \to +\infty$ for k real number fixed. In order to perform this task we prove below the following results for any fixed $k \ge 0$

$$\liminf_{\mu\to+\infty} \lim_{\epsilon\to 0} \int_0^T \int_0^t < \frac{\partial b_\epsilon(u_\epsilon)}{\partial t}, W_\mu^\epsilon > \,ds\,dt \ge 0 \qquad \text{for any } n \ge k, \tag{3.37}$$

$$\lim_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega \phi_\epsilon(x,t,u_\epsilon)S_n'(u_\epsilon)\nabla W_\mu^\epsilon\,dx\,ds\,dt = 0 \quad \text{for any } n\geq 1, \tag{3.38}$$

$$\lim_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega \phi_\epsilon(x,t,u_\epsilon)\nabla u_\epsilon\nabla W_\mu^\epsilon\,dx\,ds\,dt = 0 \qquad \text{for any } n\geq 1, \tag{3.39}$$

$$\lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S_n''(u_\epsilon)\nabla u_\epsilon\nabla W_\mu^\epsilon\,dx\,ds\,dt = 0, \tag{3.40}$$

$$\lim_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega f_\epsilon S_n'(u_\epsilon)W_\mu^\epsilon \quad dx\,ds\,dt = 0. \tag{3.41}$$

Proof of (3.37): The function S_n belongs $C^\infty(\mathbb{R})$ and is increasing. we have $n \geq k$, $S_n(r) = r$ for $|r| \leq k$ while $suppS_n'$ is compact. In view of the definition of W_μ^ϵ and lemma 3.3 applies with $S = S_n$ for fixed $n \geq k$. As a consequence (3.37) holds true.
Proof of (3.38): Let us recall the main properties of W_μ^ϵ. For fixed $\mu > 0$: W_μ^ϵ converges to $T_k(u) - (T_k(u))_\mu$ weakly in $L^p(0,T,W_0^{1,p}(\Omega))$ as $\epsilon \to 0$. Remark that

$$||W_\mu^\epsilon||_{L^\infty(Q_T)} \leq 2k \qquad \text{for any } \epsilon > 0,\ \mu > 0, \tag{3.42}$$

then we e deduce that

$$W_\mu^\epsilon \rightharpoonup T_k(u) - (T_k(u))_\mu \quad \text{a.e in } Q_T \text{ and } L^\infty(Q_T) \tag{3.43}$$

weakly-$*$ when $\epsilon \to 0$. one had $suppS_n'' \subset [-(n+1),-n] \cup [n, n+1]$ for any fixed $n \geq 1$ and $0 < \epsilon < \frac{1}{n+1}$.

$$\phi_\epsilon(x,t,u_\epsilon)S_n'(u_\epsilon)\nabla W_\mu^\epsilon = \phi_\epsilon(x,t,T_{n+1}(u_\epsilon))S_n'(u_\epsilon)\nabla W_\mu^\epsilon \quad \text{a.e. in } Q_T$$

since $suppS' \subset [-(n+1), n+1]$, on the other hand

$$\phi_\epsilon(x,t,T_{n+1}(u_\epsilon))S_n'(u_\epsilon) \to \phi(x,t,T_{n+1}(u))S_n'(u) \quad \text{a.e. in } Q_T$$

and

$$|\phi_\epsilon(x,t,T_{n+1}(u_\epsilon))S_n'(u_\epsilon)| \leq c(x,t)(n+1)^\gamma \quad \text{for } n \geq 1$$

by (3.43) and strongly convergence of $T_k(u_\epsilon)_\mu$ in $L^p(0,T,W_0^{1,p}(\Omega))$ we obtain (3.38).
Proof of (3.39): For any fixed $n \geq 1$ and $0 < \epsilon < \frac{1}{n+1}$.

$$\phi_\epsilon(x,t,u_\epsilon)S_n''(u_\epsilon)\nabla u_\epsilon W_\mu^\epsilon = \phi_\epsilon(x,t,T_{n+1}(u_\epsilon))S_n''(u_\epsilon)\nabla T_{n+1}(u_\epsilon)W_\mu^\epsilon \quad \text{a.e. in} Q_T$$

as in the previous step it is possible to pass to the limit for $\epsilon \to 0$ since by (3.42) and (3.43)

$$\phi_\epsilon(x,t,T_{n+1}(u_\epsilon))S_n''(u_\epsilon)W_\mu^\epsilon \to \phi(x,t,T_{n+1}(u))S_n''(u)W_\mu \quad \text{a.e. in } Q_T.$$

