SOME INEQUALITIES INVOLVING THE BESSEL FUNCTIONS OF
THE FIRST KIND

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Abstract. In this paper, in view of the integral representation of Bessel functions of
the first kind and the inequalities for concave and r-concave functions, we establish
some inequalities for the Bessel functions of the first kind.

INTRODUCTION

The Bessel function of the first kind of order \( \nu \), denoted by \( J_\nu(x) \), is defined as a
particular solution of the second order differential equation
\[
x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0
\]
which is also called the Bessel equation with index \( \nu \). It is known (see [4]) that
\[
J_\nu(x) = \left( \frac{x}{2} \right) ^\nu \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m! \Gamma(\nu + m + 1)}, \quad x \in \mathbb{R}.
\]
In [1], M. Abramowitz and I. A. Stegun mentioned the integral representation of the
function under the form
\[
J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(1/2) \Gamma(\nu + 1/2)} \int_0^1 (1 - t^2)^{\nu-1/2} \cos(x t) dt.
\] (1)
From this expression we can define the following function
\[
f_\nu(x) = \begin{cases} J_\nu(x), & \text{if } x \neq 0 \\ \lim_{x \to 0} \frac{J_\nu(x)}{x^\nu}, & \text{if } x = 0. \end{cases}
\] (2)
It is easy to see that \( f_\nu(-x) = f_\nu(x) \) for every \( x \in \mathbb{R} \) and \( f_\nu(0) = \frac{1}{2\nu \Gamma(\nu + 1)} \). Moreover,
\( f_\nu \) is a differentiable and continuous function on \( \mathbb{R} \). In addition, we also have
\[
f_\nu'(x) = -xf_{\nu+1}(x),
\] (3)
for all \( x \in \mathbb{R} \).
In this paper we use the equality (1) to advance some new properties and inequalities for \( f_\nu \) based on the properties of concave and \( r \)-concave function.

1. Preliminaries

Here we recall some definitions and results related our main results.

**Definition 1.1** ([9]). A function \( f \) is called to be concave on \([a, b]\) if and only if

\[
 f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)
\]

for every \( \lambda \in [0, 1] \) and \( x, y \in [a, b] \).

In [9], A. W. Roberts and D. E. Vargerg referred to the condition for a twice differentiable function \( f \) is concave on \( I \) to be

\[
 f''(x) \leq 0,
\]

for all \( x \in I \).

**Definition 1.2** ([10]). A positive valued function \( f \) is called to be \( r \)-concave on \([a, b]\), if for each \( x, y \in [a, b] \) and \( \lambda \in [0, 1] \),

\[
 f(\lambda x + (1 - \lambda)y) \geq \begin{cases} 
 [\lambda f'(x) + (1 - \lambda)f'(y)]^{1/r}, & r \neq 0, \\
 [f(x)]^\lambda [f(y)]^{1-\lambda}, & r = 0.
\end{cases}
\]

(1.2)

It is obvious 0-concave functions are simply log-concave functions and 1-concave functions are ordinary concave functions. One should note that if \( f \) is a \( r \)-concave on \([a, b]\), then \( f^r \) is concave function with \( r > 0 \).

**Definition 1.3** ([11]). A function \( f : [a, b] \subset (0, +\infty) \rightarrow (0, +\infty) \) is said to be geometrically concave if and only if

\[
 f(x^\alpha y^{1-\alpha}) \geq f^\alpha(x)f^{1-\alpha}(y)
\]

for all \( \alpha \in [0, 1] \) and \( x, y \in [a, b] \).

In [11], X. Zhang and N. Zheng referred to the condition for a twice differentiable function \( f \) is geometrically concave on interval \( I \) to be

\[
 x[f''(x)f(x) - [f'(x)]^2] + f(x)f'(x) \leq 0, \quad \text{for all } x \in I.
\]

(1.4)

**Remark 1.1.** Suppose that a positive function \( f \) defined on \([a, b]\) is to be concave. Then by using Lemma 2.5 in [10] we have the following inequalities

\[
 f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y) \geq [\lambda f'(x) + (1 - \lambda)f'(y)]^{1/r} \geq [f(x)]^\lambda [f(y)]^{1-\lambda}
\]

hold for all \( r \in (0, 1] \). This gives us the related between above functional classes.
Remark 1.2. Let $0 \leq r \leq s$ and suppose that $f$ is a $s$-concave function on $[a, b]$. Then by using Lemma 2.5 in [10], it’s easy to deduce that $f$ is also $r$-concave on the interval $[a, b]$.

S. S. Dragomir and C. E. M. Pearce [5] referred to two well-known results for a convex function. Here we present these results for concave function.

