BOUNDS FOR A ČEBYŠEV TYPE FUNCTIONAL IN TERMS OF RIEMANN-STIELTJES INTEGRAL

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ABSTRACT. Upper and lower bounds for a Čebyšev type functional in terms of Riemann-Stieltjes integral are given. Applications for functions of selfadjoint operators in Hilbert spaces are also provided.

1. Introduction

In [16], the authors have considered the following functional:

\[ D(f; u) := \int_a^b f(x) \, du(x) - \left[ u(b) - u(a) \right] \cdot \frac{1}{b - a} \int_a^b f(t) \, dt, \]

provided that the Riemann-Stieltjes integral \( \int_a^b f(x) \, du(x) \) and the Riemann integral \( \int_a^b f(t) \, dt \) exist.

In [16], the following result in estimating the above functional has been obtained:

Theorem 1. Let \( f, u : [a, b] \to \mathbb{R} \) be such that \( u \) is Lipschitzian on \([a, b]\), i.e.,

\[ |u(x) - u(y)| \leq L |x - y| \quad \text{for any } x, y \in [a, b] \quad (L > 0) \]

and \( f \) is Riemann integrable on \([a, b]\).

If \( m, M \in \mathbb{R} \) are such that

\[ m \leq f(x) \leq M \quad \text{for any } x \in [a, b], \]

then we have the inequality

\[ |D(f; u)| \leq \frac{1}{2} L (M - m) (b - a). \]

The constant \( \frac{1}{2} \) is sharp in the sense that it cannot be replaced by a smaller quantity.

In [15], the following result complementing the above has been obtained:

Theorem 2. Let \( f, u : [a, b] \to \mathbb{R} \) be such that \( u \) is of bounded variation on \([a, b]\) and \( f \) is Lipschitzian with the constant \( K > 0 \). Then we have

\[ |D(f; u)| \leq \frac{1}{2} K (b - a) \int_a^b u. \]

The constant \( \frac{1}{2} \) is sharp in the above sense.
For a function $u : [a, b] \to \mathbb{R}$, define the associated functions $\Phi, \Gamma$ and $\Delta$ by:

\begin{equation}
\Phi (t) := \frac{(t - a) u(b) + (b - t) u(a)}{b - a} - u(t), \quad t \in [a, b];
\end{equation}

\begin{equation}
\Gamma (t) := (t - a) [u(b) - u(t)] - (b - t) [u(t) - u(a)], \quad t \in [a, b]
\end{equation}

and

\begin{equation}
\Delta (t) := \frac{u(b) - u(t)}{b - t} - \frac{u(t) - u(a)}{t - a}, \quad t \in (a, b).
\end{equation}

In [9], the following subsequent bounds for the functional $D (f; u)$ have been pointed out:

**Theorem 3.** Let $f, u : [a, b] \to \mathbb{R}$.

(i) If $f$ is of bounded variation and $u$ is continuous on $[a, b]$, then

\begin{equation}
|D (f; u)| \leq \begin{cases} 
\sup_{t \in [a, b]} |\Phi (t)| \int_a^b (f), \\
\frac{1}{b - a} \sup_{t \in [a, b]} |\Gamma (t)| \int_a^b (f), \\
\frac{1}{b - a} \sup_{t \in (a, b)} [(t - a) (b - t) |\Delta (t)|] \int_a^b (f).
\end{cases}
\end{equation}

(ii) If $f$ is $L$-Lipschitzian and $u$ is Riemann integrable on $[a, b]$, then

\begin{equation}
|D (f; u)| \leq \begin{cases} 
L \int_a^b |\Phi (t)| dt, \\
\frac{L}{b - a} \int_a^b |\Gamma (t)| dt, \\
\frac{1}{b - a} \int_a^b (t - a) (b - t) |\Delta (t)| dt.
\end{cases}
\end{equation}

(iii) If $f$ is monotonic nondecreasing on $[a, b]$ and $u$ is continuous on $[a, b]$, then

\begin{equation}
|D (f; u)| \leq \begin{cases} 
\int_a^b |\Phi (t)| df (t), \\
\frac{1}{b - a} \int_a^b |\Gamma (t)| df (t), \\
\frac{1}{b - a} \int_a^b (t - a) (b - t) |\Delta (t)| df (t).
\end{cases}
\end{equation}

The case of monotonic integrators is incorporated in the following two theorems [9]:

**Theorem 4.** Let $f, u : [a, b] \to \mathbb{R}$ be such that $f$ is $L$-Lipschitzian on $[a, b]$ and $u$ is monotonic nondecreasing on $[a, b]$, then

\begin{equation}
|D (f; u)| \leq \frac{1}{2} L (b - a) [u(b) - u(a) - K (u)]
\end{equation}

\begin{equation}
\leq \frac{1}{2} L (b - a) [u(b) - u(a)],
\end{equation}

where

\begin{equation}
K (u) := \frac{4}{(b - a)^2} \int_a^b u(x) \left( x - \frac{a + b}{2} \right) dx \geq 0.
\end{equation}
The constant $\frac{1}{2}$ in both inequalities is sharp.

