ON THE MONOTONICITY AND LOG-CONVEXITY OF THE
SECOND KIND ONE-PARAMETER HOMOGENEOUS
FUNCTION

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Abstract. That $H_2f(p; a, b) := Hf(p, 1−p; a, b)$ is called a second kind one-
parameter homogeneous function, of which monotonicity and log-convexity
depend on the sign of $J = (x − y)(xI)_{xy}$, where $I = (ln f)_{xy}$. By straight-
forward computations, some conclusions on the monotonicity and log-
convexity of $H_2f(p)$ are presented, where $f(x, y) = L(x, y)$, $A(x, y)$, $E(x, y)$
and $D(x, y)$, from which other new refined inequalities involving logarith-
mic mean, exponential mean, power-exponential mean and power mean etc. are derived.

1. Introduction

The extended mean and Gini mean were generalized to the so-called two-parameter
homogeneous functions by the author, which is defined by [20]

Definition 1. Assume $f: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}_+ \cup \{0\}$ is an $n$-order homogeneous function
of variables $x$ and $y$ which is continuous and exists first partial derivatives, $(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $a \neq b$, $(p, q) \in \mathbb{R} \times \mathbb{R}$.

If $f(x, y) > 0$ for $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $x \neq y$ and $f(x, x) = 0$ for all $x \in \mathbb{R}^+$, then define that

\begin{align}
(1.1) \quad H_f(p, q; a, b) &= \left( \frac{f(a^p, b^p)}{f(a^q, b^q)} \right)^{\frac{1}{p−q}} (p \neq q, pq \neq 0), \\
(1.2) \quad H_f(p, p; a, b) &= \lim_{q \rightarrow p} H_f(p, q; a, b) = G_{f,p}(p = q \neq 0),
\end{align}

where

\begin{align}
(1.3) \quad G_{f,p} = G_f^\frac{1}{p}(a^p, b^p), \quad G_f(x, y) = \exp \left( \frac{x f_x(x, y) \ln x + y f_y(x, y) \ln y}{f(x, y)} \right),
\end{align}

$f_x(x, y)$ and $f_y(x, y)$ denote partial derivative with respect to first and second vari-
able of $f(x, y)$ respectively.

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If \( f(x, y) > 0 \) for all \((x, y) \in \mathbb{R}_+ \times \mathbb{R}_+\) then define further

\[
\mathcal{H}_f(p, 0; a, b) = \left( \frac{f(a^p, b^p)}{f(1, 1)} \right)^{\frac{1}{p}} (p \neq 0, q = 0), \tag{1.4}
\]
\[
\mathcal{H}_f(0, q; a, b) = \left( \frac{f(a^q, b^q)}{f(1, 1)} \right)^{\frac{1}{q}} (p = 0, q \neq 0), \tag{1.5}
\]
\[
\mathcal{H}_f(0, 0; a, b) = \lim_{p \to 0} \mathcal{H}_f(p, 0; a, b) = a^{f(1, 1)}_p b^{f(1, 1)}_q (p = q = 0). \tag{1.6}
\]

Since \( f(x, y) \) is a homogeneous function, \( \mathcal{H}_f(p, q; a, b) \) is also one and called a homogeneous function with parameters \( p \) and \( q \), and simply denote by \( \mathcal{H}_f(p, q) \).

Let \( q = p + 1 \) in the Definition [1]. Then \( \mathcal{H}_f(p, p + 1; a, b) \) is called a one-parameter homogeneous function and denote by \( \mathcal{H}_1f(p, a, b) \) [21], which contain one-parameter mean and Lehmer mean (see [3] [18]).

The author investigated the monotonicity and log-convexity of two-parameter and one-parameter homogeneous functions, and obtained a series of interesting and useful results (see [20] [21] [22] [23] [24]).

The one-parameter mean is relative to the two-parameter mean. In general, \( \mathcal{H}_f(p, q) \) may be called a one-parameter homogeneous function provided there exists a sort of given relations between its parameters \( p \) and \( q \). The problem is which of one-parameter functions could be investigated and worth investigating.

Let \( q = 1 - p \) in the Definition [1]. Then we can obtain another one-parameter homogeneous function \( \mathcal{H}_f(p, 1 - p; a, b) \). For avoiding confusion, we call \( \mathcal{H}_f(p, p + 1; a, b) = \mathcal{H}_1f(p, a, b) \) first kind one-parameter homogeneous function, while \( \mathcal{H}_f(p, 1 - p; a, b) \) second kind one and denote by \( \mathcal{H}_2f(p; a, b) \) or \( \mathcal{H}_2f(p) \).

Using log-convexity of two-parameter or first kind one-parameter homogeneous functions, the author obtained results involving the monotonicity of the second kind homogeneous functions, which is read as follows [21] [24]:

**Theorem 1** ([21] Corollary 1) or ([24] Corollary 3). Let \( f(x, y) \) be a positive symmetric \( n \)-order homogeneous function defined on \( \mathbb{R}_+ \times \mathbb{R}_+ \{ (x, x) : x \in \mathbb{R}_+ \} \) which is three-time differentiable. If \( I = (x - y)(xI)_x < (>)0 \) where \( I = (\ln f)_{xy} \), then

1) \( \mathcal{H}_2f(p) \) is strictly decreasing (increasing) in \( p \in (0, \frac{1}{2}) \) and increasing (decreasing) in \( p \in \left( \frac{1}{2}, 1 \right) \).

