AN OSTROWSKI TYPE INEQUALITY FOR MAPPINGS Whose SECOND DERIVATIVES ARE BOUNDED AND APPLICATIONS

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Abstract. An integral inequality of Ostrowski's type for mappings whose second derivatives are bounded is proved. Applications in Numerical Integration and for special means are pointed out.

1 Introduction

In [1], S.S. Dragomir and S. Wang obtained the following Ostrowski type inequality [2, p. 468]:

**Theorem 1.1.** Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and a differentiable on \((a, b)\). If \( f' \in L_1 (a, b) \) and there exists the constants \( \gamma, \Gamma \) so that

\[ \gamma \leq f'(x) \leq \Gamma \quad \text{for all} \quad x \in (a, b), \]

then we have the inequality:

\[ \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{4} (b-a) (\Gamma - \gamma) \]

for all \( x \in [a, b] \).

The proof used essentially the identity

\[ f(x) = \frac{1}{b-a} \int_a^b f(t) \, dt + \frac{1}{b-a} \int_a^b p(x, t) f'(t) \, dt \]

for all \( x \in [a, b] \), where \( f \) is as above and the kernel \( p(\cdot, \cdot) : [a, b]^2 \to \mathbb{R} \) is given by

\[ p(x, t) := \begin{cases} t-a & \text{if} \ t \in [a, x] \\ t-b & \text{if} \ t \in (x, b] \end{cases} \]

and Grüss’ integral inequality which says (see for example [1]) that:

\[ \left| \frac{1}{b-a} \int_a^b g(x) h(x) \, dx - \frac{1}{b-a} \int_a^b g(x) \, dx \cdot \frac{1}{b-a} \int_a^b h(x) \, dx \right| \leq \frac{1}{4} (\Phi - \varphi) (\Gamma - \gamma) \]

provided \( g, h : [a, b] \to \mathbb{R} \) are integrable and

\[ \varphi \leq g(x) \leq \Phi, \quad \gamma \leq h(x) \leq \Gamma \]

for all \( x \in [a, b] \).

The main aim of this paper is to point out a new estimation of the left membership of (1.2) and to apply it for special means and in Numerical Integration.

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2 A NEW INTEGRAL INEQUALITY

The following results holds:

**Theorem 2.1.** Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and twice differentiable on \((a, b)\), whose second derivative \( f'' : (a, b) \to \mathbb{R} \) is bounded on \((a, b)\). Then we have the inequality

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{f(b) - f(a)}{b-a} \left( x - \frac{a+b}{2} \right) \right| \leq \frac{1}{2} \left\{ \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 + \frac{1}{4} \right\} (b-a)^2 \| f'' \|_\infty
\]

for all \( x \in [a, b] \).

**Proof.** For the sake of completeness, we give a short proof of the identity (1.3) which will be used in the sequel.

Integrating by parts, we have

\[
\int_a^x (t-a) f'(t) \, dt = (x-a) f(x) - \int_a^x f(t) \, dt
\]

and

\[
\int_x^b (t-b) f'(t) \, dt = (b-x) f(x) - \int_x^b f(t) \, dt.
\]

Adding these two equalities, we get

\[
\int_a^x (t-a) f'(t) \, dt + \int_x^b (t-b) f'(t) \, dt = (b-a) f(x) - \int_a^b f(t) \, dt
\]

which is equivalent to (1.3).

Applying the identity (1.3) for \( f' (\cdot) \) we can state

\[
f'(t) = \frac{1}{b-a} \int_a^b f'(s) \, ds + \frac{1}{b-a} \int_a^b p(t,s) f''(s) \, ds
\]

which is equivalent to

\[
f'(t) = \frac{f(b) - f(a)}{b-a} + \frac{1}{b-a} \int_a^b p(t,s) f''(s) \, ds.
\]

Substituting \( f'(t) \) in the right membership of (1.3) we get

\[
f(x) = \frac{1}{b-a} \int_a^b f(t) \, dt
\]

\[
+ \frac{1}{b-a} \int_a^b p(x,t) \left[ \frac{f(b) - f(a)}{b-a} + \frac{1}{b-a} \int_a^b p(t,s) f''(s) \, ds \right] \, dt
\]
\[
\frac{1}{b-a} \int_a^b f\left(t\right) dt + \frac{f\left(b\right) - f\left(a\right)}{(b-a)^2} \int_a^b p\left(x, t\right) dt
\]