**Theorem 5.** Let $f, u : [a, b] \to \mathbb{R}$ be such that $u$ is monotonic nondecreasing on $[a, b]$, $f$ is of bounded variation on $[a, b]$ and the Stieltjes integral $\int_a^b f(x) \, du(x)$ exists. Then

$$|D(f; u)| \leq [u(b) - u(a) - Q(u)] \sqrt{f} \leq [u(b) - u(a)] \sqrt{f},$$

where

$$Q(u) := \frac{1}{b-a} \int_a^b \sgn\left(x - \frac{a+b}{2}\right) u(x) \, dx \geq 0.$$

The first inequality in (1.12) is sharp.

In the case of convex integrators, the following result may be stated [11]:

**Theorem 6.** Let $u : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ a monotonic nondecreasing function on $[a, b]$. Then

$$0 \leq D(f; u) \leq 2 \cdot \frac{u'_-(b) - u'_+(a)}{b-a} \int_a^b \left(t - \frac{a+b}{2}\right) f(t) \, dt \leq \left\{\begin{array}{l}
\frac{1}{2} \left[u'_-(b) - u'_+(a)\right] \max \{1, |f(a)|, |f(b)|\} (b-a); \\
\frac{1}{(q+1)^{1/2}} \left[u'_-(b) - u'_+(a)\right] \|f\|_p (b-a)^{1/p} \\
\left[u'_-(b) - u'_+(a)\right] \|f\|_1,
\end{array}\right.$$

if $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The following result may be stated as [11]:

**Theorem 7.** Let $u : [a, b] \to \mathbb{R}$ be a continuous convex function on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ a function of bounded variation on $[a, b]$. Then

$$|D(f; u)| \leq \frac{1}{4} \left[u'_-(b) - u'_+(a)\right] (b-a) \sqrt{f},$$

where $\sqrt{f}$ denotes the total variation of $f$ on $[a, b]$.

For other related results for the functional $D(\cdot; \cdot)$, see [1]-[5], [7]-[14] and [18].

In this paper some new lower and upper bounds for $D(\cdot; \cdot)$ are provided. Applications for functions of selfadjoint operators on complex Hilbert spaces are also given.

2. SOME NEW BOUNDS

The following lemma may be stated:

**Lemma 1.** Let $g : [a, b] \to \mathbb{R}$ and $l, L \in \mathbb{R}$ with $L > l$. The following statements are equivalent:
The function \( g + \frac{L}{2} \cdot \ell \), where \( \ell (t) = t, t \in [a, b] \) is \( \frac{1}{2} (L - l) \)-Lipschitzian.

(ii) We have the inequalities

\[ l \leq \frac{g(t) - g(s)}{t - s} \leq L \quad \text{for each } t, s \in [a, b] \quad \text{with } t \neq s; \]

(iii) We have the inequalities

\[ l (t - s) \leq g(t) - g(s) \leq L (t - s) \quad \text{for each } t, s \in [a, b] \quad \text{with } t > s. \]

Following [18], we can introduce the definition of \((l, L)\)-Lipschitzian functions:

**Definition 1.** The function \( g : [a, b] \to \mathbb{R} \) which satisfies one of the equivalent conditions (i) – (iii) from Lemma 1 is said to be \((l, L)\)-Lipschitzian on \([a, b]\).

If \( L > 0 \) and \( l = -L \), then \((-L, L)\)-Lipschitzian means \(L\)-Lipschitzian in the classical sense.

Utilising Lagrange’s mean value theorem, we can state the following result that provides examples of \((l, L)\)-Lipschitzian functions.

**Proposition 1.** Let \( g : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\). If \(-\infty < l = \inf_{t \in (a, b)} g'(t) \) and \( \sup_{t \in (a, b)} g'(t) = L < \infty \), then \( g \) is \((l, L)\)-Lipschitzian on \([a, b]\).