2) If \( f(x, y) \) is defined on \( \mathbb{R}_+ \times \mathbb{R}_+ \) and symmetric with respect to \( x \) and \( y \) further, then \( \mathcal{H}_2f(p) \) is strictly decreasing (increasing) in \( p \in (\frac{1}{2}, +\infty) \) and increasing (decreasing) in \( p \in (\frac{1}{2}, +\infty) \).

The aim of this paper is to give a new proof of [1] and investigate the log-convexity in parameter \( p \) of second kind homogeneous function. As applications, some new inequalities will be presented.

2. Basic Concepts and Main Results

we give the definition of one-parameter homogeneous functions firstly.

**Definition 2.** Assume \( f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \cup \{0\} \) is an \( n \)-order homogeneous function of variables \( x \) and \( y \) which is continuous and exist first partial derivatives, \((a, b) \in \mathbb{R}_+ \times \mathbb{R}_+ \) with \( a \neq b \), \((p, q) \in \mathbb{R} \times \mathbb{R} \).
If \( f(x, y) > 0 \) for \((x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \) with \( x \neq y \) and \( f(x, x) = 0 \) for all \( x \in \mathbb{R}^+ \), then define that

\[
(2.1) \quad \mathcal{H}_{2f}(p; a, b) = \left( \frac{f(a^p, b^p)}{f(a^{1-p}, b^{1-p})} \right)^{1/p-1} \quad \text{if} \quad p(1-p)(p-1/2) \neq 0,
\]

\[
(2.2) \quad \mathcal{H}_{2f}(\frac{1}{2}; a, b) = \lim_{p\to\frac{1}{2}} \mathcal{H}_{2f}(p; a, b) = G_{f, \frac{1}{2}},
\]

where \( G_{f,p} \) is defined by (1.3).

If \( f(x, y) > 0 \) for all \((x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \) then define further

\[
(2.3) \quad \mathcal{H}_{2f}(0; a, b) = \mathcal{H}_{2f}(1; a, b) = \frac{f(a, b)}{f(1, 1)}.
\]

Since \( f(x, y) \) is a homogeneous function, \( \mathcal{H}_{2f}(p; a, b) \) is also one and called a second kind homogeneous function with a parameter \( p \) and simply called second kind one-parameter homogeneous function.

**Example 1.** From Definition 3, put \( f(x, y) = L(x, y) = (x - y)/\ln(x/y)(x > 0, x \neq y), f(x, x) = L(x, x) = x, \) then

\[
(2.4) \quad \mathcal{H}_{2L}(p; a, b) = \begin{cases} \left( \frac{(1-p)(a^p - b^p)}{p(a^p - b^p) - p(b^p - a^p)} \right)^{1/p-1}, & p \neq \frac{1}{2}; \\ E^2(\sqrt{a}, \sqrt{b}), & p = \frac{1}{2}, \end{cases}
\]

where \( E(x, y) = e^{-1}(x^y/y^x)^{1/(x-y)}(x \neq y), E(x, x) = x. \)

**Example 2.** Put \( f(x, y) = A(x, y) = \frac{1}{2}(x + y)(x, y > 0), \) then

\[
(2.5) \quad \mathcal{H}_{2A}(p; a, b) = \begin{cases} \left( \frac{a^p + b^p}{a^p - b^p} \right)^{1/p-1}, & p \neq \frac{1}{2}; \\ Z^2(\sqrt{a}, \sqrt{b}), & p = \frac{1}{2}, \end{cases}
\]

where \( Z(x, y) = x^{x+y}/y^{x+y}. \)

**Example 3.** Put \( f(x, y) = E(x, y) = e^{-1}(x^y/y^x)^{1/(x-y)}(x, y > 0, x \neq y), f(x, x) = E(x, x) = x, \) then

\[
(2.6) \quad \mathcal{H}_{2E}(p; a, b) = \begin{cases} \left( \frac{E(a^p, b^p)}{E(a^{1-p}, b^{1-p})} \right)^{1/p-1}, & p \neq \frac{1}{2}; \\ Y^2(\sqrt{a}, \sqrt{b}), & p = \frac{1}{2}, \end{cases}
\]

where \( Y(x, y) = E(x, y) \exp(1 - G^2(x, y)/L^2(x, y))(x, x \neq y), Y(x, x) = x. \)

**Example 4.** Put \( f(x, y) = D(x, y) = |x - y|(x, y > 0, x \neq y), \) then

\[
(2.7) \quad \mathcal{H}_{2D}(p; a, b) = \begin{cases} \left| \frac{a^p - b^p}{a^{1-p} - b^{1-p}} \right|^{1/p-1}, & p \neq 0, 1, \frac{1}{2}; \\ e^2E^2(\sqrt{a}, \sqrt{b}), & p = \frac{1}{2}, \end{cases}
\]

\( \mathcal{H}_{2D}(x, y; p) \) is called second kind one-parameter logarithmic mean. In the same way, \( \mathcal{H}_{2A}(x, y; p) \) and \( \mathcal{H}_{2E}(x, y; p) \) are called second kind one-parameter arithmetic mean and exponential mean, respectively.

