ON AN INEQUALITY OF DIANANDA, II

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1. Introduction

Let $P_{n,r}(x)$ be the generalized weighted means: $P_{n,r}(x) = \left(\sum_{i=1}^{n} q_i x_i^r\right)^{\frac{1}{r}}$, where $P_{n,0}(x)$ denotes the limit of $P_{n,r}(x)$ as $r \to 0^+$, $x = (x_1, x_2, \ldots, x_n)$ and $q_i > 0$ $(1 \leq i \leq n)$ are positive real numbers with $\sum_{i=1}^{n} q_i = 1$. In this paper, we let $q = \min q_i$ and always assume $n \geq 2, 0 \leq x_1 < x_2 < \cdots < x_n$.

We define $A_n(x) = P_{n,1}(x)$, $G_n(x) = P_{n,0}(x)$, $H_n(x) = P_{n,-1}(x)$ and we shall write $P_{n,r}$ for $P_{n,r}(x)$, $A_n$ for $A_n(x)$ and similarly for other means when there is no risk of confusion.

For mutually distinct numbers $r, s, t$ and any real number $\alpha, \beta$, we define

$$\Delta_{r,s,t,\alpha,\beta} = \left| \frac{P_{n,r}^{\alpha} - P_{n,t}^{\alpha}}{P_{n,r}^{\beta} - P_{n,t}^{\beta}} \right|,$$

where we interpret $P_{n,r}^0 - P_{n,s}^0$ as $\ln P_{n,r} - \ln P_{n,s}$. When $\alpha = \beta$, we define $\Delta_{r,s,t,\alpha}$ to be $\Delta_{r,s,t,\alpha,\alpha}$. We also define $\Delta_{r,s,t}$ to be $\Delta_{r,s,t,1}$.

Bounds for $\Delta_{r,s,t,\alpha,\beta}$ have been studied by many mathematicians. For the case $\alpha \neq \beta$, we refer the reader to the articles [2, 7, 10] for the detailed discussions. In the case $\alpha = \beta$ and $r > s > t$, we seek the following bound

$$(1.1) \quad f_{r,s,t,\alpha}(q) \geq \Delta_{r,s,t,\alpha} \geq g_{r,s,t,\alpha}(q),$$

where $f_{r,s,t,\alpha}(q)$ is a decreasing function of $q$ and $g_{r,s,t,\alpha}(q)$ is an increasing function of $q$.

For $r = 1, s = 0, \alpha = 0, t = -1$ in (1.1), we can take $f_{1,0,0,0}(q) = 1/q, g_{1,0,0,0}(q) = 1/(1 - q)$, when $q_i = 1/n, 1 \leq i \leq n$, this is the well-known Sierpiński’s inequality[12](see [5] for a refinement of this). If we further require $t, \alpha > 0$, then consideration of the case $n = 2, x_1 \to 0, x_2 \to 1$ leads to the choice $f_{r,s,t,\alpha} = C_{r,s,t}(1 - q^{\alpha}), g_{r,s,t,\alpha} = C_{r,s,t}(q^{\alpha})$, where

$$C_{r,s,t}(x) = \frac{1 - x^{1/t - 1/r}}{1 - x^{1/s - 1/r}}, t > 0; C_{r,s,0}(x) = \frac{1}{1 - x^{1/s - 1/r}}.$$

We will show in Lemma 2.1 that $C_{r,s,t}(x)$ is an increasing function of $x (0 < x < 1)$ so the above choice for $f, g$ is plausible. From now on, we will assume $f, g$ to be so chosen.

Note when $t > 0$, the limiting case $\alpha \to 0$ in (1.1) leads to Liapunov’s inequality(see [8, p. 27]):

$$(1.2) \quad \Delta_{r,s,t,0} = \frac{\ln P_{n,r} - \ln P_{n,t}}{\ln P_{n,r} - \ln P_{n,s}} \leq \frac{s(r - t)}{t(r - s)} =: C(r, s, t).$$

From this(or by letting $q \to 0$ when $\alpha = 1$), one easily deduces the following result of H.Hsu[9](see also [1]): $\Delta_{r,s,t} \leq C(r, s, t), r > s > t > 0$.

The consideration of $n = 2, x_2 \to x_1$ shows that the two inequalities in (1.1) can’t hold simultaneously and from now on by saying (1.1) holds for $r > s > t \geq 0, \alpha > 0$, we mean the left-hand side

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inequality in (1.1) holds when \( C_{r,s,t}(1 - q)^\alpha \geq (r - t)/(r - s) \) and the right-hand side inequality in (1.1) holds when \((r - t)/(r - s) \geq C_{r,s,t}(q^\alpha)\).

