GEOMETRICALLY CONVEX FUNCTIONS AND SOLUTION OF A QUESTION

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ABSTRACT. From the properties of geometrically convex functions, this paper presents the solution of question 11031 in American Mathematical Monthly.

1. Definition of Geometrically Convex Functions

Throughout the paper we assume \( \alpha, \beta \in [0, 1] \) with \( \alpha + \beta = 1 \), and \( R^n \) be the n-dimensional Euclidean Space, \( R^n = \{ (x_1, x_2, \ldots, x_n) | x_i > 0, i = 1, 2, \ldots, n \} \), and \( \alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n) \), \( e^x = (e^{x_1}, e^{x_2}, \ldots, e^{x_n}) \), \( x^\alpha = (x_1^\alpha, x_2^\alpha, \ldots, x_n^\alpha) \), \( \ln x = (\ln x_1, \ln x_2, \ldots, \ln x_n) \), \( x, y = (x_1 y_1, x_2 y_2, \ldots, x_n y_n) \), where \( \alpha \in R_+ \) and \( x = (x_1, x_2, \ldots, x_n) \in R^n \), \( y = (y_1, y_2, \ldots, y_n) \in R^n \). And let \( G(x) = \sqrt[\lambda_1]{x_1} \cdot \sqrt[\lambda_2]{x_2} \cdots \sqrt[\lambda_n]{x_n} \) with \( x \in R^n \).

Paper \([1, 2] \) presents the definition of geometrically convex functions on \( R^n \).

**Definition 1.1.** \([1, 2, 3] \) Let \( f : I \subseteq (0, +\infty) \rightarrow (0, +\infty) \) be a continuous function, then \( f \) is called a geometrically convex function on \( I \), if existing \( n \geq 2 \), such that one of the following three inequalities holds for any \( x_1, x_2, \ldots, x_n \in I \) and \( \lambda_1, \lambda_2, \ldots, \lambda_n > 0 \) with \( \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1 \).

\[
\begin{align*}
(1.1) \quad f(\sqrt[\lambda_1]{x_1} \sqrt[\lambda_2]{x_2} \cdots \sqrt[\lambda_n]{x_n}) & \geq \sqrt[\lambda_1]{f(x_1)} \sqrt[\lambda_2]{f(x_2)} \cdots \sqrt[\lambda_n]{f(x_n)}, \\
(1.2) \quad f(\sqrt[\lambda_1]{x_1} \sqrt[\lambda_2]{x_2} \cdots \sqrt[\lambda_n]{x_n}) & \leq \sqrt[\lambda_1]{f(x_1)} \sqrt[\lambda_2]{f(x_2)} \cdots \sqrt[\lambda_n]{f(x_n)}, \\
(1.3) \quad f(\prod_{i=1}^{n} \lambda_i x_i) & \leq \prod_{i=1}^{n} f(\lambda_i (x_i),
\end{align*}
\]

and \( f \) is called a geometrically concave function on \( I \) if one of three inequalities \([1.1], [1.2], [1.3] \) is inverse.

References \([3] \) presents the definition of geometrically convex sets on \( R^n \).

**Definition 1.2.** \([3] \) \( H \subseteq R^n_+ \) is called a geometrically convex set if \( x^\alpha y^\beta \in H \) for any \( x, y \in H \).

Paper \([3] \) presents the definition of geometrically convex functions on \( R^n_+ \). References \([3] \) extends the definition of geometrically convex functions on geometrically convex sets.

**Definition 1.3.** \([3, 4] \) Let \( H \subseteq R^n_+ \) is a geometrically convex set, \( f : H \rightarrow (0, +\infty) \) is a continuous function, \( f \) is called a geometrically convex function if \( f(x^\alpha y^\beta) \leq f^\alpha(x) f^\beta(y) \) for any \( x, y \in H \). \( f \) is called a geometrically concave function if the above inequality is inverse.

**Definition 1.4.** \([3, 4, 5] \) Let \( x = (x_1, x_2, \ldots, x_n) \in R^n_+ \), \( y = (y_1, y_2, \ldots, y_n) \in R^n_+ \), \( (x_1, x_2, \ldots, x_n) \) and \( (y_1, y_2, \ldots, y_n) \) are the decreasing queue of \( (x_1, x_2, \ldots, x_n) \) and \( (y_1, y_2, \ldots, y_n) \) respectively. We say \( (x_1, x_2, \ldots, x_n) \) logarithm majorizes \( (y_1, y_2, \ldots, y_n) \), denoted \( \ln x > \ln y \) if

\[
\begin{align*}
(1.4) \quad \prod_{i=1}^{k} x_i & \geq \prod_{i=1}^{k} y_i, k = 1, 2, \ldots, n - 1, \\
x_1 x_2 \cdots x_n & = y_1 y_2 \cdots y_n.
\end{align*}
\]

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Lemma 1.1. (3) Let \( x \in \mathbb{R}_+^n \), then \( x \) logarithm majorizes \( \tilde{G} = (G(x), G(x), \ldots, G(x)) \).