Since \( D(x, y) \) is no a certain mean of positive numbers \( x \) and \( y \), but a difference function, we call \( \mathcal{H}_{2D}(x, y; p) \) second kind one-parameter homogeneous difference function.

Concerning the log-convexity of the second kind one-parameter homogeneous functions, we have the following results.
Theorem 2. Let $f(x, y)$ be a nonnegative $n$-order homogenous function defined on $\mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(x, x) : x \in \mathbb{R}_+\}$ which is three-time differentiable. Then $\mathcal{H}_f(p)$ is log-convex (log-concave) in $p \in (0, 1)$ if $J = (x - y)(xI)_x < (>0)$, where $I = (\ln f)_{xy}$.

For $p \neq q, p, q \in (0, 1)$, define that

$$R_2(p, q) := \left(\frac{\mathcal{H}_f(p)}{\mathcal{H}_f(q)}\right)^{\frac{1}{p-q}};$$

for $p \in (0, 1), p \neq \frac{1}{2}$, define that

$$R_2(p, p) = \lim_{q \to p} R_2(p, q) = \lim_{q \to p} \left(\frac{\mathcal{H}_f(p)}{\mathcal{H}_f(q)}\right)^{\frac{1}{p-q}}$$

$$= \exp \lim_{q \to p} \frac{\ln \mathcal{H}_f(p) - \ln \mathcal{H}_f(q)}{p-q} = \exp \frac{d\ln \mathcal{H}_f(p)}{dp}$$

$$= \exp \frac{(T'(p) + T'(1-p))(2p-1) - 2[T(p) - T(1-p)]}{(2p-1)^2}$$

$$= \left(\frac{\exp T'(p)\exp T'(1-p)}{\mathcal{H}_f(p)}\right)^{\frac{1}{p-\frac{1}{2}}}$$

$$= \left(\frac{G_f G_{f,1-p}}{\mathcal{H}_f(p) \mathcal{H}_f(1-p)}\right)^{\frac{1}{p-\frac{1}{2}}},$$

where $G_{f,p}$ is defined by (1.3).

$$R_2\left(\frac{1}{2}, \frac{1}{2}\right) = \lim_{p \to \frac{1}{2}} R_2(p, \frac{1}{2})$$

$$= \exp \left(\lim_{p \to \frac{1}{2}} \frac{d\ln \mathcal{H}_f(p)}{dp}\right)$$

$$= \exp \left(\lim_{p \to \frac{1}{2}} \frac{[T'(p) + T'(1-p)](2p-1) - 2[T(p) - T(1-p)]}{(2p-1)^2}\right)$$

$$= \exp \lim_{p \to \frac{1}{2}} \frac{(T''(p) - T''(1-p))(2p-1)}{4(2p-1)}$$

$$= \exp(0) = 1$$

Thus from Theorem 2 and by properties of convex functions, we immediately get

Corollary 1. That $R_2(p, q)$ is strictly increasing (decreasing) in either $p$ or $q$ on $(0, 1)$ if $J = (x - y)(xI)_x < (>0)$, where

$$R_2(p, q) = \begin{cases} \left(\frac{\mathcal{H}_f(p)}{\mathcal{H}_f(q)}\right)^{\frac{1}{p-q}}, & p \neq q, p, q \in (0, 1); \\ \left(\frac{G_f G_{f,1-p}}{\mathcal{H}_f(p) \mathcal{H}_f(1-p)}\right)^{\frac{1}{p-\frac{1}{2}}}, & p = q \neq \frac{1}{2}, p, q \in (0, 1); \\ 1, & p = q = \frac{1}{2}. \end{cases}$$

(2.8)
3. Proofs of the Main Results

To prove Theorem 1 and 2 we need certain lemmas of two-parameter homogeneous functions.

**Lemma 1** (Lemma 3, 4). Suppose that \( f(x, y) \) is a positive \( n \)-order homogeneous function defined on \( \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(x, x) : x \in \mathbb{R}_+ \} \) which is three-time differentiable. Set \( T(t) = \ln f(a^t, b^t), a^t = x, b^t = y \) with \( t \neq 0 \), then

\[
T'(t) = \frac{a^t f_x(a^t, b^t) \ln a + b^t f_y(a^t, b^t) \ln b}{f(a^t, b^t)} = \ln G_f^1(a^t, b^t),
\]

where \( G_f(x, y) \) is defined by (1.3);

\[
T''(t) = -xyI \ln^2(b/a), \quad \text{where } I = (\ln f)_{xy};
\]

\[
T'''(t) = -Ct^{-3} J, \quad \text{where } J = (x - y)(xI)_x, C = \frac{xy \ln^3(x/y)}{x - y} > 0.
\]

**Lemma 2** (Property 4). Suppose that \( f(x, y) \) is a positive symmetric \( n \)-order homogeneous function defined on \( \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(x, x) : x \in \mathbb{R}_+ \} \), then

\[
\mathcal{H}_f(t, -t) = G^n,
\]

\[
T(t) - T(-t) = 2nt \ln G,
\]

\[
T'(t) + T'(-t) = 2n \ln G,
\]

\[
T''(-t) = T''(t)
\]

where \( G = \sqrt{ab} \).

