HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR
DIFFERENTIABLE $m$-PREINVEX AND $(\alpha, m)$-PREINVEX
FUNCTIONS

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Abstract. In this paper, the notion of $m$-preinvex and $(\alpha, m)$-preinvex functions is introduced and then several inequalities of Hermite-Hadamard type for differentiable $m$-preinvex and $(\alpha, m)$-preinvex functions are established. The obtained inequalities for $m$-convex and $(\alpha, m)$-convex functions, are then extended to functions of several variables.

1. Introduction

A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if

$$f((tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for every $x, y \in I$ and $t \in [0,1]$.

The following celebrated double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

holds for convex functions and is well-known in literature as the Hermite-Hadamard inequality. Both of the inequalities in (1.1) hold in reversed direction if $f$ is concave.

The inequality (1.1) has been a subject of extensive research since its discovery and a number of papers have been written providing noteworthy extensions, generalizations and refinements see for example [6], [7], [25], [26] and [33].

The classical convexity that is stated above was generalized as $m$-convexity by G. Toader in [30] as follows:

Definition 1. The function $[0, b^*]$, $b^* > 0$, is said to be $m$-convex, where $m \in [0,1]$, if we have

$$f((tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b^*]$ and $t \in [0,1]$. We say that $f$ is $m$-concave if $-f$ is $m$-convex.

Obviously, for $m = 1$ the Definition 1 recaptures the concept of standard convex functions on $[0, b^*]$.

The notion of $m$-convexity has been further generalized in [14] as it is stated in the following definition:
Definition 2. The function \([0, b^*], b^* > 0\), is said to be \((\alpha, m)\)-convex , where \((\alpha, m) \in [0, 1]^2\), if we have

\[
 f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1 - t^\alpha) f(y) 
\]

for all \(x, y \in [0, b^*]\) and \(t \in [0, 1]\).

It can easily be seen that for \(\alpha = 1\), the class of \(m\)-convex functions are derived from the above definition and for \(\alpha = m = 1\) a class of convex functions are derivived.

For several results concerning Hermite-Hadamard type inequalities for \(m\)-convex and \((\alpha, m)\)-convex functions we refer the interested reader to \([8]\) and \([9]\).

More recently, a number of mathematicians have attempted to generalize the concept of classical convexity. For example in \([10]\), Hason gave the notion of in-exvexity as significant generalization of classical convexity. Ben-Israel and Mond \([4]\) introduced the concept of preinvex functions, which is a special case of invex functions. Let us first recall the definition of preinvexity and some related results.

Let \(K\) be a subset in \(\mathbb{R}^n\) and let \(f : K \to \mathbb{R}\) and \(\eta : K \times K \to \mathbb{R}^n\) be continuous functions. Let \(x \in K\), then the set \(K\) is said to be invex at \(x\) with respect to \(\eta(\cdot, \cdot)\), if

\[
x + t\eta(y, x) \in K, \forall x, y \in K, t \in [0, 1].
\]

\(K\) is said to be an invex set with respect to \(\eta\) if \(K\) is invex at each \(x \in K\). The invex set \(K\) is also called a \(\eta\)-connected set.

Definition 3. \([24]\) The function \(f\) on the invex set \(K\) is said to be preinvex with respect to \(\eta\), if

\[
f(u + t\eta(v, u)) \leq (1 - t)f(u) + tf(v), \forall u, v \in K, t \in [0, 1].
\]

The function \(f\) is said to be preconcave if and only if \(-f\) is preinvex.

It is to be noted that every convex function is preinvex with respect to the map \(\eta(x, y) = x - y\) but the converse is not true see for instance \([23]\).

In a recent paper, Noor \([17]\) obtained the following Hermite-Hadamard inequalities for the preinvex functions:

Theorem 1. \([17]\) Let \(f : [a, a + \eta(b, a)] \to (0, \infty)\) be a preinvex function on the interval of the real numbers \(K^\circ\) (the interior of \(K\)) and \(a, b \in K^\circ\) with \(a < a + \eta(b, a)\). Then the following inequality holds:

\[
 f \left( \frac{2a + \eta(b, a)}{2} \right) \leq \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

Barani, Ghazanfari and Dragomir in \([3]\), presented the following estimates of the right-side of a Hermite- Hadamard type inequality in which some preinvex functions are involved.

Theorem 2. \([3]\) Let \(K \subseteq \mathbb{R}\) be an open invex subset with respect to \(\eta : K \times K \to \mathbb{R}\). Suppose that \(f : K \to \mathbb{R}\) is a differentiable function. If \(\left| f' \right|\) is preinvex on \(K\), for
every $a, b \in K$ with $\eta(b, a) \neq 0$, then the following inequality holds:

\[
\left(\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x)dx \right) \leq \frac{\eta(b, a)}{8} \left( \left| f'(a) \right| + \left| f'(b) \right| \right) .
\]

**Theorem 3.** [3] Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. Assume $p \in \mathbb{R}$ with $p > 1$. If $\left| f' \right|^{\frac{1}{p-2}}$ is preinvex on $K$ then, for every $a, b \in K$ with $\eta(b, a) \neq 0$, the following inequality holds:

\[
\left(\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x)dx \right) \leq \frac{\eta(b, a)}{2 (1 + p)^{\frac{1}{p}}} \left[ \left| f'(a) \right|^{\frac{1}{p-2}} + \left| f'(b) \right|^{\frac{1}{p-2}} \right]^{\frac{p}{p-2}} .
\]

For several new results on inequalities for preinvex functions, we refer the interested reader to [3] and [27] and the references therein.