We have the following result:

**Theorem 8.** Let \( u : [a, b] \to \mathbb{R} \) be a convex function on \([a, b]\) and \( f : [a, b] \to \mathbb{R} \) a \((l, L)\)-Lipschitzian function on \([a, b]\). Then

\[
(2.3) \quad l \left[ \frac{u(a) + u(b)}{2} (b - a) - \int_a^b u(t) \, dt \right] \leq D(f; u) \leq L \left[ \frac{u(a) + u(b)}{2} (b - a) - \int_a^b u(t) \, dt \right].
\]

The inequalities in (2.3) are sharp.

**Proof.** Consider the auxiliary function \( f_L : [a, b] \to \mathbb{R}, f_L = L \ell - f \), where \( \ell \) is the identity function \( \ell (t) = t, t \in [a, b] \). Since \( f : [a, b] \to \mathbb{R} \) a \((l, L)\)-Lipschitzian function on \([a, b]\) then \( f(t) - f(s) \leq L (t - s) \) for each \( t, s \in [a, b] \) with \( t > s \) which shows that \( f_L \) is monotonic nondecreasing on \([a, b]\).

Utilizing the first inequality in (1.14) we have

\[
0 \leq D(L \ell - f, u) = LD(\ell, u) - D(f, u)
\]

showing that

\[
(2.4) \quad D(f, u) \leq LD(\ell, u).
\]

A similar argument applied for the auxiliary function \( f_l : [a, b] \to \mathbb{R}, f_L = f - l \ell \) produces the reverse inequality

\[
(2.5) \quad lD(\ell, u) \leq D(f, u).
\]
On the other hand, integrating by parts in the Riemann-Stieltjes integral we have
\[
D(\ell, u) = \int_a^b tdu(t) - \int_a^b \frac{1}{b-a} [u(b) - u(a)] \int_a^b tdt
= bu(b) - au(a) - \int_a^b u(t) dt - \frac{a+b}{2} [u(b) - u(a)]
= \frac{u(a) + u(b)}{2} (b-a) - \int_a^b u(t) dt,
\]
which together with (2.4) and (2.5) produce the desired result (2.3).

If we take \( f_0(t) = t \), and \( \varepsilon \in (0, 1) \) then for each \( t, s \in [a, b] \) with \( t > s \) we have
\[
(1 - \varepsilon)(t-s) \leq f_0(t) - f_0(s) = t-s \leq (1 + \varepsilon)(t-s)
\]
which shows that \( f \) is a \((1-\varepsilon, 1+\varepsilon)\)-Lipschitzian function on \([a, b]\).

Assume that there exists \( A, B > 0 \) such that
\[
L ABD(\ell, u) \leq D(f, u) \leq LBD(\ell, u)
\]
for \( u : [a, b] \to \mathbb{R} \) a convex function on \([a, b]\) and \( f : [a, b] \to \mathbb{R} \) a \((l, L)\)-Lipschitzian function on \([a, b]\).

If we write the inequality (2.6) for \( f_0 \) and \( u \) strictly convex, we get
\[
(1 - \varepsilon) AD(\ell, u) \leq D(\ell, u) \leq (1 + \varepsilon) BD(\ell, u)
\]
and dividing by \( D(\ell, u) > 0 \) we get
\[
(1 - \varepsilon) A \leq 1 \leq (1 + \varepsilon) B.
\]
Letting \( \varepsilon \to 0^+ \) in (2.7) we get \( A \leq 1 \leq B \), which proves the sharpness of the inequality (2.3).

**Remark 1.** The double inequality in (2.3) is equivalent with
\[
D(f; u) - \frac{l + L}{2} \left( \frac{u(a) + u(b)}{2} (b-a) - \int_a^b u(t) dt \right)
\leq \frac{1}{2} (L - l) \left[ \frac{u(a) + u(b)}{2} (b-a) - \int_a^b u(t) dt \right].
\]
The constant \( \frac{1}{2} \) is best possible.

**Corollary 1.** Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\).
If \(-\infty < l = \inf_{t \in (a, b)} f'(t) \) and \( \sup_{t \in (a, b)} f'(t) = L < \infty \). If \( u : [a, b] \to \mathbb{R} \) is a convex function on \([a, b]\), then the inequality (2.8) holds true.
If \( ||f'||_\infty = \sup_{t \in (a, b)} ||f'(t)|| < \infty \), then
\[
|D(f; u)| \leq ||f'||_\infty \left[ \frac{u(a) + u(b)}{2} (b-a) - \int_a^b u(t) dt \right].
\]
The inequality is sharp.