Definition 1.5. Suppose \( E \subseteq \mathbb{R}_+^n \), \( f : E \to [0, +\infty) \). Then \( f \) is called \( S \)-geometrically convex function, if the following inequality
\[
(1.5) \quad f(x) \geq f(y).
\]
holds for any \( x, y \in E \subseteq \mathbb{R}_+^n \), when \( \ln x > \ln y \). And \( f \) is called \( S \)-geometrically concave function, if the inequality \((1.5)\) is reversed.

Lemma 1.2. (3) Suppose \( E \subseteq \mathbb{R}_+^n \) is symmetric geometrically convex set, \( f : E \to [0, +\infty) \) is symmetric continuously differentiable function. Then \( f \) is \( S \)-geometrically concave function, if the following inequality
\[
(1.6) \quad (\ln x_1 - \ln x_2) \left( x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2} \right) \geq 0
\]
holds for any \( x = (x_1, x_2, \ldots, x_n) \in E \subseteq \mathbb{R}_+^n \). And \( f \) is \( S \)-geometrically concave function, if the inequality \((1.6)\) is reversed.

2. Question And Lemmas

Question 11031 of the American Mathematical Monthly: Define the monster mean \( M(a, b) \) of two positive real number to be \( \ln N(a, b) \), where \( N(a, b) \) is the fraction
\[
1 + \ln \left( \sqrt{1 + f(a, b)} + \sqrt{f(a, b)} \right) = \frac{e^{2(e^a-1)(e^b+1)^{-1}} - 1}{e^{2(e^a-1)(e^b+1)^{-1}} - 1} \cdot e^{-\left((e^a-1)(e^b+1)^{-1} + (e^a-1)(e^b+1)^{-1}\right)}.
\]
Prove or disprove: the monster mean \( M(a, b) \) is always less than or equal to the geometric mean \( \sqrt{ab} \).

To solve the question in the next section, the following lemmas are necessary.

Lemma 2.1. Let \( 0 < t < 1 \), then
\[
(2.1) \quad e^{2t} > 1 + 2t + 2t^2 + \frac{4}{3} t^3.
\]
\[
(2.2) \quad \ln \frac{1 + t}{1 - t} > 2t.
\]
\[
(2.3) \quad (2t^2 - 1) e^{4t} > -4t - 6t^2 - \frac{8}{3} t^3 + \frac{16}{3} t^4 + \frac{64}{5} t^5 + \frac{704}{45} t^6.
\]

Proof. The proof of inequalities \((2.1)\) and \((2.2)\) is easy. Suppose \( f(t) = (2t^2 - 1) e^{4t} + 1 + 4t + 6t^2 + \frac{8}{3} t^3 - \frac{16}{3} t^4 + \frac{64}{5} t^5 - \frac{704}{45} t^6 \), \( 0 < t < 1 \), then
\[
\begin{align*}
    f'(t) &= (8t^2 + 4t - 4) e^{4t} + 4 + 12t + 8t^2 - \frac{64}{3} t^3 - 64t^4 - \frac{1408}{15} t^5, \\
    f''(t) &= (32t^2 + 32t - 12) e^{4t} + 12 + 16t - 64t^2 - 256t^3 - \frac{1408}{3} t^4, \\
    \left(\frac{1}{4} f''(t)\right)' &= (32t^2 + 48t - 4) e^{4t} + 4 - 32t - 192t^2 - \frac{1408}{3} t^3, \\
    \left(\frac{1}{16} f'''(t)\right)' &= (32t^2 + 64t + 8) e^{4t} - 8 - 96t + 352t^2,
\end{align*}
\]
\[ \left( \frac{1}{128} f^{(4)}(t) \right)' = (16t^2 + 40t + 12) e^{4t} - 12 - 88t, \]
\[ \left( \frac{1}{512} f^{(5)}(t) \right)' = (16t^2 + 48t + 22) e^{4t} - 22. \]

With \( f^{(6)}(t) > 0 \) and \( \lim_{t \to 0^+} f^{(i)}(t) = 0, i = 1, 2, \cdots, 5, \) therefore
\[ f(t) = (2t^2 - 1) e^{4t} + 4t + 6t^2 + 8 \frac{t^3 - 16}{3} t^4 - \frac{64}{5} t^5 - \frac{704}{45} t^6 > 0, \]
then the inequality (2.3) holds. □

**Lemma 2.2.** Let \( h(t) = \left( 1 + \frac{2}{e^{-a-1}} \right) \cdot (1 - t^2) \cdot (\ln (1 + t) - \ln (1 - t)), \) then \( h \) is a decreasing function on \((0, 1)\).

