INEQUALITIES OF JENSEN TYPE FOR \( \varphi \)-CONVEX FUNCTIONS

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Abstract. Some inequalities of Jensen type for \( \varphi \)-convex functions defined on real intervals are given.

1. Introduction

We recall here some concepts of convexity that are well known in the literature. Let \( I \) be an interval in \( \mathbb{R} \).

Definition 1 ([38]). We say that \( f : I \to \mathbb{R} \) is a Godunova-Levin function or that \( f \) belongs to the class \( Q(I) \) if \( f \) is non-negative and for all \( x, y \in I \) and \( t \in (0, 1) \) we have

\[
(1.1) \quad f \left( tx + (1 - t)y \right) \leq \frac{1}{t} f(x) + \frac{1}{1 - t} f(y).
\]

Some further properties of this class of functions can be found in [29], [30], [32], [44], [47] and [48]. Among others, it has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

Definition 2 ([32]). We say that a function \( f : I \to \mathbb{R} \) belongs to the class \( P(I) \) if it is non-negative and for all \( x, y \in I \) and \( t \in [0, 1] \) we have

\[
(1.2) \quad f \left( tx + (1 - t)y \right) \leq f(x) + f(y).
\]

Obviously \( Q(I) \) contains \( P(I) \) and for applications it is important to note that also \( P(I) \) contain all nonnegative monotone, convex and quasi convex functions, i.e. nonnegative functions satisfying

\[
(1.3) \quad f \left( tx + (1 - t)y \right) \leq \max \{ f(x), f(y) \}
\]

for all \( x, y \in I \) and \( t \in [0, 1] \).

For some results on \( P \)-functions see [32] and [45] while for quasi convex functions, the reader can consult [31].

Definition 3 ([7]). Let \( s \) be a real number, \( s \in (0, 1] \). A function \( f : [0, \infty) \to [0, \infty) \) is said to be \( s \)-convex (in the second sense) or Breckner \( s \)-convex if

\[
(1.4) \quad f \left( tx + (1 - t)y \right) \leq t^s f(x) + (1 - t)^s f(y)
\]

for all \( x, y \in [0, \infty) \) and \( t \in [0, 1] \).

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For some properties of this class of functions see [1], [2], [7], [8], [27], [28], [39], [41] and [50].

In order to unify the above concepts for functions of real variable, S. Varoš anec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in \( \mathbb{R} \), \((0,1) \subseteq J \) and functions \( h \) and \( f \) are real non-negative functions defined in \( J \) and \( I \), respectively.

**Definition 4** ([53]). Let \( h : J \to [0, \infty) \) with \( h \) not identical to 0. We say that \( f : I \to [0, \infty) \) is an h-convex function if for all \( x, y \in I \) we have

\[
(1.4) \quad f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y)
\]

for all \( t \in (0,1) \).

For some results concerning this class of functions see [53], [6], [42], [51], [49] and [52].

We can introduce now another class of functions.

**Definition 5.** We say that the function \( f : I \to [0, \infty) \) is of s-Godunova-Levin type, with \( s \in [0,1] \), if

\[
(1.5) \quad f(tx + (1-t)y) \leq \frac{1}{t^s}f(x) + \frac{1}{(1-t)^s}f(y),
\]

for all \( t \in (0,1) \) and \( x, y \in I \).

We observe that for \( s = 0 \) we obtain the class of P-functions while for \( s = 1 \) we obtain the class of Godunova-Levin. If we denote by \( Q_s(I) \) the class of s-Godunova-Levin functions defined on \( I \), then we obviously have

\[
P(I) = Q_0(I) \subseteq Q_{s_1}(I) \subseteq Q_{s_2}(I) \subseteq Q_1(I) = Q(I)
\]

for \( 0 \leq s_1 \leq s_2 \leq 1 \).

The following inequality holds for any convex function \( f \) defined on \( \mathbb{R} \)

\[
(1.6) \quad (b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x)dx < (b-a)\frac{f(a) + f(b)}{2}, \quad a, b \in \mathbb{R}.
\]

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [43]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite’s result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite’s note in *Mathesis* [43]. Since (1.6) was known as Hadamard’s inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [10]-[19], [22]-[26], [33]-[36] and [46].

The following inequality of Hermite-Hadamard type for h-convex function holds [49].