The proof follows from (2.8) by taking \( L = ||f'||_\infty \) and \( l = - ||f'||_\infty \).
For two Lebesgue integrable functions $f$ and $g$ we can define the Čebyšev functional:

$$C(f, g) := \frac{1}{b-a} \int_a^b f(t) g(t) \, dt - \frac{1}{b-a} \int_a^b f(t) \, dt \cdot \frac{1}{b-a} \int_a^b g(t) \, dt.$$  

**Corollary 2.** Let $w : [a, b] \to \mathbb{R}$ be a monotonic nondecreasing function on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ a $(l, L)$-Lipschitzian function on $[a, b]$. Then

$$\frac{l}{b-a} \int_a^b \left( t - \frac{a + b}{2} \right) w(t) \, dt \leq C(f, w) \leq \frac{L}{b-a} \int_a^b \left( t - \frac{a + b}{2} \right) w(t) \, dt.$$  

The inequalities in (2.10) are sharp.

**Proof.** Choose $u(t) := \int_t^b w(s) \, ds$, $t \in [a, b]$. Since $w : [a, b] \to \mathbb{R}$ is a monotonic nondecreasing function on $[a, b]$, then $u$ is convex on $[a, b]$.

We also have

$$\frac{u(a) + u(b)}{2} (b - a) - \int_a^b u(t) \, dt = \frac{1}{2} (b - a) \int_a^b w(s) \, ds - \left[ t \int_a^t w(s) \, ds \bigg|_a^b - \int_a^b sw(s) \, ds \right] = \int_a^b \left( s - \frac{a + b}{2} \right) w(s) \, ds.$$

Writing the inequalities (2.3) for these functions we deduce the desired result (2.10).

**Remark 2.** The inequalities (2.10) are equivalent with

$$|C(f, w)| \leq \frac{1}{2} (L - l) \int_a^b \left( t - \frac{a + b}{2} \right) w(t) \, dt.$$  

The constant $\frac{1}{2}$ is best possible.

If $\|f'\|_{\infty} = \sup_{t \in [a, b]} |f'(t)| < \infty$, then

$$\frac{1}{b-a} \int_a^b \left( t - \frac{a + b}{2} \right) w(t) \, dt.$$

The inequality is sharp.

**Definition 2.** For two constants $\delta, \Delta$ with $\delta < \Delta$, we say that the function $g : [a, b] \to \mathbb{R}$ is $(\delta, \Delta)$-convex (see also [6] for more general concepts) if $g - \frac{1}{2} \delta t^2$ and $\frac{1}{2} \Delta t^2 - g$ are convex functions on $[a, b]$.

It is easy to see that, if $g$ is twice differentiable on $(a, b)$ and the second derivative satisfies the condition

$$\delta \leq g''(t) \leq \Delta$$  

for any $t \in (a, b)$, then $g$ is $(\delta, \Delta)$-convex.

The following result also holds:
Theorem 9. Let \( f : [a, b] \to \mathbb{R} \) be a monotonic nondecreasing function on \([a, b]\) and for \( \delta, \Delta \) with \( \delta < \Delta \), a \((\delta, \Delta)\)-convex function \( u : [a, b] \to \mathbb{R} \). Then we have the double inequality

\[
\delta \int_a^b \left( t - \frac{a + b}{2} \right) f(t) \, dt \leq D(f; u) \leq \Delta \int_a^b \left( t - \frac{a + b}{2} \right) f(t) \, dt.
\]

The inequalities are sharp.

Proof. Since the function \( f \) is monotonic nondecreasing and \( u - \frac{1}{2} \delta \ell^2 \) is convex, then from the first inequality in (1.14) we have

\[
D\left(f; u - \frac{1}{2} \delta \ell^2\right) \geq 0,
\]

which is equivalent with

\[
\frac{1}{2} \delta D\left(f; \ell^2\right) \leq D(f; u).
\]

From the convexity of \( \frac{1}{2} \Delta \ell^2 - g \) we also have

\[
D(f; u) \leq \frac{1}{2} \Delta D\left(f; \ell^2\right).
\]

However

\[
D\left(f; \ell^2\right) = \int_a^b f(t) \, d\ell^2(t) - \frac{\ell^2(b) - \ell^2(a)}{b - a} \int_a^b f(t) \, dt
\]

\[
= 2 \int_a^b f(t) \, d(t) - (b + a) \int_a^b f(t) \, dt
\]

\[
= 2 \int_a^b \left( t - \frac{a + b}{2} \right) f(t) \, dt.
\]

If we take \( u_0(t) := \frac{1}{2} \ell^2 \), and \( \varepsilon \in (0, 1) \), then for \( \delta = 1 - \varepsilon \) and \( \Delta = 1 + \varepsilon \) we have that \( u_0 \) is \((1 - \varepsilon, 1 + \varepsilon)\)-convex on \([a, b]\).