**Proof.**
\[ h'(t) = -2 \cdot \frac{1 - e^{4t} + (2e^{2t} - t + te^{4t} - 2t^2 e^{2t}) (\ln (1 + t) - \ln (1 - t))}{(e^{2t} - 1)^2}, \]
\[ -\frac{1}{2} (e^{2t} - 1)^2 h'(t) = 1 - e^{4t} + (2t^2 - t + te^{4t} - 2t^2 e^{2t}) (\ln (1 + t) - \ln (1 - t)). \]
Form Lemma 2.1
\[ -\frac{1}{2} (e^{2t} - 1)^2 h'(t) > 1 - e^{4t} + \left( 2 \left( 1 - t^2 \right) \left( 1 + 2t + 2t^2 + \frac{4}{3} t^3 \right) - t + te^{4t} \right) \cdot 2t \]
\[ = (2t^2 - 1) e^{4t} + 4t + 6t^2 + 8 \frac{t^3 - 16}{3} t^4 - \frac{64}{5} t^5 - \frac{704}{45} t^6 \]
\[ > \frac{4}{3} t^3 + \frac{8}{3} t^4 + \frac{24}{5} t^5 + \frac{464}{45} t^6 > 0. \]
So \( h'(t) < 0, h \) is a decreasing function on open interval \((0, 1)\). □

3. **Solution of the Question**

**Proof.** Let \( T(t) = \frac{e^t - 1}{e^t + 1}, t \in R, \) and \((a, b) \in R^2_+\), then
\[ f(a, b) = \frac{1}{4} \left( e^{T(a)} - e^{-T(a)} \right) \left( e^{T(b)} - e^{-T(b)} \right), \]
\[ f'_1(a, b) = \frac{1}{4} \left( e^{T(a)} + e^{-T(a)} \right) \left( e^{T(b)} - e^{-T(b)} \right) \frac{2e^a}{(e^a + 1)^2}, \]
\[ f'_2(a, b) = \frac{1}{4} \left( e^{T(a)} - e^{-T(a)} \right) \left( e^{T(b)} + e^{-T(b)} \right) \frac{2e^b}{(e^b + 1)^2}. \]
And
\[ \left( \ln a - \ln b \right) (af'_1 - bf'_2) \]
\[ = \left( \ln a - \ln b \right) \left( e^{T(a)} + e^{-T(a)} \right) \left( e^{T(b)} - e^{-T(b)} \right) \frac{ae^a}{(e^a + 1)^2} - \left( e^{T(a)} - e^{-T(a)} \right) \left( e^{T(b)} + e^{-T(b)} \right) \frac{be^b}{(e^b + 1)^2}. \]

Let \( a > b > 0, T(a) = u = \frac{e^a - 1}{e^a + 1}, T(b) = v = \frac{e^b - 1}{e^b + 1}, \) then \( u > v, 0 < u, v < 1, a = \ln (1 + u) - \ln (1 - u), b = \ln (1 + v) - \ln (1 - v), \) so
\[ \left( \ln a - \ln b \right) (af'_1 - bf'_2) \leq 0 \]
\[ \iff \frac{e^{T(a)} + e^{-T(a)}}{e^{T(a)} - e^{-T(a)}} \cdot \frac{ae^a}{(e^a + 1)^2} - \frac{e^{T(b)} + e^{-T(b)}}{e^{T(b)} - e^{-T(b)}} \cdot \frac{be^b}{(e^b + 1)^2} \leq 0 \]
By Lemma 2.2 and $u > v$, the previous inequality holds. So $f$ is a $S$-geometrically concave function. Let $g(a, b) = \sqrt{1 + f(a, b)}$, $(a, b) \in R^2_+$, from Definition 1.5, we know $g$ is a $S$-geometrically concave function. Then from Definition 1.5 and Lemma 1.1, we immediately get

$$\sqrt{1 + f(a, b)} + \sqrt{f(a, b)} \leq \sqrt{1 + f(\sqrt{ab}, \sqrt{ab})} + \sqrt{f(\sqrt{ab}, \sqrt{ab})} = \exp\left(\frac{e^{\sqrt{ab}} - 1}{e^{\sqrt{ab}} + 1}\right)$$

Thus, the solution of the question is completed.

REFERENCES


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