**Theorem 1.** Assume that the function \( f : I \to [0, \infty) \) is an h-convex function with \( h \in L[0,1] \). Let \( y, x \in I \) with \( y \neq x \) and assume that the mapping \([0,1] \ni t \mapsto f((1-t)x+ty)\) is Lebesgue integrable on \([0,1]\). Then

\[
(1.7) \quad \frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)du \leq [f(x) + f(y)]\int_0^1 h(t)dt.
\]
If we write (1.7) for \( h(t) = t \), then we get the classical Hermite-Hadamard inequality for convex functions

\[
(1.8) \quad f \left( \frac{x + y}{2} \right) \leq \frac{1}{y - x} \int_x^y f(u) \, du \leq \frac{f(x) + f(y)}{2}.
\]

If we write (1.7) for the case of \( P \)-type functions \( f : I \rightarrow [0, \infty) \), i.e., \( h(t) = 1, t \in [0, 1] \), then we get the inequality

\[
(1.9) \quad \frac{1}{2} f \left( \frac{x + y}{2} \right) \leq \frac{1}{y - x} \int_x^y f(u) \, du \leq f(x) + f(y),
\]

that has been obtained for functions of real variable in [32].

If \( f \) is Breckner \( s \)-convex on \( I \), for \( s \in (0, 1) \), then by taking \( h(t) = t^s \) in (1.7) we get

\[
(1.10) \quad 2^{s-1} f \left( \frac{x + y}{2} \right) \leq \frac{1}{y - x} \int_x^y f(u) \, du \leq \frac{f(x) + f(y)}{s + 1},
\]

that was obtained for functions of a real variable in [27].

If \( f : I \rightarrow [0, \infty) \) is of \( s \)-Godunova-Levin type, with \( s \in [0, 1) \), then

\[
(1.11) \quad \frac{1}{2^{s+r}} f \left( \frac{x + y}{2} \right) \leq \frac{1}{y - x} \int_x^y f(u) \, du \leq \frac{f(x) + f(y)}{1 - s}.
\]

We notice that for \( s = 1 \) the first inequality in (1.11) still holds, i.e.

\[
(1.12) \quad \frac{1}{4} f \left( \frac{x + y}{2} \right) \leq \int_0^1 f \left( (1 - t)x + ty \right) \, dt.
\]

The case for functions of real variables was obtained for the first time in [32].

2. \( \varphi \)-Convex Functions

We introduce the following class of \( h \)-convex functions.

**Definition 6.** Let \( \varphi : (0, 1) \rightarrow (0, \infty) \) a measurable function. We say that the function \( f : I \rightarrow [0, \infty) \) is a \( \varphi \)-convex function on the interval \( I \) if for all \( x, y \in I \) we have

\[
(2.1) \quad f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)
\]

for all \( t \in (0, 1) \).

If we denote \( \ell(t) = t \), the identity function, then it is obvious that \( f \) is \( h \)-convex with \( h = \ell \varphi \). Also, all the examples from the introduction can be seen as \( \varphi \)-convex functions with appropriate choices of \( \varphi \).

If we take \( \varphi(t) = \frac{1}{t^s} \) with \( s \in [0, 1] \), then we get the class of \( s \)-Godunova-Levin functions. Also, if we put \( \varphi(t) = t^{s-1} \) with \( s \in (0, 1) \), then we get the concept of Breckner \( s \)-convexity. We notice that for all these examples we have

\[
\varphi(+0) := \lim_{t \to 0^+} \varphi(t) = \infty.
\]

The case of convex functions, i.e. when \( \varphi(t) = 1 \) is the only example from above for which \( \varphi(+0) \) is finite, namely \( \varphi(+0) = 1 \).

Consider the family of functions, for \( p > 1 \) and \( k > 0 \)

\[
(2.2) \quad \delta(p,k) : [0, 1] \rightarrow \mathbb{R}, \quad \delta(p,k)(t) = k(1-t)^p + 1.
\]
We observe that $\delta_+(p, k) (0) = \delta(p, k) (0) = k + 1$, $\delta(p, k)$ is strictly decreasing on $[0, 1]$ and $\delta(p, k)(t) \geq \delta(p, k)(1) = 1$.

**Definition 7.** We say that the function $f : I \to [0, \infty)$ is a $\delta(p, k)$-convex function on the interval $I$ if for all $x, y \in I$ we have
\[ f(tx + (1-t)y) \leq t[k(1-t)^p + 1]f(x) + (1-t)(kt^p + 1)f(y) \tag{2.3} \]
for all $t \in (0, 1)$.

It is obvious that any nonnegative convex function is a $\delta(p, k)$-convex function for any $p > 1$ and $k > 0$.

For $m > 0$ we consider the family of functions
\[ \eta(m) : [0, 1] \to \mathbb{R}_+, \eta(m)(t) := \exp[m(1-t)]. \]
We observe that $\eta_+(m)(0) = \eta(m)(0) = \exp(m)$, $\eta(m)$ is strictly decreasing on $[0, 1]$ and $\eta(m)(t) \geq \eta(m)(1) = 1$.

**Definition 8.** We say that the function $f : I \to [0, \infty)$ is a $\eta(m)$-convex function on the interval $I$ if for all $x, y \in I$ we have
\[ f(tx + (1-t)y) \leq t \exp[m(1-t)]f(x) + (1-t)\exp(mt) f(y) \tag{2.4} \]
for all $t \in (0, 1)$.