\[
+ \frac{1}{(b-a)^2} \int_a^b \int_a^b p\left(x, t\right) p\left(t, s\right) f''\left(s\right) ds dt
\]

and as
\[
\int_a^b p\left(x, t\right) dt = \int_a^x \left(t-a\right) dt + \int_x^b \left(t-b\right) dt
\]

\[
= \left(b-a\right) \left(x-a+b\right)
\]

we get the integral identity:
\[
\left| f\left(x\right) - \frac{1}{b-a} \int_a^b f\left(t\right) dt - \frac{f\left(b\right) - f\left(a\right)}{b-a} \left(x-a+b\right) \right|
\]

\[
\leq \frac{1}{(b-a)^2} \int_a^b \int_a^b \left| p\left(x, t\right) p\left(t, s\right) \right| \left| f''\left(s\right) \right| ds dt
\]

\[
\leq \left\| f'' \right\|_{\infty} \int_a^b \int_a^b \left| p\left(x, t\right) \right| \left| p\left(t, s\right) \right| ds dt.
\]

We have
\[
\int_a^b \left| p\left(t, s\right) \right| ds = \frac{(t-a)^2 + (b-t)^2}{2}.
\]

Also
\[
A := \int_a^b \left| p\left(x, t\right) \right| \left[ \frac{(t-a)^2 + (b-t)^2}{2} \right] dt
\]

\[
= \frac{1}{2} \left[ \int_a^x \left(t-a\right) \left[ (t-a)^2 + (b-t)^2 \right] dt + \int_x^b \left( b-t \right) \left[ (t-a)^2 + (b-t)^2 \right] dt \right]
\]
\[= \frac{1}{2} \left[ \int_a^x (t - a)^3 + (t - a)(b - t)^2 \, dt + \int_x^b (t - a)^2 (b - t) + (b - t)^3 \, dt \right].\]

Note that

\[\int_a^x (t - a)^3 \, dt = \frac{(x - a)^4}{4},\]

\[\int_a^x (t - a)(b - t)^2 \, dt = -\frac{1}{3} (b - x)^3 (x - a) - \frac{1}{12} (b - x)^4 + \frac{1}{12} (b - x)^4;\]

\[\int_x^b (t - b)(t - a)^2 \, dt = \frac{1}{3} (x - a)^3 (b - x) - \frac{1}{12} (b - a)^4 + \frac{1}{12} (x - a)^4;\]

\[\int_x^b (t - b)^3 \, dt = \frac{(x - b)^4}{4}.\]

Consequently, we have

\[A = \frac{1}{2} \left[ \frac{(x - a)^4}{4} - \frac{1}{3} (b - x)^3 (x - a) - \frac{1}{12} (b - x)^4 + \frac{1}{12} (b - a)^4 - \frac{1}{3} (x - a)^3 (b - x) + \frac{1}{12} (b - a)^4 - \frac{1}{12} (x - a)^4 + \frac{(x - b)^4}{4} \right] = \frac{1}{12} \left[ (x - a)^4 - 2 (b - x)^3 (x - a) - 2 (x - a)^3 (b - x) + (b - x)^4 + (b - a)^4 \right].\]

Now, observe that,

\[(x - a)^4 + (b - x)^4 = \left[ (x - a)^2 + (b - x)^2 \right]^2 - 2 (x - a)^2 (b - x)^2\]

and

\[-2 (b - x)^3 (x - a) - 2 (x - a)^3 (b - x)\]

\[= -2 (x - a) (b - x) \left[ (x - a)^2 + (b - x)^2 \right].\]
then
\[ B := 12A = \left[ (x - a)^2 + (b - x)^2 \right]^2 - 2(x - a)(b - x)\left[ (x - a)^2 + (b - x)^2 \right] \\
-2(x - a)^2(b - x)^2 + (b - a)^4 \\
= \left[ (x - a)^2 + (b - x)^2 - (x - a)(b - x) \right]^2 - 3(x - a)^2(b - x)^2 + (b - a)^4. \]

But a simple calculation shows that
\[ (x - a)^2 + (b - x)^2 = \frac{1}{2} (b - a)^2 + 2\left( x - \frac{a + b}{2} \right)^2 \]
and as
\[ (x - a)^2 + (b - x)^2 + 2(x - a)(b - x) = (b - a)^2 \]
we get
\[ 2(x - a)(b - x) = (b - a)^2 - \left[ (x - a)^2 + (b - x)^2 \right] \]
i.e.,
\[ (x - a)(b - x) = \frac{1}{2} (b - a)^2 - \frac{1}{2} \left[ (x - a)^2 + (b - x)^2 \right] \]
\[ = \frac{1}{4} (b - a)^2 - \left( x - \frac{a + b}{2} \right)^2. \]