Since $|\phi(x,t,T_{n+1}(u))S_n''(u)W_\mu| \leq 2k|c(x,t)|(n+1)^\gamma$ a.e. in Q_T and $(T_k(u))_\mu$ converges to 0 in $L^p(0,T;W_0^{1,p}(\Omega))$, we obtain (3.39).
Proof of (3.40): In view of the definition of S_n we have $suppS' \subset [-(n+1),-n] \cup [n, n+1]$ for any $n \geq 1$, as a consequence

$$\Big|\int_0^T\int_0^t\int_\Omega a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S_n''(u_\epsilon)W_\mu^\epsilon\,dx\,ds\,dt\Big|$$

$$\leq T\|S_n''(u_\epsilon)\|_{L^\infty(\mathbb{R})}\|W_\mu^\epsilon\|_{L^\infty(Q_T)}\int_{n\leq|u_\epsilon|\leq n+1} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon\,dx\,ds\,dt$$

for any $n \geq 1$, any $0 < \epsilon < \frac{1}{n+1}$ any $\mu > 0$. By (3.35) it is possible to establish (3.40).
Proof of (3.41): By (3.9), the pointwise convergence of u_ϵ and W_μ^ϵ and its boundness it is possible to pass the limit for $\epsilon \to 0$ for any $\mu > 0$ and any $n \geq 1$

$$\lim_{\epsilon \to 0} \int_0^T \int_0^t \int_\Omega f_\epsilon S_n'(u)(T_k(u) - (T_k(u))_\mu)\, dx\, ds\, dt = \int_0^T \int_0^t \int_\Omega f S_n'(u)(T_k(u) - (T_k(u))_\mu).$$

Now for fixed $n \geq 1$, using that $||(T_k(u))_\mu||_{L^\infty(Q)} \leq max(||(T_k(u))||_{L^\infty(Q)}, ||\nu_0^\mu||_{L^\infty(\Omega)}) \leq k,\ \forall \mu > 0,\ \forall k > 0$ (see[15]), it possible to pass to the limit as μ tends to $+\infty$ in the above inequality.
Now we turn back to the proof of lemma 3.4. Due to (3.37)-(3.41) we can to pass to the limit-sup when μ tends to $+\infty$ and to the limit as n tends to $+\infty$ in (3.36). using the definition of W_μ^ϵ we deduce that for any $k \geq 0$

$$\lim_{n \to +\infty} \limsup_{\mu \to +\infty} \limsup_{\epsilon \to 0} \int_0^T \int_0^t \int_\Omega S_n'(u_\epsilon) a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)(\nabla T_k(u_\epsilon) - \nabla(T_k(u)_\mu)\, dx\, ds\, dt \leq 0.$$

Since $S_n'(u_\epsilon) a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon) = a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon)$ for $k \leq \frac{1}{\epsilon}$ and $k \leq n$, using the properties of S_n' the above inequality implies that for $k \leq n$:

$$\limsup_{\epsilon \to 0} \int_0^T \int_0^t \int_\Omega a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)(\nabla T_k(u_\epsilon))\, dx\, ds\, dt \tag{3.44}$$
$$\leq \lim_{n \to +\infty} \limsup_{\mu \to +\infty} \limsup_{\epsilon \to 0} \int_0^T \int_0^t \int_\Omega S_n'(u_\epsilon) a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla(T_k(u)_\mu\, dx\, ds\, dt$$

On the other hand, for $\epsilon < \frac{1}{n+1}$

$$S_n'(u_\epsilon) a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon) = S_n'(u_\epsilon) a(x,t,T_{n+1}(u_\epsilon),\nabla T_{n+1}(u_\epsilon)) \quad \text{a.e. in } Q_T.$$

Furthermore we have

$$a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon) \rightharpoonup \sigma_k \quad \text{weakly in } (L^{p'}(Q_T))^N \tag{3.45}$$

it follows that for a fixed $n \geq 1$

$$S_n'(u_\epsilon) a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon) \to S_n'(u_\epsilon)\sigma_{n+1} \quad weakly \quad in \quad L^{p'}(Q_T)$$

when ϵ tends to 0. Finally, using the strong convergence of $(T_k(u)_\mu)$ to $T_k(u)$ in $L^p(0,T,W_0^{1,p}(\Omega))$ as μ tends to $+\infty$, we get

$$\lim_{\mu \to +\infty} \lim_{\epsilon \to 0} \int_0^T \int_0^t \int_\Omega S_n'(u_\epsilon) a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla(T_k(u_\epsilon)_\mu\, dx\, ds\, dt \tag{3.46}$$
$$= \int_0^T \int_0^t \int_\Omega S_n'(u_\epsilon)\sigma_{n+1}\nabla T_k(u)\, dx\, ds\, dt$$