It is obvious that any nonnegative convex function is a $\eta(m)$-convex function for any $m > 0$.

There are many other examples one can consider. In fact any continuous function $\varphi : [0, 1] \to [1, \infty)$ can generate a class of $\varphi$-convex function that contains the class of nonnegative convex functions.

Utilising Theorem 1 we can state the following result.

**Theorem 2.** Assume that the function $f : I \to [0, \infty)$ is a $\varphi$-convex function with $\ell \varphi \in L [0, 1]$. Let $y, x \in I$ with $y \neq x$ and assume that the mapping $[0, 1] \ni t \mapsto f([(1-t) x + ty]$ is Lebesgue integrable on $[0, 1]$. Then
\[ \frac{1}{\varphi\left(\frac{1}{2}\right)} \frac{1}{y-x} \int_x^y f(u) du \leq \int_0^1 t \varphi(t) dt. \tag{2.5} \]

The proof follows from (1.7) by taking $h(t) = t \varphi(t)$, $t \in (0, 1)$.

**Remark 1.** We notice that, since $\int_0^1 t \varphi(t) dt$ can be seen as the expectation of a random variable $X$ with the density function $\varphi$, the inequality (2.5) provides a connection to Probability Theory and motivates the introduction of $\varphi$-convex function as a natural concept, having available many examples of density functions $\varphi$ that arise in applications.

For different inequalities related to these classes of functions, see [1]-[4], [6], [9]-[37], [40]-[42] and [45]-[52].

A function $h : J \to \mathbb{R}$ is said to be supermultiplicative if
\[ h(ts) \geq h(t)h(s) \tag{2.6} \]
for any $t, s \in J$. If the inequality (2.6) is reversed, then $h$ is said to be submultiplicative. If the equality holds in (2.6) then $h$ is said to be a multiplicative function on $J$. 
In [53] it has been noted that if \( h : [0, \infty) \to [0, \infty) \) with \( h(t) = (x + c)^p - 1 \), then for \( c = 0 \) the function \( h \) is multiplicative. If \( c \geq 1 \), then for \( p \in (0, 1) \) the function \( h \) is supermultiplicative and for \( p > 1 \) the function is submultiplicative.

We observe that, if \( h, g \) are nonnegative and supermultiplicative, the same is their product. In particular, if \( h \) is supermultiplicative then its product with a power function \( f_r(t) = t^r \) is also supermultiplicative.

The case of \( h \)-convex function with \( h \) supermultiplicative is of interest due to several Jensen type inequalities one can derive.

The following results were obtained in [53] for functions of a real variable.

**Theorem 3.** Let \( h : J \to [0, \infty) \) be a supermultiplicative function on \( J \). If the function \( f : I \to [0, \infty) \) is \( h \)-convex on the interval \( I \), then for any \( w_i \geq 0, x_i \in I, \ i \in \{1, \ldots, n\}, n \geq 2 \) with \( W_n := \sum_{i=1}^{n} w_i > 0 \) we have

\[
f\left( \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \right) \leq \sum_{i=1}^{n} h\left( \frac{w_i}{W_n} \right) f(x_i).
\]

In particular, we have the unweighted inequality

\[
f\left( \frac{1}{n} \sum_{i=1}^{n} x_i \right) \leq h\left( \frac{1}{n} \right) \sum_{i=1}^{n} f(x_i).
\]

Let \( h(z) = \sum_{n=0}^{\infty} a_n z^n \) be a power series with complex coefficients and convergent on the open disk \( D(0, R) \subset \mathbb{C}, R > 0 \). We have the following examples

\[
h(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = \ln \frac{1}{1 - z}, \ z \in D(0, 1);
\]
\[
h(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \ z \in \mathbb{C};
\]
\[
h(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \ z \in \mathbb{C};
\]
\[
h(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}, \ z \in D(0, 1).
\]

Other important examples of functions as power series representations with non-negative coefficients are:

\[
h(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = \exp(z) \quad z \in \mathbb{C},
\]
\[
h(z) = \sum_{n=1}^{\infty} \frac{1}{2n - 1} z^{2n-1} = \frac{1}{2} \ln \left( \frac{1 + z}{1 - z} \right), \quad z \in D(0, 1);
\]
\[
h(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left( n + \frac{1}{2} \right)}{\sqrt{\pi} (2n + 1) n!} z^{2n+1} = \sin^{-1}(z), \quad z \in D(0, 1);
\]
and

\[ h(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1} = \tanh^{-1}(z), \quad z \in D(0,1) \]

\[ h(z) = F_1(\alpha, \beta, \gamma, z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha) \Gamma(n+\beta) \Gamma(\gamma)}{n! \Gamma(\alpha) \Gamma(\beta) \Gamma(n+\gamma)} z^n, \quad \alpha, \beta, \gamma > 0, \]

\[ z \in D(0,1) \]

where \( \Gamma \) is the Gamma function.