**Remark 1.** If \( f(1, 1) := \lim_{x \to 1} f(x, 1) > 0 \), then define that \( T'(0) := \lim_{t \to 0} T'(t) = n \ln G \). Thus (2.6) can be written as

\[
T'(0) + T'(-0) = 2T'(0).
\]

Based on the above lemmas, we prove main results next.

A New Proof of Theorem 7. Without loss generality, we assume that \( J = (x - y)(xI)_x < 0 \). From (3.3) we see that \( T''(t) \) is strictly increasing on \((0, \infty)\) and decreasing on \((-\infty, 0)\).

1) For \( p \in (0, 1) \). Since \( \ln \mathcal{H}_{2f}(p) = \frac{T(p) - T(1 - p)}{2p - 1} \), a simply derivative calculation yields

\[
\frac{d \ln \mathcal{H}_{2f}(p)}{dp} = \frac{\{T'(p) + T'(1 - p)\}(2p - 1) - 2[T(p) - T(1 - p)]}{(2p - 1)^2}
\]

(3.9)

where

\[
l(p) = (T'(p) + T'(1 - p))(2p - 1) - 2(T(p) - T(1 - p)) \cdot
\]

Note \( l(\frac{1}{2}) = 0 \) and

\[
l'(p) = (2p - 1)(T''(p) - T''(1 - p)) = (2p - 1)^2T'''(\xi),
\]

(3.11)
where \( \xi = 1 - p + \theta(2p - 1), \theta \in (0, 1) \). By Lemma 1 we see \( T'''(\xi) > 0 \) if \( J = (x - y)(xI)x < 0 \) and \( p \in (0, 1) \), hence \( l'(p) > 0 \) on \((0, 1)\). It follows that \( l(p) < l\left(\frac{1}{2}\right) = 0 \) for \( p \in (0, \frac{1}{2}) \) and \( l(p) > l\left(\frac{1}{2}\right) = 0 \) for \( p \in \left(\frac{1}{2}, 1\right) \), and then

\[
\frac{d \ln \mathcal{H}_2(p)}{dp} = \frac{l(p)}{(2p - 1)^2} \begin{cases} < 0, & p \in (-\infty, \frac{1}{2}); \\ > 0, & p \in \left(\frac{1}{2}, \infty\right). \end{cases}
\]

which shows that \( \mathcal{H}_2(p) \) is strictly decreasing on \((0, \frac{1}{2})\) and increasing on \(\left(\frac{1}{2}, 1\right)\).

2) For the symmetric function \( f(x, y) \) and \( p \in (-\infty, \infty) \), we present some conclusions about \( J \).

By part one of this Theorem, we understand that \( l'(p) > 0 \) for \( p \in (0, 1) \).

By (3.7) we have

\[
l'(p) = T'''(p) - T'''(1 - p),
\]

which shows that \( l'(p) > 0 \) is always true for \( p \in (-\infty, \infty) \), i.e. that \( l(p) \) is strictly increasing. It follows that \( l(p) < l\left(\frac{1}{2}\right) = 0 \) for \( p \in (-\infty, \frac{1}{2}) \) and \( l(p) > l\left(\frac{1}{2}\right) = 0 \) for \( p \in \left(\frac{1}{2}, \infty\right) \), and then

\[
\frac{d \ln \mathcal{H}_2(p)}{dp} = \frac{l(p)}{(2p - 1)^2} \begin{cases} < 0, & p \in (-\infty, \frac{1}{2}); \\ > 0, & p \in \left(\frac{1}{2}, \infty\right). \end{cases}
\]

which completes the proof. \( \square \)

Proof of Theorem 2 For \( J = (x - y)(xI)x < 0 \). By (3.9)

\[
\frac{d^2 \ln \mathcal{H}_2(p)}{dp^2} = \frac{l'(p)(2p - 1) - 4l(p)}{(2p - 1)^3} = \frac{m(p)}{(2p - 1)^3},
\]

where

\[
m(p) = l'(p)(2p - 1) - 4l(p) = (T'''(p) - T'''(1 - p))(2p - 1)^2 - 4l(p).
\]

A derivative calculation leads to

\[
m'(p) = (T'''(p) + T'''(1 - p))(2p - 1)^2.
\]

For \( p \in (0, 1) \), we have \( T'''(p) > 0 \) and \( T'''(1 - p) > 0 \) if \( J = (x - y)(xI)x < 0 \). This shows that \( m'(p) > 0 \) i.e. \( m(p) \) is strictly increasing in \( p \in (0, 1) \). It follows that \( m(p) < m\left(\frac{1}{2}\right) = 0 \) for \( p \in (0, \frac{1}{2}) \) and \( m(p) > m\left(\frac{1}{2}\right) = 0 \) for \( p \in \left(\frac{1}{2}, 1\right) \). Consequently, there always exists

\[
\frac{d^2 \ln \mathcal{H}_2(p)}{dp^2} = \frac{m(p)}{(2p - 1)^3} > 0 \text{ for } p \in (0, 1),
\]

i.e. \( \mathcal{H}_2(p) \) is strictly log-convex in \( p \in (0, 1) \).