Assume that there exists the constants \( P, Q > 0 \) such that

\[
\delta P \int_a^b \left( t - \frac{a + b}{2} \right) f(t) \, dt \leq D(f; u) \leq \Delta Q \int_a^b \left( t - \frac{a + b}{2} \right) f(t) \, dt,
\]

for \( f : [a, b] \to \mathbb{R} \) a monotonic nondecreasing function on \([a, b]\) and \((\delta, \Delta)\)-convex function \( u : [a, b] \to \mathbb{R} \).

Since

\[
D(f; u_0) = \int_a^b \left( t - \frac{a + b}{2} \right) f(t) \, dt
\]

then by replacing \( u_0, \delta = 1 - \varepsilon \) and \( \Delta = 1 + \varepsilon \) in (2.15) we get

\[
(1 - \varepsilon) P \int_a^b \left( t - \frac{a + b}{2} \right) f(t) \, dt \leq \int_a^b \left( t - \frac{a + b}{2} \right) f(t) \, dt
\]

\[
\leq (1 + \varepsilon) Q \int_a^b \left( t - \frac{a + b}{2} \right) f(t) \, dt,
\]

which by division with \( \int_a^b (t - \frac{a + b}{2}) f(t) \, dt \) that is positive for many functions \( f \) (for instance \( f(t) = t - \frac{a + b}{2} \)), we obtain

\[
(1 - \varepsilon) P \leq 1 \leq (1 + \varepsilon) Q.
\]
Letting $\varepsilon \to 0+$ we deduce $P \leq 1 \leq Q$, and the sharpness of the inequalities are proved.

**Remark 3.** Integrating by parts in the Riemann-Stieltjes integral we have

\begin{equation}
D(f; u) = f(b)u(b) - f(a)u(a) - \int_a^b u(t)df(t)
- \frac{u(b) - u(a)}{b-a} \int_a^b f(t)dt
= u(b)\left(f(b) - \frac{1}{b-a} \int_a^b f(t)dt\right) + u(a)\left(\frac{1}{b-a} \int_a^b f(t)dt - f(a)\right)
- \int_a^b u(t)df(t).
\end{equation}

The inequality (2.3) is then equivalent with

\begin{equation}
l \left[\frac{u(a) + u(b)}{2} (b-a) - \int_a^b u(t)dt\right]
\leq u(b)\left(f(b) - \frac{1}{b-a} \int_a^b f(t)dt\right) + u(a)\left(\frac{1}{b-a} \int_a^b f(t)dt - f(a)\right)
- \int_a^b u(t)df(t)
\leq L \left[\frac{u(a) + u(b)}{2} (b-a) - \int_a^b u(t)dt\right].
\end{equation}

while (2.14) is equivalent with

\begin{equation}
\delta \int_a^b \left(t - \frac{a+b}{2}\right)f(t)dt
\leq u(b)\left(f(b) - \frac{1}{b-a} \int_a^b f(t)dt\right) + u(a)\left(\frac{1}{b-a} \int_a^b f(t)dt - f(a)\right)
- \int_a^b u(t)df(t)
\leq \Delta \int_a^b \left(t - \frac{a+b}{2}\right)f(t)dt.
\end{equation}

3. Applications for Selfadjoint Operators

Let $A \in \mathcal{B}(H)$ be selfadjoint and let $\varphi_\lambda$ defined for all $\lambda \in \mathbb{R}$ as follows

\[
\varphi_\lambda(s) := \begin{cases} 
1, & \text{for } -\infty < s \leq \lambda, \\
0, & \text{for } \lambda < s < +\infty.
\end{cases}
\]

Then for every $\lambda \in \mathbb{R}$ the operator

\begin{equation}
E_\lambda := \varphi_\lambda(A)
\end{equation}
is a projection which reduces $A$. The properties of these projections are summed up in the following fundamental result concerning the spectral decomposition of bounded selfadjoint operators in Hilbert spaces, see for instance [17, p. 256]