The following result may provide many examples of supermultiplicative functions.

**Lemma 1.** Let \( h(z) = \sum_{n=0}^{\infty} a_n z^n \) be a power series with complex coefficients and convergent on the open disk \( D(0,R) \subset \mathbb{C}, \ R > 0 \). Assume that \( 0 < r < R \) and define \( h_r: [0,1] \to [0,\infty), \ h_r(t) := \frac{h(rt)}{h(t)} \). Then \( h_r \) is supermultiplicative on \( [0,1] \).

**Proof.** We use the Čebyshev inequality for synchronous (the same monotonicity) sequences \( (c_i)_{i \in \mathbb{N}}, (b_i)_{i \in \mathbb{N}} \) and nonnegative weights \( (p_i)_{i \in \mathbb{N}} \):

\[ \sum_{i=0}^{n} p_i c_i b_i \geq \sum_{i=0}^{n} p_i c_i \sum_{i=0}^{n} p_i b_i, \]

for any \( n \in \mathbb{N} \).

Let \( t, s \in (0,1) \) and define the sequences \( c_i := t^i, \ b_i := s^i \). These sequences are decreasing and if we apply Čebyshev’s inequality for these sequences and the weights \( p_i := a_i r^i \geq 0 \) we get

\[ \sum_{i=0}^{n} a_i r^i \sum_{i=0}^{n} a_i (rs)^i \geq \sum_{i=0}^{n} a_i (rt)^i \sum_{i=0}^{n} a_i (rs)^i \]

for any \( n \in \mathbb{N} \).

Since the series

\[ \sum_{i=0}^{\infty} a_i r^i, \ \sum_{i=0}^{\infty} a_i (rs)^i, \ \sum_{i=0}^{\infty} a_i (rt)^i \text{ and } \sum_{i=0}^{\infty} a_i (rs)^i \]

are convergent, then by letting \( n \to \infty \) in (2.13) we get

\[ h(r) h(rs) \geq h(rt) h(rs) \]

i.e.

\[ h_r(ts) \geq h_r(t) h_r(s). \]

This inequality is also obviously satisfied at the end points of the interval \([0,1]\) and the proof is completed. \( \square \)

**Remark 2.** Utilising the above theorem, we then conclude that the functions

\[ h_r: [0,1] \to [0,\infty), \ h_r(t) := \frac{1-r}{1-rt}, \ r \in (0,1) \]

and

\[ h_r: [0,1] \to [0,\infty), \ h_r(t) := \exp[-r(1-t)], \ r > 0 \]

are supermultiplicative.
We say that the function \( f : I \to [0, \infty) \) is \( r \)-resolvent convex with \( r \) fixed in \((0,1)\), if \( f \) is \( h \)-convex with \( h(t) = \frac{1}{1-rt} \), i.e.

\[
(2.14) \quad f(tx + (1-t)y) \leq (1-r) \left[ \frac{1}{1-rt} f(x) + \frac{1}{1-r+rt} f(y) \right]
\]

for any \( x, y \in I \) and \( t \in [0,1] \).

In particular, for \( r = \frac{1}{2} \) we have \( \frac{1}{2} \)-resolvent convex functions defined by the condition

\[
(2.15) \quad f(tx + (1-t)y) \leq \frac{1}{2-t} f(x) + \frac{1}{1+t} f(y)
\]

for any \( t \in [0,1] \) and \( x, y \in I \).

Since

\[
t < \frac{1}{2-t} < \frac{1}{t} \quad \text{and} \quad 1-t < \frac{1}{1+t} < \frac{1}{1-t} \quad \text{for} \quad t \in (0,1)
\]

it follows that any nonnegative convex function is \( \frac{1}{2} \)-resolvent convex which, in its turn, is of Godunova-Levin type.

We say that the function \( f : I \to [0, \infty) \) is \( r \)-exponential convex with \( r \) fixed in \((0,\infty)\), if \( f \) is \( h \)-convex with \( h(t) = \exp(-r(1-t)) \), i.e.

\[
(2.16) \quad f(tx + (1-t)y) \leq \exp[-r(1-t)] f(x) + \exp(-rt) f(y)
\]

for any \( t \in [0,1] \) and \( x, y \in I \).

Since

\[
t \leq \exp[-r(1-t)] \quad \text{and} \quad 1-t \leq \exp(-rt) \quad \text{for} \quad t \in [0,1]
\]

it follows that any nonnegative convex function is \( r \)-exponential convex with \( r \in (0,\infty) \).