For \( J = (x - y)(xI)x > 0 \) it can be proved in the same way.

The proof ends. \( \square \)

4. SOME APPLICATIONS

By Theorem 1 and 2, the monotonicity and log-convexity of \( \mathcal{H}_2(p) \) depend on the sign of \( J = (x - y)(xI)x \). In this section, by some straightforward computations, we present some conclusions about \( \mathcal{H}_2(p) \), where \( f(x, y) = L(x, y), A(x, y), E(x, y) \) and \( D(x, y) \).
(i) For \( f(x, y) = L(x, y) = \frac{x - y}{\ln x - \ln y} \), where \( x, y > 0 \) with \( x \neq y \), there are

\[
I = (\ln f)_{xy} = \frac{1}{(x - y)^2} - \frac{1}{xy(\ln x - \ln y)^2} \\
J = (x - y)(xI)_x = (x - y)\left(-\frac{x + y}{(x - y)^3} + \frac{2}{xy(\ln x - \ln y)^3}\right) \\
= \frac{2}{xy(x - y)^2} \left(L^3(x, y) - \frac{x + y}{2} (\sqrt{xy})^2\right).
\]

By the well-known inequalities \( L(x, y) > \sqrt{xy} \) \([17]\) and \( L(x, y) > (\frac{x + y}{2})^\frac{3}{2} (\sqrt{xy})^\frac{3}{2} \) \([10]\), we have \( I < 0, J > 0 \).

(ii) For \( f(x, y) = A(x, y) = \frac{x + y}{2} \), where \( x, y > 0 \) with \( x \neq y \), there are

\[
I = (\ln f)_{xy} = -\frac{1}{(x + y)^2} < 0, \\
J = (x - y)(xI)_x = \frac{(x - y)^2}{(x + y)^2} > 0.
\]

(iii) For \( f(x, y) = E(x, y) = e^{-1} (x^y / y^x)^{1/(x - y)} \), where \( x, y > 0 \) with \( x \neq y \), there are

\[
I = (\ln f)_{xy} = \frac{1}{(x - y)^3} \left[2(x - y) - (x + y)(\ln x - \ln y)\right] \\
J = (x - y)(xI)_x = \frac{-3(x^2 - y^2) + (x^2 + 4xy + y^2)(\ln x - \ln y)}{(x - y)^3} \\
= -\frac{6(\ln x - \ln y)}{(x - y)^3} \left(\frac{x^2 - y^2}{\ln x^2 - \ln y^2} - \frac{x^2 + y^2}{2} + 2xy\right).
\]

By the well-known inequalities \( L(x, y) < \frac{x + y}{2} \) \([17]\) and \( L(x, y) < \frac{x + y}{2} + 2\sqrt{xy} \) \([7]\), we have \( I < 0, J > 0 \).

(iii) For \( f(x, y) = D(x, y) = |x - y| \), where \( x, y > 0 \) with \( x \neq y \), there are

\[
I = (\ln f)_{xy} = \frac{1}{(x - y)^2} > 0, \\
J = (x - y)(xI)_x = -\frac{x + y}{(x - y)^2} < 0.
\]

Notice that \( L(x, y), A(x, y) \) and \( E(x, y) \) are all symmetric with respect to \( x \) and \( y \) and \( L(1, 1) = A(1, 1) = E(1, 1) = 1 \) but \( D(1, 1) = 0 \), using Theorem \([1]\) and \([2]\) we easily obtain the following corollaries:

**Corollary 2.** That \( \mathcal{H}_{2L}(p; a, b) \), \( \mathcal{H}_{2A}(p; a, b) \) and \( \mathcal{H}_{2E}(p; a, b) \) are strictly increasing in \( p \in (-\infty, \frac{1}{3}) \) and decreasing in \( p \in (\frac{1}{3}, \infty) \) and log-concave in \( p \in [0, 1] \). That \( R_{2L}(p), R_{2A}(p) \) and \( R_{2E}(p) \) are strictly decreasing on \([0, 1];\)

Corollary 3. That $\mathcal{H}_2D(p; a, b)$ is strictly decreasing in $p \in (0, \frac{1}{2})$ and increasing in $p \in (\frac{1}{2}, 1)$ and log-convex in $(0, 1)$. That $R_2D(p)$ is strictly increasing in $p \in (0, 1)$.