More generally, for any set \( \{a, b, c\} \) with \( a, b, c \) mutually distinct and non-negative, we let \( r = \max\{a, b, c\}, t = \min\{a, b, c\}, s = \{a, b, c\}/\{r, t\} \). By saying (1.1) holds for the set \( \{a, b, c\}, \alpha > 0 \) we mean (1.1) holds for \( r > s > t \geq 0, \alpha > 0 \).

In the case \( \alpha = 1 \), a result of P. Diananda[3], [4])(see also [1], [11]) shows (1.1) holds for \( \{1, 1/2, 0\} \) and recently, the author[6] has shown that (1.1) holds for \( \{1, 0\} \) and \( \{r, 1, 1/2\} \), where \( r \in (0, \infty) \). It is the goal of this paper to further extend the result in [6].

2. Lemmas

Lemma 2.1. For \( 0 < x < 1, 0 \leq t < s < r, C_{r,s,t}(x) \) is a strictly increasing function of \( x \). In particular, for \( 0 < q \leq 1/2, C_{r,s,t}(1 - q) \geq C_{r,s,t}(q) \).

Proof. We may assume \( t > 0 \). Note \( C_{r,s,t}(x) = C_{1,t/r-s/r}(x^{1/r}) \), thus it suffices to prove the lemma for \( C_{1,r,s} \) with \( 1 > r > s > 0 \). By the mean value theorem,

\[
\frac{1/s - 1}{1/r - 1} \cdot \frac{1 - x^{1/r - 1}}{1 - x^{1/s - 1}} = 1^{1/r - 1/s} < x^{1/r - 1/s}
\]

for some \( x < \eta < 1 \) and this implies \( C_{1,r,s}'(x) > 0 \) which completes the proof. \( \Box \)

Lemma 2.2. For \( 1/2 < r < 1, C_{1,1-r,1}(1/2) > r/(1 - r) \).

Proof. By setting \( x = r/(1 - r) > 1 \), it suffices to show \( f(x) > 0 \) where \( f(x) = 1 - 2^{-x} - x(1 - 2^{-1/x}) \). Now \( f''(x) = (\ln 2)^2 2^{-x} x^{-3} (2^{x-1/x} - x^{-3}) \) and let \( g(x) = (x - 1/x) \ln 2 - 3 \ln x \). Note \( g'(x) \) has one root in \( (1, \infty) \) and \( g(1) = 0 \). It follows that \( g(x) \) hence \( f''(x) \) has only one root \( x_0 \) in \( (1, \infty) \). Note when \( f'(x) > 0 \) for \( x > x_0 \), this together with the observation that \( f(1) = 0, f'(1) = 0 \), \( 1/2 > 0, \lim_{x \to \infty} f(x) = 1 - \ln 2 = 0 \) shows \( f(x) > 0 \) for \( x > 1 \). \( \Box \)

Lemma 2.3. Let \( 0 < q \leq 1/2 \). For \( 0 < s < r < 1, r + s \geq 1, C_{r,s,1}(1 - q) > (1 - s)/(1 - r) \). For \( 0 \leq s < 1 < r, C_{r,1,s}(1 - q) > (r - s)/(r - 1) \) and for \( 1 < s < r, C_{r,s,1}(1 - q) > (r - 1)/(r - s) \).

Proof. We shall give a proof for the case \( 1 > r > s > 0, r + s \geq 1 \) and the proofs for the other cases are similar. We note first that in this case \( 1/2 < r < 1 \). By Lemma 2.1, it suffices to prove \( C_{r,s,1}(1/2) > (1 - s)/(1 - r) \). Consider

\[
f(s) = (1 - r)(1 - \left(\frac{1}{2}\right)^{1/s - 1}) - (1 - s)(1 - \left(\frac{1}{2}\right)^{1/r - 1}).
\]

We have \( f(r) = 0 \) and Lemma 2.2 implies \( f(1 - r) > 0 \). Now \( f'(r) = 2^{1-1/r} g(1/r) \) where \( g(x) = - \ln 2(x^2 - x) + 2^{x-1} - 1 \) with \( 1 < x < 2 \). One checks easily \( g(1) = g'(1) = 0, g''(x) < 0 \) which implies \( g(x) < 0 \). Hence \( f'(r) < 0 \), this combined with the observation that

\[
f''(s) = (1 - r) g(1/s - 1) (2 s - \ln 2) / s^4
\]

has at most one root and \( f''(r) > 0, f'(1 - r) > 0, f(r) = 0 \) implies \( f(s) > 0 \) for \( 1 - r \leq s < r \). \( \Box \)

3. The Main Theorems

Theorem 3.1. Let \( \alpha = 1 \). For the set \( \{1, r, s\} \), with \( 1, r, s \) mutually distinct and \( r > s \geq 0, r + s \geq 1 \), the left-hand side inequality of (1.1) holds. The equality holds if and only if \( n = 2, x_1 = 0, q_1 = q \).