**Corollary 1.** Let \( h(z) = \sum_{n=0}^{\infty} a_n z^n \) be a power series with complex coefficients and convergent on the open disk \( D(0,R) \subseteq \mathbb{C}, R > 0 \). Assume that \( 0 < r < R \) and define \( h_r : [0,1] \to [0, \infty), \) \( h_r(t) := \frac{h(rt)}{h(r)} \). If the function \( f : I \to [0, \infty) \) is \( h_r \)-convex on the on the interval \( I \), namely

\[
(2.17) \quad f(tx + (1-t)y) \leq \frac{1}{h(r)} [h(rt) f(x) + h(r(1-t)) f(y)]
\]

for any \( t \in [0,1] \) and \( x, y \in I \), then for any \( x_i \in I, \) \( w_i \geq 0, i \in \{1, \ldots, n\}, n \geq 2 \) with \( W_n := \sum_{i=1}^{n} w_i > 0 \) we have

\[
(2.18) \quad f \left( \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \right) \leq \frac{1}{h(r)} \sum_{i=1}^{n} h_i \left( r \frac{w_i}{W_n} \right) f(x_i).
\]

**Remark 3.** If the function \( f : I \to [0, \infty) \) is \( \frac{1}{2} \)-resolvent convex on \( I \), then for any \( x_i \in I, \) \( w_i \geq 0, i \in \{1, \ldots, n\}, n \geq 2 \) with \( W_n := \sum_{i=1}^{n} w_i > 0 \) we have

\[
f \left( \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \right) \leq W_n \sum_{i=1}^{n} \frac{1}{2W_n - w_i} f(x_i).
\]

If the function \( f : I \to [0, \infty) \) is \( r \)-exponential convex with \( r \) fixed in \((0,\infty)\), then for any \( x_i \in I, \) \( w_i \geq 0, i \in \{1, \ldots, n\}, n \geq 2 \) with \( W_n := \sum_{i=1}^{n} w_i > 0 \) we have

\[
f \left( \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \right) \leq \sum_{i=1}^{n} \exp \left[ -r \left( 1 - \frac{w_i}{W_n} \right) \right] f(x_i).
\]
We have the following Jensen type inequality for $\varphi$-convex functions.

**Corollary 2.** Let $\varphi : J \to [0, \infty)$ be a supermultiplicative function on $J$. If the function $f : I \to [0, \infty)$ is $\varphi$-convex on the interval $I$, then for any $w_i \geq 0$, $x_i \in I$, $i \in \{1, ..., n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^n w_i > 0$ we have

$$
(2.19) \quad f \left( \frac{1}{W_n} \sum_{i=1}^n w_i x_i \right) \leq \frac{1}{W_n} \sum_{i=1}^n w_i \varphi \left( \frac{w_i}{W_n} \right) f(x_i).
$$

In particular, we have the unweighted inequality

$$
(2.20) \quad f \left( \frac{1}{n} \sum_{i=1}^n x_i \right) \leq \varphi \left( \frac{1}{n} \right) \frac{1}{n} \sum_{i=1}^n f(x_i).
$$

The proof follows by Theorem 3 for the supermultiplicative function $h(t) = t \varphi(t)$, $t \in J$.

The inequality (2.19) will be used further to obtain an integral Jensen type inequality.

### 3. Some Results for Differentiable Functions

If we assume that the function $f : I \to [0, \infty)$ is differentiable on the interior of $I$, denoted by $\hat{I}$, then we have the following "gradient inequality" that will play an essential role in the following.

**Lemma 2.** Let $\varphi : (0,1) \to (0, \infty)$ be a measurable function and such that the right limit $\varphi_+(0)$ exists and is finite, the left limit $\varphi_-(1) = 1$ and the left derivative in 1 denoted $\varphi_-(1)$ exists and is finite. If the function $f : I \to [0, \infty)$ is differentiable on $\hat{I}$ and $\varphi$-convex, then

$$
(3.1) \quad \varphi_+(0) f(x) - \left[ \varphi_-(1) + 1 \right] f(y) \geq f'(y) (x-y)
$$

for any $x, y \in \hat{I}$ with $x \neq y$.

**Proof.** Since $f$ is $\varphi$-convex on $I$, then

$$
\begin{align*}
t \varphi(t) f(x) + (1-t) \varphi(1-t) f(y) &\geq f(tx + (1-t)y) \\
t \varphi(t) f(x) + [(1-t) \varphi(1-t) - 1] f(y) &\geq f(tx) + (1-t) f(y) - f(y)
\end{align*}
$$

for any $t \in (0,1)$ and for any $x, y \in \hat{I}$, which is equivalent to

$$
\begin{align*}
t \varphi(t) f(x) + [(1-t) \varphi(1-t) - 1] f(y) &\geq f(tx) + (1-t) f(y) - f(y)
\end{align*}
$$

and by dividing by $t > 0$ we get

$$
(3.2) \quad \varphi(t) f(x) + \left[ \frac{(1-t) \varphi(1-t) - 1}{t} \right] f(y) \geq \frac{f(tx) + (1-t) f(y) - f(y)}{t}
$$

for any $t \in (0,1)$.