Corollary 4. For $p_1, p_2 \in [\frac{1}{2}, 1)$ with $p_1 < p_2$, we have

\begin{equation}
1 < \frac{\mathcal{H}_{2L}(p_1; a, b)}{\mathcal{H}_{2L}(p_2; a, b)} < e^{\frac{1}{L(p_2; 1-p_2)} - \frac{1}{L(p_1; 1-p_1)}}.
\end{equation}

Proof. For $p_1, p_2 \in [\frac{1}{2}, 1)$ with $p_1 < p_2$, by Corollary 2 and 3 there are

\begin{align*}
\mathcal{H}_{2L}(p_1; a, b) &> \mathcal{H}_{2L}(p_2; a, b), \\
\mathcal{H}_{2D}(p_1; a, b) &< \mathcal{H}_{2D}(p_2; a, b).
\end{align*}

Note

\begin{equation}
\mathcal{H}_{2D}(p; a, b) = \left(\frac{p}{1-p}\right)^{\frac{1}{p-1}} \mathcal{H}_{2L}(p) = e^{\frac{1}{\mathcal{H}_{2L}(1-p; a, b)} - \mathcal{H}_{2L}(p)}.
\end{equation}

then (4.3) can be changed as

\begin{equation}
e^{\frac{1}{\mathcal{H}_{2L}(1-p; a, b)} - \mathcal{H}_{2L}(p)} \mathcal{H}_{2L}(p_1; a, b) < e^{\frac{1}{\mathcal{H}_{2L}(1-p; a, b)} - \mathcal{H}_{2L}(p)} \mathcal{H}_{2L}(p_2; a, b).
\end{equation}

From (4.5) and (4.2), we immediately get (4.1).

The proof ends.

Corollary 5. For $p_1, p_2, q \in (0, 1)$ with $p_1 < p_2$, we have

\begin{equation}
1 < \frac{R_{2L}(p_1; q)}{R_{2L}(p_2; q)} < e^{\frac{1}{\mathcal{H}_{2L}(1-p; a, b)} - \mathcal{H}_{2L}(q)}.
\end{equation}

Proof. For $p, q \in (0, 1)$, by (4.4) we have

\begin{equation}
R_{2D}(p; q) = \left(\frac{e^{\frac{1}{\mathcal{H}_{2L}(1-p; a, b)} - \mathcal{H}_{2L}(p)}}{e^{\frac{1}{\mathcal{H}_{2L}(1-p; a, b)} - \mathcal{H}_{2L}(q)}}\right)^{\frac{1}{p}} = e^{\frac{1}{\mathcal{H}_{2L}(1-p; a, b)} - \mathcal{H}_{2L}(q)} R_{2L}(p; q).
\end{equation}

If $p_1, p_2 \in (0, 1)$ with $p_1 < p_2$, then by Corollary 2 and 3 there are

\begin{align*}
R_{2D}(p_1; q) &< R_{2D}(p_2; q), \\
R_{2L}(p_1; q) &> R_{2L}(p_2; q),
\end{align*}

it follows from (4.7) and (4.8) that

\begin{equation}
e^{\frac{1}{\mathcal{H}_{2L}(1-p; a, b)} - \mathcal{H}_{2L}(q)} R_{2L}(p_1; q) < e^{\frac{1}{\mathcal{H}_{2L}(1-p; a, b)} - \mathcal{H}_{2L}(q)} R_{2L}(p_2; q).
\end{equation}

Combining (4.9) and (4.10), we immediately get (4.6).

The proof ends.

For $f(x, y) = L(x, y), A(x, y), E(x, y)$ and $D(x, y)$, according to the monotonicity of $\mathcal{H}_{2L}(p)$, we can get some interesting inequalities involving the logarithmic mean, exponential mean (identic mean), power mean and Heron mean etc., for instance
Example 5. For \( f(x, y) = L(x, y) \), \( R_{2L}(p, q) \) is strictly decreasing in either \( p \) or \( q \) on \([0, 1]\.  
1) From \( R_{2L}(\frac{1}{2}, \frac{1}{2}) > R_{2L}(\frac{1}{3}, \frac{1}{3}) > R_{2L}(\frac{1}{2}, 1) \) i.e. 
\[
\left( \frac{A_{\frac{1}{2}}}{E_{\frac{1}{2}}} \right)^6 > \left( \frac{L}{E_{\frac{1}{2}}} \right)^4 > \left( \frac{H_{\frac{1}{2}}}{E_{\frac{1}{2}}} \right)^2
\]

it follows that 
\[
E_{\frac{1}{2}} < A_{\frac{1}{2}}^3 H_{\frac{1}{2}}^{-2}, \\
L E_{\frac{1}{2}} < H_{\frac{1}{2}}^2. 
\]

2) From \( R_{2L}(\frac{2}{3}, \frac{1}{2}) > R_{2L}(\frac{2}{3}, \frac{2}{3}) > R_{2L}(\frac{2}{3}, \frac{3}{4}) \) i.e. 
\[
\left( \frac{A_{\frac{1}{2}}}{E_{\frac{1}{2}}} \right)^6 > \left( \frac{E_{\frac{1}{2}} E_{\frac{1}{2}}}{A_{\frac{1}{2}}} \right)^3 > \left( \frac{H_{\frac{1}{2}}}{A_{\frac{1}{2}}} \right)^{12}
\]