Theorem 1.3 ([5]). Let $p, q$ be given positive numbers and $f$ is a continuous concave function on $[a_1, b_1]$. Then for $a_1 \leq a < b \leq b_1$ the following inequalities
\[
f\left(\frac{pa + qb}{p + q}\right) \geq \frac{1}{2y} \int_{A-y}^{A+y} f(t)dt \geq \frac{1}{2} (f(A-y) + f(A+y)) \geq \frac{pf(a) + qf(b)}{p + q}
\] (1.6)
hold for $A = \frac{pa + qb}{p + q}$ and $0 < y \leq \frac{b-a}{p+q} \min\{p, q\}$.

Theorem 1.4 ([5]). Let $f$ be a concave function on $[a, b]$. Then for all $t \in [a, b]$ we have the following inequality
\[
\frac{1}{b-a} \int_a^b f(x)dx \geq \frac{f(t)}{2} + \frac{1}{2} \frac{bf(b) - af(a) - t[f(b) - f(a)]}{b-a}.
\] (1.7)

2. Main results

In this section, firstly we advance some properties of the function $f_\nu$. Then we use it to advance some new inequalities.

Theorem 2.1. For $\nu \geq 0$ we have the following statements:

(i) $f_\nu$ is concave on $[-\frac{\pi}{2}, \frac{\pi}{2}]$;
(ii) $f_\nu$ is $r$-concave on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $r \in [0, 1]$;
(iii) $f_\nu$ is geometrically concave on $(0, \frac{\pi}{2})$.

Proof. (i) It’s easy to check that for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ we have
\[
f_\nu''(x) = \frac{-1}{2^\nu \Gamma(1/2)\Gamma(\nu + 1/2)} \int_0^1 t^2(1 - t^2)^{\nu-1/2} \cos(\nu t)dt \leq 0.
\] (2.1)
So the function $f_\nu$ is concave on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(ii) This case is directly consequence of the statement (i) and the inequality (1.5).

(iii) For every $x \in (0, \frac{\pi}{2})$, we have $f_\nu(x) \geq 0$ and
\[
f_\nu(x) = \frac{-1}{2^\nu \Gamma(1/2)\Gamma(\nu + 1/2)} \int_0^1 t(1 - t^2)^{\nu-1/2} \cos(\nu t)dt \leq 0.
\] (2.2)
Thus, combining (2.1) and (2.2) give us, for all $x \in (0, \frac{\pi}{2})$,
\[
x[f_\nu''(x)f_\nu(x) - [f_\nu'(x)]^2] + f_\nu(x)f_\nu'(x) \leq 0.
\]
Hence $f_\nu$ satisfies the condition (1.4) and therefore is geometrically concave on $(0, \frac{\pi}{2})$. □
Remark 2.2. For $\nu \geq 0$ we have, in view of Jensen inequality,
\[
\frac{f_\nu(x) + f_\nu(y)}{2} \leq f_\nu\left(\frac{x + y}{2}\right), \quad x, y \in [-\pi/2, \pi/2]. \tag{2.3}
\]

Theorem 2.3. Suppose that $-\frac{\pi}{2} \leq a < b \leq \frac{\pi}{2}$ and $p, q > 0$. Let $A = \frac{pa + qb}{p + q}$ and $0 < y \leq \frac{b - a}{p + q} \min\{p, q\}$. Then, for all $\nu \geq 0$ and $r \in (0, 1]$, we have
\[
f_\nu^{r} \left(\frac{pa + qb}{p + q}\right) \geq \frac{1}{(2y)^r} \left(\int_{A - y}^{A + y} f_\nu(t)dt\right)^r \geq \frac{1}{2y} \int_{A - y}^{A + y} f_\nu'(t)dt \geq \frac{1}{2} [f_\nu'(A - y) + f_\nu'(A + y)] \geq \frac{pf_\nu'(a) + qf_\nu'(b)}{p + q}. \tag{2.4}
\]

Proof. It’s easy to see that the function $f_\nu(t) \geq 0$ for all $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore, by Hölder inequality for $r \in (0, 1]$, we have
\[
\frac{1}{2y} \int_{A - y}^{A + y} f_\nu'(t)dt \leq \frac{1}{(2y)^r} \left(\int_{A - y}^{A + y} f_\nu(t)dt\right)^r. \tag{2.5}
\]
By (ii) of Theorem 2.1, the function $f_\nu$ is $r$-concave on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $r \in (0, 1]$. Hence, applying the inequalities (1.14) in [5] and Theorem 2.1, we get
\[
\frac{1}{2y} \int_{A - y}^{A + y} f_\nu'(t)dt \leq \frac{1}{(2y)^r} \left(\int_{A - y}^{A + y} f_\nu(t)dt\right)^r \leq f_\nu^{r} \left(\frac{pa + qb}{p + q}\right), \tag{2.6}
\]
and
\[
\frac{1}{2y} \int_{A - y}^{A + y} f_\nu'(t)dt \geq \frac{1}{2} [f_\nu'(A - y) + f_\nu'(A + y)] \geq \frac{pf_\nu'(a) + qf_\nu'(b)}{p + q}. \tag{2.7}
\]
Combining (2.6) and (2.7) give us the proved. \hfill \Box