Proof. The case \( s = 0 \) was treated in [6], so we may assume \( s > 0 \) here. We shall give a proof for the case \( 1 > r > s > 0 \) here and the proofs for the other cases are similar. Define

\[
D_n(x) = A_n - P_n r - C(1 - q)(A_n - P_n s), C(x) = \frac{1 - x^{1/r - 1}}{1 - x^{1/s - 1}}.
\]
By Lemma 2.3, we need to show $D_n \geq 0$ and we have

$$\frac{1}{q_n} \frac{\partial D_n}{\partial x_n} = 1 - P_{n,r}^{1-r}x^{r-1} - C(1-q)(1 - P_{n,s}^{1-s}x^{s-1}).$$

By a change of variables: $\frac{x_i}{n} \rightarrow x_i, 1 \leq i \leq n$, we may assume $0 \leq x_1 < x_2 < \cdots < x_n = 1$ in (3.1) and rewrite it as

$$g_n(x_1, \cdots, x_{n-1}) := 1 - P_{n,r}^{1-r} - C(1-q)(1 - P_{n,s}^{1-s}).$$

We want to show $g_n \geq 0$. Let $a = (a_1, \cdots, a_{n-1}) \in [0,1]^{n-1}$ be the point in which the absolute minimum of $g_n$ is reached. We may assume $a_1 \leq a_2 \leq \cdots \leq a_{n-1}$. If $a_i = a_{i+1}$ for some $1 \leq i \leq n-2$ or $a_{n-1} = 1$, by combing $a_i$ with $a_{i+1}$ and $q_i$ with $q_{i+1}$ or $a_{n-1}$ with 1 and $q_{n-1}$ with $g_n$, it follows from Lemma 2.1 that we can reduce the determination of the absolute minimum of $g_n$ to that of $g_{n-1}$ with different weights. Thus without loss of generality, we may assume $a_1 < a_2 < \cdots < a_{n-1} < 1$.

If $a$ is a boundary point of $[0,1]^{n-1}$, then $a_1 = 0$, and we can regard $g_n$ as a function of $a_2, \cdots, a_{n-1}$, then we obtain

$$\nabla g_n(a_2, \cdots, a_{n-1}) = 0.$$

Otherwise $a_1 > 0$, $a$ is an interior point of $[0,1]^{n-1}$ and

$$\nabla g_n(a_1, \cdots, a_{n-1}) = 0.$$

In either case $a_2, \cdots, a_{n-1}$ solve the equation

$$(r - 1)P_{n,r}^{1-2r}x^{r-1} + C(1-q)(1-s)P_{n,s}^{1-2s}x^{s-1} = 0.$$

The above equation has at most one root (regarding $P_{n,r}, P_{n,s}$ as constants), so we only need to show $g_n \geq 0$ for the case $n = 3$ with $0 = a_1 < a_2 = x < a_3 = 1$ in (3.2). In this case we regard $g_3$ as a function of $x$ and we get

$$\frac{1}{q_2} g_3'(x) = P_{3,r}^{1-2r}x^{r-1}h(x),$$

where

$$h(x) = r - 1 + (1-s)C(1-q)(q_2x^{s/2} + q_3x^{r-s/2})(1-2s)/(q_2x^{r/2} + q_3x^{r-s/2})^{(2r-1)/r}. $$

If $q_2 = 0$ (note $q_3 > 0$), then

$$h(x) = r - 1 + (1-s)C(1-q)q_3^{1/s-1/r}x^{s-r}.$$ 

One easily checks that in this case $h(x)$ has exactly one root in $(0, 1)$. Now assume $q_2 > 0$, then

$$h'(x) = (1-s)C(1-q)P_{3,s}^{1-3s}P_{3,r}^{1-3r}x^{-r+s/2}p(x),$$

where

$$p(x) = (r-s)(q_2^{2}x^{r+s} - q_3^{2}) + (r+s-1)q_2q_3(x^{r} - x^{s}).$$

Now

$$p'(x) = x^{s-1}((r^2 - s^2)q_2^{2}x^{r} + (r+s-1)q_2q_3(rx^{r-s}) := x^{s-1}q(x).$$