it follows that 
\[
H_{\frac{1}{2}} A_{\frac{1}{2}}^{-1} < \sqrt{E_{\frac{1}{2}} E_{\frac{1}{2}}} < A_{\frac{1}{2}}^2 E_{\frac{1}{2}}^{-1}. 
\]
3) From \( R_{2L}(\frac{3}{4}, \frac{1}{2}) > R_{2L}(\frac{3}{4}, \frac{3}{4}) > R_{2L}(\frac{3}{4}, \frac{1}{4}) > R_{2L}(\frac{3}{4}, \frac{3}{4}) \) \( \text{i.e.} \)
\[
\left( \frac{\sqrt{2}}{E_{\frac{1}{2}}} \right)^4 > \left( \frac{\sqrt{2}}{A_{\frac{1}{2}}} \right)^{12} > \left( \frac{E_{\frac{1}{2}}E_{\frac{1}{2}}}{H_{\frac{1}{2}}} \right)^2 > \left( \frac{L}{H_{\frac{1}{2}}} \right)^4
\]

it follows that

(4.19) \( L < \sqrt{E_{\frac{1}{2}}E_{\frac{1}{2}}} < H_{\frac{1}{2}}A^{-3}_{\frac{1}{2}} \).

4) From \( R_{2}(1, \frac{1}{2}) > R_{2}(1, \frac{3}{4}) > R_{2}(1, \frac{1}{4}) > R_{2}(1, 1) \) \( \text{i.e.} \)
\[
\left( \frac{L}{E_{\frac{1}{2}}} \right)^2 > \left( \frac{L}{A_{\frac{1}{2}}} \right)^3 > \left( \frac{L}{H_{\frac{1}{2}}} \right)^4 \] \( \frac{E}{L} \)

it follows that

(4.20) \( LE_{\frac{1}{2}}^2 < A_{\frac{3}{2}} \),
(4.21) \( (\sqrt{EG})^3 H_{\frac{2}{2}}^\frac{3}{2} < L < H_{\frac{1}{2}}A^{-3}_{\frac{1}{2}} \).

**Remark 2.** Since \( L < H_{\frac{1}{2}} < A_{\frac{1}{2}} < E_{\frac{1}{2}} \) that second inequality of (4.21) is stronger than \( L < H_{\frac{1}{2}} < A_{\frac{1}{2}} \); while first inequality gives a new lower bound of logarithmic mean \( L \). Inequality (4.16) gives a new upper bound of exponential mean \( E \).

Note the corresponding relations of \( f(x, y) = L(x, y) \) and \( f(x, y) = A(x, y), E(x, y) \) (see Table 1), we can derive corresponding inequalities with (4.16)-(4.21).

<table>
<thead>
<tr>
<th>( f )</th>
<th>( G_{f,p} )</th>
<th>( G_{f,0} )</th>
<th>( H_{2f}(\frac{1}{2}) )</th>
<th>( H_{2f}(\frac{3}{4}) )</th>
<th>( H_{2f}(\frac{5}{4}) )</th>
<th>( H_{2f}(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L )</td>
<td>( E_p )</td>
<td>( G )</td>
<td>( E_{\frac{1}{2}} )</td>
<td>( A_{\frac{1}{2}} )</td>
<td>( H_{\frac{1}{2}} )</td>
<td>( L )</td>
</tr>
<tr>
<td>( A )</td>
<td>( Z_p )</td>
<td>( G )</td>
<td>( Z_{\frac{1}{2}} )</td>
<td>( A^2_{\frac{1}{2}}A^{-1}_{\frac{3}{2}} )</td>
<td>( A^2_{\frac{3}{4}}A^{-1}_{\frac{1}{4}} )</td>
<td>( A )</td>
</tr>
<tr>
<td>( E )</td>
<td>( Y_p )</td>
<td>( G )</td>
<td>( Y_{\frac{1}{2}} )</td>
<td>( Z_{\frac{3}{4}} )</td>
<td>( E^2_{\frac{1}{4}}E^{-1}_{\frac{1}{4}} )</td>
<td>( E )</td>
</tr>
<tr>
<td>( D )</td>
<td>( e^\frac{1}{2}E_p )</td>
<td>does not exist</td>
<td>( e^2E_{\frac{1}{2}} )</td>
<td>( 8A_{\frac{1}{2}} )</td>
<td>( 9H_{\frac{1}{2}} )</td>
<td>does not exist</td>
</tr>
</tbody>
</table>

**Table 1.** Comparison Table of \( L, A, E \) and \( D \)

**Example 6.** For \( f(x, y) = A(x, y), R_2A(p, q) \) is strictly decreasing in either \( p \) or \( q \) on \([0, 1]\), so we have

(4.22) \( Z_{\frac{1}{2}} < A^6_{\frac{1}{2}}A^{-3}_{\frac{3}{4}}A^{-3}_{\frac{1}{4}} \),
(4.23) \( AZ_{\frac{1}{2}} < A^3_{\frac{1}{4}}A^{-1}_{\frac{1}{4}} \).
(4.24) \( A^3_{\frac{1}{4}}A^{-1}_{\frac{3}{4}}A^{-2}_{\frac{1}{4}}A_{\frac{1}{4}} < \sqrt{Z_{\frac{1}{2}}Z_{\frac{3}{4}}} < A^4_{\frac{1}{4}}A^{-2}_{\frac{3}{4}}Z^{-1}_{\frac{1}{2}} \)
(4.25) \( A < \sqrt{Z_{\frac{1}{2}}Z_{\frac{3}{4}}} < A^6_{\frac{1}{4}}A^{-2}_{\frac{3}{4}}A^{-6}_{\frac{1}{4}}A_{\frac{3}{4}} \)
(4.26) \( AZ_{\frac{2}{2}} < \left( A^2_{\frac{3}{4}}A^{-1}_{\frac{1}{4}} \right)^3 \),
(4.27) \( (\sqrt{ZG})^{\frac{1}{2}}A^3_{\frac{1}{4}}A^{-3}_{\frac{3}{4}} < A < A^6_{\frac{1}{4}}A^{-2}_{\frac{3}{4}}A^{-6}_{\frac{1}{4}}A_{\frac{3}{4}} \)
Example 7. For \( f(x, y) = E(x, y) \), note \( E(a^2, b^2)/E(a, b) = Z(a, b) \) \[20\] Remark 4.1. \( R_{2E}(p, q) \) is strictly decreasing in either \( p \) or \( q \) on \([0, 1]\), so we have