Now, since $f$ is differentiable on $y \in \hat{I}$, then we have

$$
(3.3) \quad \lim_{t \to 0^+} \frac{f(tx + (1-t)y) - f(y)}{t} = \lim_{t \to 0^+} \frac{f(y + t(x-y)) - f(y)}{t} = (x-y) \lim_{t \to 0^+} \frac{f(y + t(x-y)) - f(y)}{t(x-y)} = (x-y) f'(y)
$$

for any $x \in \hat{I}$ with $x \neq y$. 


Also since \( \varphi_- (1) = 1 \) and \( \varphi'_- (1) \) exists and is finite, we have

\[
(3.4) \quad \lim_{t \to 0^+} \frac{(1-t) \varphi'(1-t) - 1}{t} = \lim_{s \to 1^-} \frac{s \varphi(s) - 1}{1-s} = -\lim_{s \to 1^-} \frac{s \varphi(s) - 1}{s-1} = -\varphi'_- (1) - 1.
\]

Taking the limit over \( t \to 0^+ \) in (3.2) and utilizing (3.3) and (3.4) we get the desired result (3.1).

**Remark 4.** If we assume that

\[
(3.5) \quad \varphi_+ (0) \geq \varphi'_- (1) + 1,
\]

then the inequality (3.1) also holds for \( x = y \).

There are numerous examples of such functions, for instance, if, as above we take \( \varphi(t) = k(1-t)^p + 1, t \in [0,1] (p > 1, k > 0) \) then \( \varphi_+ (0) = k + 1, \varphi_- (1) = 1 \) and \( \varphi'_- (1) = 0 \), which satisfy the condition (3.5).

If we take \( \varphi(t) = \exp \left[ m(1-t) \right] \) \( (m > 0) \), then \( \varphi_+ (0) = \exp m, \varphi_- (1) = 1 \) and \( \varphi'_- (1) = -m \). We have

\[
\varphi_+ (0) - \varphi_- (1) - \varphi'_- (1) = e^m - 1 + m > 0
\]

for \( m > 0 \).

The following result holds:

**Theorem 4.** Let \( \varphi : (0,1) \to (0,\infty) \) a measurable function and such that the right limit \( \varphi_+ (0) \) exists and is finite, the left limit \( \varphi_- (1) = 1 \) and the left derivative in 1 denoted \( \varphi'_- (1) \) exists and is finite. Assume also that \( \varphi'_- (1) > -1 \). If the function \( f : I \to [0,\infty) \) is differentiable on \( I \) and \( \varphi \)-convex, then

\[
(3.6) \quad \frac{\varphi_+ (0)}{\varphi'_- (1) + 1} \cdot \frac{f(x) + f(y)}{2} \geq \frac{1}{y-x} \int_x^y f(u) \, du \geq \frac{\varphi'_- (1) + 1}{\varphi_+ (0)} f \left( \frac{x+y}{2} \right)
\]

for any \( x, y \in I \).

**Remark 5.** It has been shown in [25] that the inequalities (2.5) and (3.6) are not comparable, meaning that some time one is better then the other, depending on the \( \varphi \)-convex function involved.

The following discrete Jensen type inequality holds:

**Theorem 5.** Let \( \varphi : (0,1) \to (0,\infty) \) be a measurable function and such that the right limit \( \varphi_+ (0) \) exists and is finite, the left limit \( \varphi_- (1) = 1 \) and the left derivative in 1 denoted \( \varphi'_- (1) \) exists and is finite. Assume also that

\[
(3.7) \quad \varphi_+ (0) \geq \varphi'_- (1) + 1 > 0.
\]

If the function \( f : I \to [0,\infty) \) is differentiable on \( I \) and \( \varphi \)-convex, then for any \( w_i \geq 0, x_i \in I, i \in \{1,\ldots,n\}, n \geq 2 \) with \( W_n := \sum_{i=1}^n w_i > 0 \) we have

\[
(3.8) \quad \frac{\varphi_+ (0)}{\varphi'_- (1) + 1} \cdot \frac{1}{W_n} \sum_{i=1}^n w_i f (x_i) \geq \frac{1}{W_n} \sum_{i=1}^n w_i x_i.
\]

If \( \frac{1}{W_n} \sum_{i=1}^n w_i x_i \neq x_j \) for any \( j \in \{1,\ldots,n\} \), then the first condition in 3.7 can be dropped.
Proof. From (3.1) we have
\[
\varphi_+(0) f(x_j) \geq f' \left( \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \right) \left( x_j - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \right)
\]
for any \( j \in \{1, \ldots, n\} \).