as soon as $k \leq n$. Now for $k \leq n$ we have

$$a(x,t,T_{n+1}(u_\epsilon),\nabla T_{n+1}(u_\epsilon))\chi_{\{|u_\epsilon| \leq k\}} = a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))\chi_{\{|u_\epsilon| \leq k\}} \quad \text{a.e. in } Q_T$$

which implies that, by (3.25), (3.45), and by passing to the limit when ϵ tends to 0,

$$\sigma_{n+1}\chi_{|u| \leq k} = \sigma_k \chi_{\{|u| \leq k\}} \quad \text{a.e. in } Q_T - \{|u| = k\} \quad \text{for } k \leq n \tag{3.47}$$

Finally, by (3.47) and (3.45) we have for $k \leq n$: $\sigma_{n+1}\nabla T_k(u) = \sigma_k \nabla T_k(u)$ a.e. in Q_T. Recalling (3.44), (3.46) the proof of the lemma is complete. ∎

Step 6: In this step we prove that the weak limit σ_k of $a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))$ can be identified with $a(x,t,T_k(u),\nabla T_k(u))$. In order to prove this result we recall the following lemma:

Lemma 3.5. *the subsequence of u_ϵ defined in Step 1 satisfies for any $k \geq 0$*

(3.48)
$$\lim_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega \Big(a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))-a(x,t,T_k(u_\epsilon),\nabla T_k(u))\Big)\Big(\nabla T_k(u_\epsilon)-\nabla T_k(u)\Big)=0$$

Proof. Using (2.3) we have

(3.49)
$$\lim_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega \Big(a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))-a(x,t,T_k(u_\epsilon),\nabla T_k(u))\Big)\Big(\nabla T_k(u_\epsilon)-\nabla T_k(u)\Big)\geq 0.$$

Furthermore, by (2.2), (3.25) we have

$$a(x,t,T_k(u_\epsilon),\nabla T_k(u)))\to a(x,t,T_k(u),\nabla T_k(u)))\quad \text{a.e. in } Q_T,$$

and

$$|a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon)))|\leq \nu[h(x,t)+|\nabla T_k(u_\epsilon)|^{p-1}]\quad \text{a.e. in } Q_T,$$

uniformly with respect to ϵ. As a consequence

(3.50) $$a(x,t,T_k(u_\epsilon),\nabla T_k(u)))\to a(x,t,T_k(u),\nabla T_k(u)))\text{ strongly in }(L^{p'}(Q_T))^N.$$

Finally, using (3.25), (3.45) and (3.50) make it possible to pass to the $limit-sup$ as ϵ tends to 0 in (3.49) and we have (3.48). ∎

Lemma 3.6. *For fixed $k\geq 0$, we have*

(3.51) $$\sigma_k = a(x,t,T_k(u),\nabla T_k(u)))\quad \textit{a.e. in } Q_T,$$

and as ϵ tends to 0

(3.52) $$a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))\nabla T_k(u_\epsilon)\rightharpoonup a(x,t,T_k(u),\nabla T_k(u)))\nabla T_k(u)$$

weakly in $L^1(Q_T)$.

Proof. We observe that for for any $k>0$, any $0<\epsilon<\frac{1}{k}$ and any $\xi\in\mathbb{R}^N$:

$$a_\epsilon(x,t,T_k(u_\epsilon),\xi)=a(x,t,T_k(u_\epsilon),\xi)=a_{\frac{1}{k}}(x,t,T_k(u_\epsilon),\xi)\quad \text{a.e. in } Q_T.$$

Since

(3.53) $$T_k(u_\epsilon)\rightharpoonup T_k(u)\quad \text{weakly in } L^p((0,T),W_0^p(\Omega)),$$

and by (3.48) we obtain

(3.54)
$$\lim_{\epsilon\to 0}\int_0^T\int_0^t\int_\Omega a_{\frac{1}{k}}(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))\nabla T_k(u_\epsilon)\,dx\,ds\,dt=\int_0^T\int_0^t\int_\Omega \sigma_k\nabla T_k(u)\,dx\,ds\,dt.$$

Since, for fixed $k>0$, the function $a_{\frac{1}{k}}(x,t,s,\xi)$ is continuous and bounded with respect to s, the usual Minty's argument applies in view of (3.53), (3.45) and (3.54). It follows that (3.51) holds true. In order to prove (3.54), by (2.3), (3.48) and proceeding as in [5, 8] it's easy to show (3.52). ∎

Taking the limit as ϵ tends to 0 in (3.35) and using (3.52) show that u satisfies (3.3). Our aim is to prove that u satisfies (3.4) and (3.5). Now we want to prove that u satisfies the equation (3.4).

Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that $suppS'\subset[-k,k]$ where k is a real positive number. Pointwise multiplication of the approximate equation (3.11) by $S'(u_\epsilon)$ leads to

(3.55) $$\frac{\partial B_S^\epsilon(u_\epsilon)}{\partial t}-div\Big(a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S'(u_\epsilon)\Big)+S''(u_\epsilon)a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon$$

$$+ div\Big(\phi_\epsilon(x,t,u_\epsilon)S'(u_\epsilon)\Big) - S''(u_\epsilon)\phi_\epsilon(x,t,u_\epsilon)\nabla u_\epsilon = f^\epsilon S'(u_\epsilon) \quad \text{in } D'(\Omega).$$

In what follows we pass to the limit as ϵ tends to O in each term of (3.55).

Since u_ϵ converges to u a.e. in Q_T implies that $B_S^\epsilon(u_\epsilon)$ converge to $B_S(u)$ a.e. in Q_T and $L^\infty(Q_T)$ weak-$*$, Then $\dfrac{\partial B_S^\epsilon}{\partial t}$ converges to $\dfrac{\partial B_S}{\partial t}$ in $D'(\Omega)$. We observe that the term $a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S'(u_\epsilon)$ can be identified with $a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))S'(u_\epsilon)$ for $\epsilon \leq \frac{1}{k}$, so using the pointwise convergence of $u_\epsilon \to u$ in Q_T, the weakly convergence of $T_k(u_\epsilon) \rightharpoonup T_k(u)$ in $L^p((0,T),W_0^p(\Omega))$, we get

$$a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S'(u_\epsilon) \rightharpoonup a(x,t,T_k(u_\epsilon),\nabla T_k(u))S'(u) \quad \text{in } L^{p'}(Q_T),$$

and

$$S''(u_\epsilon)a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon \rightharpoonup S''(u)a(x,t,T_k(u_\epsilon),\nabla T_k(u))\nabla T_k(u) \quad \text{in } L^1(Q_T).$$

Furthermore, since $\phi_\epsilon(x,t,u_\epsilon)S'(u_\epsilon) = \phi_\epsilon(x,t,T_k(u_\epsilon))S'(u_\epsilon)$ a.e. in Q_T. By (3.8) we obtain $|\phi_\epsilon(x,t,T_k(u_\epsilon))S'(u_\epsilon)| \leq |c(x,t)|k^\gamma$, it follows that

$$\phi_\epsilon(x,t,T_k(u_\epsilon))S'(u_\epsilon) \to \phi_\epsilon(x,t,T_k(u))S'(u) \quad \text{strongly in } L^{p'}(Q_T).$$

In a similar way, it results

$$S''(u_\epsilon)\phi_\epsilon(x,t,u_\epsilon)\nabla u_\epsilon = S''(T_k(u_\epsilon))\phi_\epsilon(x,t,T_k(u_\epsilon))\nabla T_k(u_\epsilon) \quad \text{a.e. in } Q_T.$$

Using the weakly convergence of $T_k(u_\epsilon)$ in $L^p((0,T);W_0^p(\Omega))$ it is possible to prove that

$$S''(u_\epsilon)\phi_\epsilon(x,t,u_\epsilon)\nabla u_\epsilon \to S''(u)\phi(x,t,u)\nabla u \quad \text{in } L^1(Q_T).$$

Finally by (3.9) we deduce that $f_\epsilon S'(u_\epsilon)$ converges to $fS'(u)$ in $L^1(Q_T)$.

It remains to prove that $B_S(u)$ satisfies the initial condition $B_S(t=0) = B_S(u_0)$ in Ω. To this end, firstly remark that S being bounded, $B_S^\epsilon(u_\epsilon)$ is bounded in $L^\infty(Q)$. Secondly the above considerations of the behavior of the terms of this equation show that $\frac{\partial B_S^\epsilon(u_\epsilon)}{\partial t}$ is bounded in $L^1(Q_T) + L^{p'}(0,T;W^{-1,p'}(\Omega))$. As a consequence, an Aubin's type lemma (See e.g [20]) implies that $B_S^\epsilon(u_\epsilon)$ lies in a compact set of $C^0([0,T],L^1(\Omega))$. On the other hand, the smoothness of of S implies that $B_S(t=0) = B_S(u_0)$ in Ω. ∎

REFERENCES

[1] L. AHAROUCH, E. AZROUL and M. RHOUDAF, Strongly nonlinear variational degenerated parabolic problems in weighted sobolev spaces. *The Australian journal of Mathematical Analysis and Applications*, **Vol. 5**, Article 13, pp. 1-25, 2008.