Theorem 2.4. Suppose that $-\frac{\pi}{2} \leq a < b \leq \frac{\pi}{2}$ and $r \in (0, 1]$. Then, for every $t \in [a, b]$ and $\nu \geq 0$ we have the following inequalities
\[
\frac{1}{(b - a)^r} \left(\int_{a}^{b} f_\nu(t)dt\right)^r \geq \frac{1}{b - a} \int_{a}^{b} f_\nu'(t)dt \geq \frac{f_\nu'(t)}{2} + \frac{1}{2} \frac{bf_\nu'(b) - af_\nu'(a) - tf_\nu'(b) - f_\nu'(a)}{b - a}. \tag{2.8}
\]

Proof. The proof is run analogously to Theorem 2.3 but applying Theorem 19 in [5] and Theorem 2.1. \hfill \Box

Theorem 2.5. Let $-\frac{\pi}{2} \leq a < b \leq \frac{\pi}{2}$. Then for all $t \in [a, b]$ one has the inequality
\[
f_\nu(t) + tf_\nu'(t) \left(t - \frac{a + b}{2}\right) \geq \frac{1}{b - a} \int_{a}^{b} f_\nu(x)dx. \tag{2.9}
\]

Proof. Directly applying Theorem 18 in [5] and Theorem 2.1. \hfill \Box
Theorem 2.6. Let $-\frac{\pi}{2} \leq a < b \leq \frac{\pi}{2}$ and $q = \frac{p}{p - 1}$ where $p > 1$. Then one has the inequality
\[
\left| \frac{f_\nu(a) + f_\nu(b)}{2} - \frac{1}{b - a} \int_a^b f_\nu(x)dx \right| \geq \frac{1}{2} \left( b-a \right)^{1/p} \left( \int_a^b |x|^q |f_{\nu + 1}(x)|^q dx \right)^{1/q} . \tag{2.10}
\]


Corollary 2.7. For $-\frac{\pi}{2} \leq a < b \leq \frac{\pi}{2}$ and $q = \frac{p}{p - 1}$ where $p > 1$ we have the inequality
\[
\left| \frac{1}{b - a} \int_a^b f_\nu(x)dx - \frac{f_\nu(a) + f_\nu(b)}{2} \right| \geq \frac{1}{2} \left( b-a \right)^{1/p} \left( \int_a^b |x|^q |f_{\nu + 1}(x)|^q dx \right)^{1/q} . \tag{2.11}
\]

Proof. Directly applying the inequality (2.10) and Theorem 2.1.

Corollary 2.8. For $-\frac{\pi}{2} \leq a < b \leq \frac{\pi}{2}$ we have the inequality
\[
\left| \frac{1}{b - a} \int_a^b f_\nu(x)dx - \frac{f_\nu(a) + f_\nu(b)}{2} \right| \geq \frac{1}{4} \frac{q(f_\nu(a) - b f_{\nu+1}(b))(b-a)}{a f_{\nu+1}(a)}. \tag{2.12}
\]


Theorem 2.9. Let $-\frac{\pi}{2} \leq a < b \leq \frac{\pi}{2}$. Then for all $y \in [a, b]$ we have the following inequalities
\[
\frac{1}{b-a} \int_a^b f_\nu(x)dx \geq \frac{1}{b-a} \int_y^b f_\nu(x)dx + \frac{y-a}{b-a} f_\nu(a) + \frac{f_\nu(y)}{2} \geq \frac{f_\nu(a)}{2} + \frac{f_\nu(b)}{2} . \tag{2.13}
\]
and
\[
f_\nu \left( \frac{a+b}{2} \right) \geq f_\nu \left( \frac{a+b}{2} \right) - \frac{y-a}{b-a} f_\nu \left( \frac{a+y}{2} \right) + \frac{1}{b-a} \int_a^y f_\nu(x)dx \geq \frac{1}{b-a} \int_a^b f_\nu(x)dx. \tag{2.14}
\]


Theorem 2.10. Let $-\frac{\pi}{2} \leq a < b \leq \frac{\pi}{2}$ and $0 \leq s \leq r \leq 1$. Then for $\nu \geq 0$ we have the following inequalities
\[
\frac{1}{b-a} \int_a^b f_\nu(x)dx \geq \frac{r}{r+1} \frac{f_\nu^{s+1}(b) - f_\nu^{s+1}(a)}{f_\nu^s(b) - f_\nu^s(a)} \geq \frac{s}{s+1} \frac{f_\nu^{s+1}(b) - f_\nu^{s+1}(a)}{f_\nu^s(b) - f_\nu^s(a)} \geq \left[ f_\nu(b) - f_\nu(a) \right] \ln f_\nu(b) - \ln f_\nu(a) . \tag{2.15}
\]

Proof. Directly applying Theorem 2.6 in [10] and Theorem 2.1.
REFERENCES


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