If we multiply (3.9) by \( w_i \) and sum over \( j \) from 1 to \( n \) we get
\[
\varphi_+(0) \sum_{j=1}^{n} w_j f(x_j) \geq f' \left( \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \right) \sum_{j=1}^{n} w_j \left( x_j - \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i \right) = 0,
\]
which proves the desired result (3.8). \( \square \)

4. INTEGRAL INEQUALITIES

We have the following Jensen inequality for the Riemann integral:

**Theorem 6.** Let \( u : [a, b] \to [m, M] \) be a Riemann integrable function. Suppose that \( \varphi : J \to [0, \infty) \) is a supermultiplicative function on \( J \) and the function \( f : [m, M] \to [0, \infty) \) is \( \varphi \)-convex and continuous on the interval \( [m, M] \). If the right limit \( \varphi_+(0) \) exists and is finite, then
\[
f \left( \frac{1}{b-a} \int_{a}^{b} u(t) \, dt \right) \leq \varphi_+(0) \frac{1}{b-a} \int_{a}^{b} f(u(t)) \, dt.
\]

**Proof.** Consider the sequence of divisions
\[
d_n : x_i^{(n)} = a + \frac{i}{n} (b-a), \quad i \in \{0, \ldots, n\}
\]
and the intermediate points
\[
\xi_i^{(n)} = a + \frac{i}{n} (b-a), \quad i \in \{0, \ldots, n\}.
\]
We observe that the norm of the division \( \Delta_n := \max_{i \in \{0, \ldots, n-1\}} (x_i^{(n)} - x_{i+1}^{(n)}) = \frac{b-a}{n} \to 0 \) as \( n \to \infty \) and since \( u \) is Riemann integrable on \( [a, b] \), then
\[
\int_{a}^{b} u(t) \, dt = \lim_{n \to \infty} \sum_{i=0}^{n-1} u \left( \xi_i^{(n)} \right) \left[ x_{i+1}^{(n)} - x_i^{(n)} \right]
\]
\[
= \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=0}^{n-1} u \left( a + \frac{i}{n} (b-a) \right).
\]
Also, since \( f : [m, M] \to [0, \infty) \) is Riemann integrable, then \( f \circ u \) is Riemann integrable and
\[
\int_{a}^{b} f \left( u(t) \right) \, dt = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=0}^{n-1} f \left( a + \frac{i}{n} (b-a) \right).
\]
Utilising the inequality (2.19) for \( w_i := \frac{b-a}{n} \) and \( x_i := u \left( a + \frac{i}{n} (b-a) \right) \) we have

\[
\frac{1}{b-a} \varphi \left( \frac{1}{n} \right) \frac{b-a}{n} \sum_{i=0}^{n-1} f \left( u \left( a + \frac{i}{n} (b-a) \right) \right) \]

for any \( n \geq 1 \).

Since \( f \) is continuous, then

\[
\lim_{n \to \infty} f \left( \frac{1}{b-a} \sum_{i=0}^{n-1} u \left( a + \frac{i}{n} (b-a) \right) \right) = f \left( \frac{1}{b-a} \int_a^b u(t) \, dt \right).
\]

Also

\[
\lim_{n \to \infty} \varphi \left( \frac{1}{n} \right) = \varphi_+ (0) < \infty.
\]

Therefore, taking the limit over \( n \to \infty \) in the inequality (4.2) we deduce the desired result (4.1).

We have the following Hermite-Hadamard type inequality:

**Corollary 3.** Suppose that \( \varphi : J \to [0, \infty) \) is a supermultiplicative function on \( J \) and the function \( f : I \to [0, \infty) \) is \( \varphi \)-convex and continuous on the interval \( I \). If the right limit \( \varphi_+ (0) \) exists and is finite with \( \varphi_+ (0) > 0 \), then for any \( x, y \in I \) with \( x \neq y \) we have

\[
\frac{1}{\varphi_+ (0)} f \left( \frac{x+y}{2} \right) \leq \frac{1}{y-x} \int_x^y f (u(t)) \, dt.
\]

**Remark 6.** If the function \( f : [m, M] \to [0, \infty) \) is a \( \delta (p,k) \)-convex and continuous function on the interval \( [m, M] \) \( (p > 1 \text{ and } k > 0, \text{ see Definition 7}) \) then for any \( u : [a, b] \to [m, M] \) a Riemann integrable function on \( [a, b] \) we have

\[
\frac{1}{k+1} f \left( \frac{1}{b-a} \int_a^b u(t) \, dt \right) \leq \frac{1}{b-a} \int_a^b f (u(t)) \, dt.
\]