[2] Y. AKDIM, J. BENNOUNA, M. MEKKOUR, Renormalized solutions of nonlinear degenerated parabolic equations with natural growth terms and L^1 data. *International journal of evolution equations*, **Vol. 5**, Number 4, pp. 421-446, 2011.

[3] Y. AKDIM, J. BENNOUNA, M. MEKKOUR and M. RHOUDAF, Renormalized solution of nonlinear degenerated parabolic problems with L^1-data: existence and uniquness, *Recent Developments in Nonlinear Analysis*, Proccedings of the conference in Mathematics and Mathematical Physics World, Scientific Publishing C.O Ltd.

[4] Y. AKDIM, J. BENNOUNA, M. MEKKOUR, Solvability of degenerate parabolic equations without sign condition and three unbounded nonlinearities, *Electronic Journal of Differetial Equations*, No. 03, pp. 1-25, 2011.

[5] D. BLANCHARD, F. MURAT, H. REDWANE, Existence and uniqueness of renormalized solution for fairly general class of non linear parabolic problems, *J. Differential Equations*, No. 177, pp. 331-374, 2001.

[6] D. BLANCHARD, F. MURAT, Renormalized solutions of nonlinear parabolic with L^1 data: existence and uniqueness, *Proc. Roy. Soc. Edinburgh Sect.*, A 127, pp. 1137-1152, 1997.

[7] D. BLANCHARD, Truncations and monotonicity methods for parabolic equations, *Nonlinear Anal.*, 21, pp. 725-43, 1993.

[8] D. BLANCHARD, H. REDWANE, Renormalized solutions for class of nonlinear evolution problems, *J. Math. Pure.*, 77, pp. 117-151, 1998.

[9] L. BOCCARDO, T. GOLLOU$\ddot{E}$T, Nonlinear elliptic and parabolic equations involving measure data, *J. Funct. Anal.*, pp. 87, 149-169, 1989.

[10] L. BOCCARDO, D. GIACHETTI, J.-I. DIAZ, F. MURAT, Existence and regularity of renormalized solutions of some elliptic problems involving derivatives of nonlinear terms, *Journal of differential equations*, 106, pp. 215-237, 1993.

[11] A. DALL'AGLIO, Approximated solutions of equations with L^1 data. Application to the H-convergence of quasi-linear parabolic equations, *Ann. Mat. Pura Appl*. **(4)** 170, pp. 207-240, 1996

[12] R.-DI NARDO, Nonlinear parabolic equations with a lower order term, Prepint N. 60-2008, Departimento di Matematica e Applicazioni "R.Caccippoli", Universita degli Studi di Napoli "FedericoII".

[13] G. DAL MASO, F. MURAT, L. ORSINA and A. PRIGNET, Renormalized solutions of elliptic equations with general measure data, *Ann. Scuala Norm. Sup. Pisa Cl. Sci*., **(4)** 28, 1999.

[14] R.-J. DIPERNA, P.-L. LIONS, On the Cauchy Problem for the Boltzmann Equations: Global existence and weak stability, *Ann. of Math*., 130, pp. 285-366, 1989.

[15] R. LANDES, On the existence of weak solutions for quasilinear parabolic initial-boundary problems, *Proc. Roy. Soc. Edinburgh Sect*., A89, pp. 321-366, 1981.

[16] J.-L. LIONS, *Quelques Méthodes de Resolution des Problémes aux Limites non Linéaires*, Dunod et Gauthier-Villars, Paris, 1969.

[17] H. REDWANE, Existence of solution for a class of a parabolic equation with three unbounded nonlinearities, *Adv. Dyn. Syt. A2PL*. 2, pp. 241-264, 2007.

[18] H. REDWANE, **Solution Renormalisé de Problémes Paraboliques et Elliptiques non Linéaires**, Ph.D., Rouen 1997.

[19] J.-M. RAKOTOSON, Resolution of the critical cases for the problems with L^1 data, Asymptotic Analysis when right hand side is in $L^1(\Omega)$. 6, pp. 285-293, 1993.

[20] J. SIMON, Compact set in the space $L^p(0, T, B)$, *Ann. Mat. Pura, Appl*., 146, pp. 65-96, 1987.