Consequently,
\[ B = \left[ \frac{1}{2} (b - a)^2 + 2\left( x - \frac{a + b}{2} \right)^2 - \frac{1}{4} (b - a)^2 + \left( x - \frac{a + b}{2} \right)^2 \right]^2 - \\
-3 \left[ \frac{1}{4} (b - a)^2 - \left( x - \frac{a + b}{2} \right)^2 \right]^2 + (b - a)^4 \]
\[ = 6\left( x - \frac{a + b}{2} \right)^2 + 3(b - a)^2\left( x - \frac{a + b}{2} \right)^2 + \frac{7}{8} (b - a)^4 \]
and then
\[ A = \frac{1}{12} \left[ 6\left( x - \frac{a + b}{2} \right)^4 + 3(b - a)^2\left( x - \frac{a + b}{2} \right)^2 + \frac{7}{8} (b - a)^4 \right]. \]

Now, using the inequality (2.3) and simple algebraic manipulations, we get the first result in (2.1).

The second part is obvious by the fact that
\[ \left| x - \frac{a + b}{2} \right| \leq \frac{b - a}{2} \]
for all \( x \in [a, b] \).
3 Applications in Numerical Integration

Let \( I_h : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \) be a division of the interval \([a, b]\), \( \xi_i \in [x_i, x_{i+1}] \) \((i = 0, 1, \ldots, n - 1)\) a sequence of intermediate points and \( h_i := x_{i+1} - x_i \) \((i = 0, 1, \ldots, n - 1)\). As in [1], consider the perturbed Riemann’s sum defined by

\[
(3.1) \quad A_G (f, I_h, \xi) := \sum_{i=0}^{n-1} f (\xi_i) h_i - \sum_{i=0}^{n-1} \left( \xi_i - \frac{x_i + x_{i+1}}{2} \right) (f (x_{i+1}) - f (x_i)).
\]

In that paper Dragomir and Wang proved the following result:

**Theorem 3.1.** Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\), whose derivative \( f' : (a, b) \to \mathbb{R} \) is bounded on \((a, b)\) and assume that

\[
(3.2) \quad \gamma \leq f' (x) \leq \Gamma \quad \text{for all} \quad x \in (a, b).
\]

Then we have the quadrature formula:

\[
(3.3) \quad \int_a^b f (x) \, dx = A_G (f, I_h, \xi) + R_G (f, I_h, \xi)
\]

and the remainder \( R_G (f, I_h, \xi) \) satisfies the estimation

\[
(3.4) \quad |R_G (f, I_h, \xi)| \leq \frac{1}{4} (\Gamma - \gamma) \sum_{i=0}^{n-1} h_i^2,
\]

for all \( \xi = (\xi_0, \ldots, \xi_{n-1}) \) as above.

Here, we prove another type of estimation for the remainder \( R_G (f, I_h, \xi) \) in the case when \( f \) is twice differentiable.

**Theorem 3.2.** Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and twice differentiable on \((a, b)\), whose second derivative \( f'' : (a, b) \to \mathbb{R} \) is bounded on \((a, b)\). Denote \( \|f''\|_\infty := \sup_{t \in (a, b)} |f'' (t)| < \infty \). Then we have the quadrature formula (3.3) and the remainder \( R_G (f, I_h, \xi) \) satisfies the estimation:

\[
(3.5) \quad |R_G (f, I_h, \xi)| \leq \frac{\|f''\|_\infty}{2} \sum_{i=0}^{n-1} \left[ \left( \frac{\xi_i - x_i + x_{i+1}}{2} \right)^2 + \frac{1}{4} \right] \left( \frac{1}{12} \right) \left( h_i^3 \right)
\]

\[
\leq \frac{\|f''\|_\infty}{6} \sum_{i=0}^{n-1} h_i^3
\]

for all \( \xi_i \) as above.

**Proof.** Apply Theorem 2.1 on the interval \([x_i, x_{i+1}]\) \((i = 0, \ldots, n - 1)\) to obtain

\[
\left| f (\xi_i) h_i - \int_{x_i}^{x_{i+1}} f (t) \, dt - \left( \xi_i - \frac{x_i + x_{i+1}}{2} \right) (f (x_{i+1}) - f (x_i)) \right|
\]

\[
\leq \frac{\|f''\|_\infty}{2} \left[ \left( \frac{\xi_i - x_i + x_{i+1}}{2} \right)^2 + \frac{1}{4} \right] \left( \frac{1}{12} \right) \left( h_i^3 \right) \leq \frac{\|f''\|_\infty}{6} h_i^3
\]

for all \( \xi_i \in [x_i, x_{i+1}] \) and \( i \in \{0, \ldots, n - 1\} \).