If $r+s \geq 1$ then $q'(x) > 0$ which implies there can be at most one root for $p'(x) = 0$. Since $p(0) < 0$ and $\lim_{x \to \infty} p(x) = +\infty$, we conclude that $p(x)$ hence $h'(x)$ has at most one root. Since $h(1) < 0$ by Lemma 2.3 and $\lim_{x \to 0^+} h(x) = +\infty$, this implies $h(x)$ has exactly one root in $(0, 1)$.

Thus $g_3'(x)$ has only one root $x_0$ in $(0, 1)$. Since $g_3'(1) < 0$, $g_3(x)$ takes its maximum value at $x_0$. Thus $g_3(x) \geq \min\{g_3(0), g_3(1)\} = 0$.

Thus we have shown $g_n \geq 0$, hence $\frac{\partial D_n}{\partial x_n} \geq 0$ with equality holding if and only if $n = 1$ or $n = 2, x_1 = 0, q_1 = q$. By letting $x_n$ tend to $x_{n-1}$, we have $D_n \geq D_{n-1}$ (with weights $q_1, \cdots, q_{n-2}, q_{n-1} + q_n$). Since $C$ is an increasing function of $q$, it follows by induction that $D_n > D_{n-1} > \cdots > D_2 = 0$ when $x_1 = 0, q_1 = q$ in $D_2$. Else $D_n > D_{n-1} > \cdots > D_1 = 0$. Since we assume $n \geq 2$ in this paper, this completes the proof. \qed
The relations between (1.1) and (1.2) seems to suggest that if the left-hand side inequality of (1.1) holds for \( r > s > t \geq 0, \alpha > 0 \), then the left-hand side inequality of (1.1) also holds for \( r > s > t \geq 0, k\alpha \) with \( k < 1 \) and if the right-hand side inequality of (1.1) holds for \( r > s > t \geq 0, \alpha > 0 \), then the right-hand side inequality of (1.1) also holds for \( r > s > t \geq 0, k\alpha \) with \( k > 1 \).

We don’t know the answer in general but for a special case, we have the following:

**Theorem 3.2.** Let \( r > s > 0 \), if the right-hand side inequality of (1.1) holds for \( \{r, s, 0\}, \alpha > 0 \), then it also holds for \( \{r, s, 0\}, k\alpha \) with \( k > 1 \). If the left-hand side inequality of (1.1) holds for \( \{r, s, 0\}, \alpha > 0 \), then it also holds for \( \{r, s, 0\}, k\alpha \) with \( 0 < k < 1 \).

**Proof.** We will only prove the first assertion here and the second can be proved similarly. By the assumption, we have

\[
P_n^\alpha - G_n^\alpha \geq \frac{1}{1 - (q^\alpha)^{\frac{1}{s} - \frac{1}{r}}} (P_{n,r}^\alpha - P_{n,s}^\alpha).
\]

We write the above as

\[
(3.3) \quad P_{n,s}^\alpha \geq (q^\alpha)^{\frac{1}{s} - \frac{1}{r}} P_{n,r}^\alpha + (1 - (q^\alpha)^{\frac{1}{s} - \frac{1}{r}}) G_n^\alpha.
\]

We now need to show for \( k > 1 \),

\[
P_{n,s}^{k\alpha} \geq (q^{k\alpha})^{\frac{1}{s} - \frac{1}{r}} P_{n,r}^{k\alpha} + (1 - (q^{k\alpha})^{\frac{1}{s} - \frac{1}{r}}) G_n^{k\alpha}.
\]

Note by (3.3), via setting \( w = (q^{k\alpha})^{1/s-1/r}, x = G_n/P_{n,r} \), it suffices to show

\[
f(x) = (w + (1-w)x^k)^{1/k} - w^{1/k} - (1-w^{1/k})x \leq 0,
\]

for \( 0 \leq w, x \leq 1 \). Note

\[
f'(x) = (1-w)(wx^{-k} + (1-w))^{1/k-1} - (1-w^{1/k}),
\]

thus \( f'(x) \) can have at most one root in \((0,1)\), note also \( f(0) = f(1) = 0 \) and \( f'(1) > 0 \), we then conclude \( f(x) \leq 0 \) for \( 0 \leq x \leq 1 \) and this completes the proof. \(\Box\)

**References**


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