**Theorem 10** (Spectral Representation Theorem). Let $A$ be a bounded selfadjoint operator on the Hilbert space $H$ and let $m = \min \{ \lambda \mid \lambda \in \text{Sp}(A) \}$ and $M = \max \{ \lambda \mid \lambda \in \text{Sp}(A) \}$, then there exists a family of projections $(E_{\lambda})_{\lambda \in \mathbb{R}}$, called the spectral family of $A$, with the following properties:

1. $E_{\lambda} \leq E_{\lambda'}$ for $\lambda \leq \lambda'$;
2. $E_{m-0} = 0$, $E_{M} = 1_H$ and $E_{\lambda} = E_{\lambda}$ for all $\lambda \in \mathbb{R}$;
3. We have the representation

$$A = \int_{m-0}^{M} \lambda dE_{\lambda}. \quad (3.2)$$

More generally, for every continuous complex-valued function $\varphi$ defined on $\mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_{k}') [E_{\lambda_{k}} - E_{\lambda_{k}-1}] \right\| \leq \varepsilon \quad (3.3)$$

whenever

$$\begin{cases}
\lambda_0 < m = \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n = M, \\
\lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\
\lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n
\end{cases} \quad (3.4)$$

this means that

$$\varphi(A) = \int_{m-0}^{M} \varphi(\lambda) dE_{\lambda}, \quad (3.5)$$

where the integral is of Riemann-Stieltjes type.

**Corollary 3.** With the assumptions of Theorem 10 for $A, E_{\lambda}$ and $\varphi$ we have the representations

$$\varphi(A)x = \int_{m-0}^{M} \varphi(\lambda) dE_{\lambda}x \quad \text{for all } x \in H \quad (3.6)$$

and

$$\langle \varphi(A)x, y \rangle = \int_{m-0}^{M} \varphi(\lambda) d \langle E_{\lambda}x, y \rangle \quad \text{for all } x, y \in H. \quad (3.7)$$

In particular,

$$\langle \varphi(A)x, x \rangle = \int_{m-0}^{M} \varphi(\lambda) d \langle E_{\lambda}x, x \rangle \quad \text{for all } x \in H. \quad (3.8)$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2 d \|E_{\lambda}x\|^2 \quad \text{for all } x \in H. \quad (3.9)$$
Utilising the Spectral Representation Theorem we can prove the following inequalities for functions of selfadjoint operators:

**Theorem 11.** Let $A$ be a bonded selfadjoint operator on the Hilbert space $H$ and let $m = \min \{ \lambda \mid \lambda \in \text{Sp}(A) \} =: \min \text{Sp}(A)$ and $M = \max \{ \lambda \mid \lambda \in \text{Sp}(A) \} =: \max \text{Sp}(A)$. Assume that the function $f : I \to \mathbb{R}$ is differentiable on the interior of $I$ denoted $I$ and $[m, M] \subset \bar{I}$. If the derivative $f'$ is $(\delta, \Delta)$-Lipschitzian with $\delta < \Delta$, then

\[
\frac{1}{2} \delta (M1_H - A) (A - m1_H) \leq \frac{1}{M - m} \left[ f(M) (A - m1_H) + f(m) (M1_H - A) - f(A) \right] \leq \frac{1}{2} \Delta (M1_H - A) (A - m1_H)
\]

in the operator order of $B(H)$.

**Proof.** Let $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ the spectral family of $A$ and $x \in H$. Utilising the inequality (2.10) for the $(\delta, \Delta)$-Lipschitzian function $f'$ and the monotonic nondecreasing function $w(t) = \langle E_t x, x \rangle$, $t \in [m - \varepsilon, M]$ for a small positive $\varepsilon$, we have

\[
\frac{\delta}{M - m + \varepsilon} \int_{m - \varepsilon}^{M} \left( t - \frac{m - \varepsilon + M}{2} \right) \langle E_t x, x \rangle dt \leq \frac{1}{M - m + \varepsilon} \int_{m - \varepsilon}^{M} f'(t) \langle E_t x, x \rangle dt - \frac{1}{M - m + \varepsilon} \int_{m - \varepsilon}^{M} f'(t) dt \cdot \frac{1}{M - m + \varepsilon} \int_{m - \varepsilon}^{M} \langle E_t x, x \rangle dt \leq \frac{\Delta}{M - m + \varepsilon} \int_{m - \varepsilon}^{M} \left( t - \frac{a + b}{2} \right) w(t) dt.
\]

Letting $\varepsilon \to 0+$ in (3.11) we get

\[
\delta \int_{m - 0}^{M} \left( t - \frac{m + M}{2} \right) \langle E_t x, x \rangle dt \leq \int_{m - 0}^{M} f'(t) \langle E_t x, x \rangle dt - \frac{1}{M - m} \int_{m - 0}^{M} f'(t) dt \cdot \int_{m - 0}^{M} \langle E_t x, x \rangle dt \leq \Delta \int_{m - 0}^{M} \left( t - \frac{a + b}{2} \right) w(t) dt
\]

for any $x \in H$. 