Summing over \( i \) from 0 to \( n - 1 \) and using the generalized triangle inequality, we get the desired inequality (3.5).

We omit the details.
4 Applications for Special Means

Recall the following means:

(a) The arithmetic mean
\[ A = A(a, b) := \frac{a + b}{2}, \quad a, b \geq 0; \]

(b) The geometric mean:
\[ G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0; \]

(c) The harmonic mean:
\[ H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a, b \geq 0; \]

(d) The logarithmic mean:
\[ L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b - a}{\ln b - \ln a} & \text{if } a \neq b \end{cases}, \quad a, b > 0; \]

(e) The identric mean:
\[ I := I(a, b) = \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases}, \quad a, b > 0; \]

(f) The \( p \)-logarithmic mean:
\[ L_p = L_p(a, b) := \begin{cases} \left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b - a)} \right]^\frac{1}{p} & \text{if } a \neq b; \\ a & \text{if } a = b \end{cases}, \quad a, b > 0. \]

where \( p \in \mathbb{R} \setminus \{-1, 0\} \) and \( a, b > 0 \).

The following simple relationships are well known in the literature:

\[ H \leq G \leq L \leq I \leq A \quad (4.1) \]

and

\[ L_p \text{ is monotonically increasing in } p \in \mathbb{R} \text{ with } L_0 := I \text{ and } L_{-1} := L. \quad (4.2) \]

1. Consider the mapping \( f(x) = x^p \) \( (p \geq 2) \) on \([a, b] \subset (0, \infty)\).

Applying the inequality \( (2.1) \) for \( f(x) = x^p \), we get:

\[ |x^p - L_p^p - pL_{p-1}^{p-1}(x - A)| \leq \frac{p(p-1)b^{p-2}}{2} \left\{ \left(\frac{(x - A)^2}{(b - a)^2} + \frac{1}{4} \right)^2 + \frac{1}{12} \right\} (b - a)^2 \quad (4.3) \]
\[ \leq \frac{p(p-1)b^{p-2}}{6} (b-a)^2 \]

for all \( x \in [a,b] \).

Choosing in (4.3), \( x = A \), we get

\[ (4.4) \quad 0 \leq L_p^p - A^p \leq \frac{7}{96} p(p-1)b^{p-2} (b-a)^2. \]

2. Consider the mapping \( f(x) = \frac{1}{x} \) \( (x \in [a,b] \subset (0,\infty)) \).

Applying the inequality (2.1) for this mapping we get:

\[ (4.5) \quad \left| \frac{1}{x} - \frac{1}{L - \frac{x-A}{G^2}} \right| \]

\[ \leq \frac{1}{3a^3} \left\{ \left[ \frac{(x-A)^2}{(b-a)^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} (b-a)^2 \leq \frac{1}{3a^3} (b-a)^2 \]

for all \( x \in [a,b] \).

Choosing in (4.5) \( x = A \), we get

\[ (4.6) \quad 0 \leq \frac{A-L}{AL} \leq \frac{7}{48a^3} (b-a)^2. \]

Also, choosing in (4.5) \( x = L \), we get

\[ (4.7) \quad 0 \leq \frac{A-L}{G^2} \leq \frac{1}{3a^3} \left\{ \left[ \frac{(x-A)^2}{(b-a)^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} (b-a)^2 \]

\[ \leq \frac{1}{3a^3} (b-a)^2. \]

3. Finally, let us consider the mapping \( f(x) = -\ln x \) \( (x \in [a,b] \subset (0,\infty)) \). Then, by (2.1), we get:

\[ (4.8) \quad \ln \left( \frac{I \left( \frac{b}{a} \right)}{x} \right) \leq \frac{1}{2a^2} \left\{ \left[ \frac{(x-A)^2}{(b-a)^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} (b-a)^2 \leq \frac{1}{6a^2} (b-a)^2 \]

for all \( x \in [a,b] \).

Putting \( x = A \) in (4.8) we get

\[ (4.9) \quad 1 \leq \frac{A}{I} \leq \exp \left[ \frac{7}{96a^2} (b-a)^2 \right]. \]

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