If the function \( f : [m, M] \to [0, \infty) \) is a \( \eta (s) \)-convex and continuous function on the interval \( [m, M] \) \( (s > 0, \text{ see Definition 8}) \) then for any \( u : [a, b] \to [m, M] \) a Riemann integrable function on \( [a, b] \) we have

\[
\frac{1}{e^n} f \left( \frac{1}{b-a} \int_a^b u(t) \, dt \right) \leq \frac{1}{b-a} \int_a^b f (u(t)) \, dt.
\]

Let \( (\Omega, \mathcal{A}, \mu) \) be a measurable space consisting of a set \( \Omega \), a \( \sigma \)-algebra \( \mathcal{A} \) of parts of \( \Omega \) and a countably additive and positive measure \( \mu \) on \( \mathcal{A} \) with values in \( \mathbb{R} \cup \{\infty\} \).
For a $\mu$-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for $\mu$-a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R} | f \text{ is } \mu\text{-measurable and } \int_{\Omega} w(x) |f(x)| \, d\mu(x) < \infty\}.$$ 

For simplicity of notation we write everywhere in the sequel $\int_{\Omega} w(x) \, d\mu(x)$ instead of $\int_{\Omega} w(x) \, d\mu(x)$.

**Theorem 7.** Let $\varphi : (0, 1) \to (0, \infty)$ be a measurable function and such that the right limit $\varphi'_+(0)$ exists and is finite, the left limit $\varphi'_-(1) = 1$ and the left derivative in 1 denoted $\varphi'_-(1)$ exists and is finite. Assume also that

$$\varphi'_+(0) \geq \varphi'_-(1) + 1 > 0. \tag{4.6}$$

If the function $f : I \to [0, \infty)$ is differentiable on $I$ and $\varphi$-convex, then for any $u : \Omega \to [m, M] \subset I$ so that $f \circ u$, $u \in L_w(\Omega, \mu)$, where $w \geq 0$ $\mu$-a.e. (almost everywhere) on $\Omega$ with $\int_{\Omega} w \, d\mu = 1$ we have

$$\frac{\varphi'_+(0)}{\varphi'_-(1) + 1} \int_{\Omega} w(f \circ u) \, d\mu \geq f \left( \int_{\Omega} w \, d\mu \right). \tag{4.7}$$

If $\int_{\Omega} w \, d\mu \neq u(x)$ for $\mu$-a.e. $x \in \Omega$, then we can drop the first condition in (4.6).

**Proof.** From (3.1) and since $\int_{\Omega} w \, d\mu \in [m, M] \subset I$ we have

$$\varphi'_+(0) f(u(x)) - [\varphi'_-(1) + 1] f \left( \int_{\Omega} w \, d\mu \right) \geq f' \left( \int_{\Omega} w \, d\mu \right) \left( u(x) - \int_{\Omega} w \, d\mu \right) \tag{4.8}$$

for any $x \in \Omega$.

If we multiply (4.8) by $w \geq 0$ $\mu$-a.e. on $\Omega$ and integrate over the positive measure $\mu$ we get

$$\varphi'_+(0) \int_{\Omega} w(x) f(u(x)) \, d\mu(x) - [\varphi'_-(1) + 1] f \left( \int_{\Omega} w \, d\mu \right) \int_{\Omega} w(x) \, d\mu(x) \geq f' \left( \int_{\Omega} w \, d\mu \right) \int_{\Omega} w(x) \, d\mu(x) - \int_{\Omega} w(x) \, d\mu(x) = 0,$$

which produces the desired result (4.7). \qed

**Remark 7.** If the function $f : [m, M] \to [0, \infty)$ is a $\delta(p,k)$-convex and continuous function on the interval $[m, M]$, then for any $u : \Omega \to [m, M] \subset I$ so that $f \circ u$, $u \in L_w(\Omega, \mu)$, where $w \geq 0$ $\mu$-a.e. on $\Omega$ with $\int_{\Omega} w \, d\mu = 1$ we have

$$\int_{\Omega} w(f \circ u) \, d\mu \geq \frac{1}{k + 1} f \left( \int_{\Omega} w \, d\mu \right). \tag{4.9}$$

If the function $f : [m, M] \to [0, \infty)$ is a $\eta(s)$-convex and continuous function on the interval $[m, M]$ then for any $u : \Omega \to [m, M] \subset I$ so that $f \circ u$, $u \in L_w(\Omega, \mu)$, where $w \geq 0$ $\mu$-a.e. on $\Omega$ with $\int_{\Omega} w \, d\mu = 1$ we have

$$\int_{\Omega} w(f \circ u) \, d\mu \geq \frac{1}{e^s} f \left( \int_{\Omega} w \, d\mu \right). \tag{4.10}$$

These results generalize the inequalities (4.4) and (4.5).
References


[17] S. S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral \( \int_a^b f(t)u(t) dt \) where \( f \) is of Hölder type and \( u \) is of bounded variation and applications, *J. KSIAM*, 5(1) (2001), 35-45.


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