Utilising the integration by parts formula for the Riemann-Stieltjes integral, we have

\[
\int_{m-0}^{M} \left( t - \frac{m + M}{2} \right) (E_t, x) \, dt
\]

\[
= \frac{1}{2} \int_{m-0}^{M} (E_t, x) \, d\left( \left( t - \frac{m + M}{2} \right)^2 \right)
\]

\[
= \frac{1}{2} \left[ (E_t, x) \left( t - \frac{m + M}{2} \right)^2 \right]_{m-0}^{M} - \int_{m-0}^{M} \left( t - \frac{m + M}{2} \right)^2 \, d((E_t, x))
\]

\[
= \frac{1}{2} \left[ \|x\|^2 \left( \frac{M - m}{2} \right)^2 - \int_{m-0}^{M} \left( t - \frac{m + M}{2} \right)^2 \, d((E_t, x)) \right]
\]

\[
= \frac{1}{2} \left[ \int_{m-0}^{M} \left( \frac{M - m}{2} \right)^2 - \left( t - \frac{m + M}{2} \right)^2 \right] \, d((E_t, x))
\]

\[
= \frac{1}{2} \int_{m-0}^{M} (M - t) (t - m) \, d((E_t, x)) = \frac{1}{2} ((M1_H - A) (A - m1_H) x, x)
\]

for any \( x \in H \).

We also have

\[
\int_{m-0}^{M} f'(t) \, (E_t, x) \, dt = f(t) \, (E_t, x) \bigg|_{m-0}^{M} - \int_{m-0}^{M} f(t) \, d((E_t, x))
\]

\[
= f(M) \|x\|^2 - \int_{m-0}^{M} f(t) \, d((E_t, x))
\]

\[
= \int_{m-0}^{M} [f(M) - f(t)] \, d((E_t, x))
\]

\[
= \langle [f(M) 1_H - f(A)] x, x \rangle
\]

and, similarly

\[
\int_{m-0}^{M} (E_t, x) \, dt = \langle (M1_H - A) x, x \rangle
\]

for any \( x \in H \).

Utilising (3.14) and (3.15) we have

\[
\int_{m-0}^{M} f'(t) \, (E_t, x) \, dt - \frac{1}{M - m} \int_{m-0}^{M} f'(t) \, dt \cdot \int_{m-0}^{M} (E_t, x) \, dt
\]

\[
= \langle [f(M) 1_H - f(A)] x, x \rangle - \frac{f(M) - f(m)}{M - m} \langle (M1_H - A) x, x \rangle
\]

\[
= \left< \left[ \frac{(M - m) f(M) 1_H - [f(M) - f(m)] (M1_H - A)}{M - m} - f(A) \right] x, x \right>
\]

\[
= \left< \left[ \frac{f(m) (M1_H - A) + f(M) (A - m1_H)}{M - m} - f(A) \right] x, x \right>
\]

for any \( x \in H \).

From (3.12) we deduce the desired result (3.10). \( \square \)
From Theorem 6, we have for \( h : [a, b] \to \mathbb{R} \) a convex function on \([a, b]\) and \( g : [a, b] \to \mathbb{R} \) a monotonic nondecreasing function on \([a, b]\),

\[
(3.17) \quad 0 \leq D(g; h) \leq 2 \cdot \frac{h'_-(b) - h'_+(a)}{b - a} \int_a^b \left( t - \frac{a + b}{2} \right) g(t) dt.
\]