\[
Y_{\frac{1}{3}} < Z_{\frac{3}{1}}E^{-3}E_{\frac{1}{3}},
\]

\[
EY_{\frac{2}{3}} < E^3E^{-1}_4,
\]

\[
E^3E^{-1}_3Z^{-1}_{\frac{1}{3}} < \sqrt{Y_{\frac{1}{3}}Y_{\frac{2}{3}}} < Z^2_{\frac{3}{1}}Y^{-1}_{\frac{1}{3}},
\]

\[
E < \sqrt{Y_{\frac{1}{3}}Y_{\frac{2}{3}}} < E^6E^{-2}_4Z^{-3}_{\frac{1}{3}},
\]

\[
EY^2_{\frac{2}{3}} < Z^1_{\frac{1}{3}},
\]

\[
(\sqrt{Y_{G}})^3E_{\frac{1}{3}}E^{-\frac{4}{3}} < E < E^6E^{-2}_4Z^{-3}_{\frac{1}{3}}.
\]

Example 8. For \( f(x, y) = D(x, y) \), noticing \( f(1, 1) \) doesn’t exist, \( R_{2D}(p, q) \) is strictly increasing in either \( p \) or \( q \) on \((0, 1)\), so we have the following inequalities corresponding with \[4.16\], \[4.18\] and \[4.19\]

\[
e^2E_{\frac{1}{3}} > 8^3A^3_{\frac{3}{1}}9^{-2}H^{-2}_{\frac{1}{3}},
\]

\[
9^2H^2_{\frac{1}{3}}8^{-1}A^{-1}_{\frac{1}{3}} > \sqrt{e^3E_{\frac{3}{1}}e^2E_{\frac{2}{3}}} > 8^2A^2_{\frac{3}{1}}e^{-2}E^{-1}_{\frac{1}{3}}.
\]

\[
\sqrt{e^4E_{\frac{1}{3}}e^2E_{\frac{2}{3}}} > 9^4H^4_{\frac{1}{3}}8^{-3}A^{-3}_{\frac{1}{3}},
\]

which can be rewritten as

\[
E_{\frac{1}{3}} > (e^{-2}8^39^{-2})A^3_{\frac{3}{1}}H^{-2}_{\frac{1}{3}},
\]

\[
(e^{-\pi}9^28^{-1})H^2_{\frac{1}{3}}A^{-1}_{\frac{1}{3}} > \sqrt{E_{\frac{1}{3}}E_{\frac{2}{3}}},
\]

\[
\sqrt{E_{\frac{1}{3}}E_{\frac{2}{3}}} > (e^{-\pi}9^48^{-3})H^4_{\frac{1}{3}}A^{-3}_{\frac{1}{3}}.
\]

Combining \[4.16\], \[4.18\] and \[4.19\] with \[4.37\], \[4.38\] and \[4.39\], we get

\[
1 > E_{\frac{1}{3}}/(A^3_{\frac{3}{1}}H^{-2}_{\frac{1}{3}}) > e^{-2}8^39^{-2} \approx 0.855452655,
\]

\[
1 < \sqrt{E_{\frac{1}{3}}E_{\frac{2}{3}}/(H^2_{\frac{1}{3}}A^{-1}_{\frac{1}{3}})} < e^{-\pi}9^28^{-1} \approx 1.067167149,
\]

\[
1 > \sqrt{E_{\frac{1}{3}}E_{\frac{2}{3}}/(H^4_{\frac{1}{3}}A^{-3}_{\frac{1}{3}})} > e^{-\pi}9^48^{-3} \approx 0.8903924291.
\]

Remark 3. The author gave a left estimation of exponential mean \( E \) by \( A^2_{\frac{2}{3}} \), which is read as follows:

\[
1 > A^2_{\frac{2}{3}}/E > e/\sqrt{\phi} \approx 0.9610577571.
\]

Replace \( a^2 \), \( b^2 \) with \( a, b \) in \[4.40\], then inequalities \[4.40\] can be rewritten as

\[
1 < (A^3_{\frac{3}{1}}H^{-2})/E < e\sqrt{e^28^39^2} \approx 1.081189977,
\]

which shows that the relative error estimating exponential mean \( E \) by \( A^3_{\frac{3}{1}}H^{-2} \) is approximate to 8%.
References


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