Since, by (2.17) we have

\[
(3.18) \quad 0 \leq D(g; h) = h(b) \left( g(b) - \frac{1}{b - a} \int_a^b g(t) dt \right) + h(a) \left( \frac{1}{b - a} \int_a^b g(t) dt - g(a) \right) - \int_a^b h(t) df(t)
\]

and

\[
(3.19) \quad \int_a^b \left( t - \frac{a + b}{2} \right) g(t) dt
\]

\[
= \frac{1}{2} \int_a^b g(t) d \left[ \left( t - \frac{a + b}{2} \right)^2 \right]
\]

\[
= \frac{1}{2} \left[ g(t) \left( t - \frac{a + b}{2} \right)^2 \right]_a^b - \int_a^b \left( t - \frac{a + b}{2} \right)^2 dg(t)
\]

\[
= \frac{1}{2} \left[ g(b) - g(a) \right] \left( \frac{b - a}{2} \right)^2 - \int_a^b \left( t - \frac{a + b}{2} \right)^2 dg(t)
\]

\[
= \frac{1}{2} \int_a^b \left( \left( \frac{b - a}{2} \right)^2 - \left( t - \frac{a + b}{2} \right)^2 \right) dg(t)
\]

\[
= \frac{1}{2} \int_a^b (b - t)(t - a) dg(t),
\]

then by (3.17) we have

\[
(3.20) \quad 0 \leq h(b) \left( g(b) - \frac{1}{b - a} \int_a^b g(t) dt \right) + h(a) \left( \frac{1}{b - a} \int_a^b g(t) dt - g(a) \right)
\]

\[
- \int_a^b h(t) df(t) \leq \frac{h'_-(b) - h'_+(a)}{b - a} \int_a^b (b - t)(t - a) dg(t)
\]

We can state the following result as well:

**Theorem 12.** Let \( A \) be a bounded selfadjoint operator on the Hilbert space \( H \) and let \( m = \min \{ \lambda | \lambda \in \text{Sp}(A) \} =: \min \text{Sp}(A) \) and \( M = \max \{ \lambda | \lambda \in \text{Sp}(A) \} =: \max \text{Sp}(A) \). Assume that the function \( f : I \to \mathbb{R} \) is convex on the interior of \( I \)
denoted $\tilde{I}$ and $[m, M] \subset \tilde{I}$. Then

$$0 \leq \frac{1}{M - m} \left[ f(M) (A - m1_H) + f(m) (M1_H - A) - f(A) \right]$$

$$\leq \frac{f'_{L}(M) - f'_{R}(m)}{M - m} (M1_H - A) (A - m1_H).$$

The proof follows by (3.20) by choosing $h = f$ and $g = \langle E_t x, x \rangle$, $t \in \mathbb{R}$, where $\{E_{\lambda}\}_{\lambda \in \mathbb{R}}$ is the spectral family of $A$.

Consider the exponential function $f : \mathbb{R} \to \mathbb{R}$, and let $A$ be a bonded self-adjoint operator on the Hilbert space $H$ and let $m = \min \{\lambda | \lambda \in \text{Sp}(A)\}$ and $M = \max \{\lambda | \lambda \in \text{Sp}(A)\}$. Then by (3.10) we have

$$\frac{1}{2} \exp(m) (M1_H - A) (A - m1_H)$$

$$\leq \frac{1}{M - m} \left[ \exp(M) (A - m1_H) + \exp(m) (M1_H - A) - \exp(A) \right]$$

$$\leq \frac{1}{2} \exp(M) (M1_H - A) (A - m1_H).$$

Consider the function $f : [m, M] \to \mathbb{R}$, $f(t) = -\ln t$ and $[m, M] \subset (0, \infty)$. Then by (3.10) we have

$$\frac{1}{2M^2} (M1_H - A) (A - m1_H)$$

$$\leq \ln (A) - \frac{1}{M - m} \left[ \ln(M) (A - m1_H) + \ln(m) (M1_H - A) \right]$$

$$\leq \frac{1}{2m^2} (M1_H - A) (A - m1_H).$$

If we take the power function $f : [m, M] \to \mathbb{R}$, $f(t) = t^p$, $p \geq 2$ and $[m, M] \subset [0, \infty)$ then by (3.10) we have

$$\frac{1}{2} p(p - 1) m^{p-2} (M1_H - A) (A - m1_H)$$

$$\leq \frac{1}{M - m} \left[ M^p (A - m1_H) + m^p (M1_H - A) - A^p \right]$$

$$\leq \frac{1}{2} p(p - 1) M^{p-2} (M1_H - A) (A - m1_H).$$

Consider the convex function $f : \mathbb{R} \to \mathbb{R}$, $f(t) = |t - \frac{m + M}{2}|$. Utilizing the inequality (3.21) we have

$$0 \leq \frac{M - m}{2} - \left| A - \frac{m + M}{2} \right| \leq \frac{2}{M - m} (M1_H - A) (A - m1_H).$$

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