In the present paper we first give the concept of $m$-preinvex and $(\alpha, m)$-preinvex functions, which generalize the concept of preinvex functions, and then we will present new inequalities of Hermite-Hadamard for functions whose derivatives in absolute value are $m$-preinvex and $(\alpha, m)$-preinvex. Our results generalize those results presented in very recent paper [3] concerning Hermite-Hadamard type inequalities for preinvex functions. We also present extensions to several variables of some inequalities for $m$-convex and $(\alpha, m)$-convex functions which are special cases of our established results.

### 2. Main Results

To establish our main results we first give the following essential definitions and Lemmas:

**Definition 4.** The function $f$ on the invex set $K \subseteq [0, b^*], b^* > 0$, is said to be $m$-preinvex with respect to $\eta$ if

\[
f(u + t \eta(v, u)) \leq (1 - t) f(u) + mf \left( \frac{v}{m} \right)
\]

holds for all $u, v \in K$, $t \in [0, 1]$ and $m \in (0, 1]$. The function $f$ is said to be $m$-preconcave if and only if $-f$ is $m$-preinvex.

**Definition 5.** The function $f$ on the invex set $K \subseteq [0, b^*], b^* > 0$, is said to be $(\alpha, m)$-preinvex with respect to $\eta$ if

\[
f(u + t \eta(v, u)) \leq (1 - t^\alpha) f(u) + mt^\alpha f \left( \frac{v}{m} \right)
\]

holds for all $u, v \in K$, $t \in [0, 1]$ and $(\alpha, m) \in (0, 1] \times (0, 1]$. The function $f$ is said to be $(\alpha, m)$-preconcave if and only if $-f$ is $(\alpha, m)$-preinvex.
Remark 1. If in definition 4, \( m = 1 \), then one obtain the usual definition of preinvexity. If \( \alpha = m = 1 \), then definition 5 recaptures the usual definition of the preinvex functions. It is to be noted that every \( m \)-preinvex function and \((\alpha, m)\)-preinvex functions are \( m \)-convex and \((\alpha, m)\)-convex with respect to \( (v; u) = v - u \) respectively.

Lemma 1. [3] Let \( K \subseteq \mathbb{R} \) be an open invex subset with respect to \( \eta : K \times K \rightarrow \mathbb{R} \) and \( a, b \in K \) with \( a < a + \eta(b, a) \). Suppose \( f : K \rightarrow \mathbb{R} \) is a differentiable mapping on \( K \) such that \( f' \in L([a, a + \eta(b, a)]) \), then the following equality holds:

\[
(2.1) \quad f(a) + f(a + \eta(b, a)) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x) \, dx = \frac{\eta(b,a)}{2} \int_0^1 (1 - 2t)f'(a + t\eta(b,a)) \, dt.
\]

Now we establish results for functions whose derivatives in absolute values raise to some certain power are \( m \)-preinvex and \((\alpha, m)\)-preinvex.

Theorem 4. Let \( K \subseteq [0, b^*] \), \( b^* > 0 \) be an open invex subset with respect to \( \eta : K \times K \rightarrow \mathbb{R} \) and \( a, b \in K \) with \( a < a + \eta(b, a) \). Suppose \( f : K \rightarrow \mathbb{R} \) is a differentiable mapping on \( K \) such that \( f' \in L([a, a + \eta(b, a)]) \). If \( |f'| \) is \( m \)-preinvex on \( K \), then we have the following inequality:

\[
(2.2) \quad \left| f(a) + f(a + \eta(b, a)) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x) \, dx \right| \leq \frac{\eta(b,a)}{8} \left[ |f'(a)| + m |f'\left(\frac{b}{m}\right)| \right].
\]

Proof. From lemma 1, we obtain

\[
(2.3) \quad \left| f(a) + f(a + \eta(b, a)) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x) \, dx \right| \leq \frac{\eta(b,a)}{2} \int_0^1 |1 - 2t| |f'(a + t\eta(b,a))| \, dt.
\]

Since \( |f'| \) is \( m \)-preinvex on \( K \), for every \( a, b \in K \) and \( t \in [0, 1] \), \( m \in (0, 1] \), we have

\[
(2.4) \quad |f'(a + t\eta(b,a))| \leq (1 - t) |f'(a)| + mt |f'\left(\frac{b}{m}\right)|.
\]

Hence we have

\[
(2.5) \quad \left| f(a) + f(a + \eta(b, a)) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x) \, dx \right| \leq \frac{\eta(b,a)}{2} \left[ |f'(a)| \int_0^1 |1 - 2t| (1 - t) \, dt + m |f'\left(\frac{b}{m}\right)| \int_0^1 |1 - 2t| t \, dt \right].
\]
Since
\[ \int_0^1 |1 - 2t| (1 - t) \, dt = \int_0^1 |1 - 2t| t \, dt \]
\[ = \int_0^{\frac{1}{2}} (1 - 2t) (1 - t) \, dt - \int_{\frac{1}{2}}^1 (1 - 2t) (1 - t) \, dt = \frac{1}{4}. \]
We get the desired inequality from (2.5). This completes the proof of theorem 4.

**Corollary 1.** If \( \eta(b, a) = b - a \) in theorem 4, then (2.2) reduces to the following inequality:

(2.6) \[
\left| f(a) + f(a + \eta(b, a)) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{8} \left[ \left| f'(a) \right| + m \left| f' \left( \frac{b}{m} \right) \right| \right].
\]

**Theorem 5.** Let \( K \subseteq [0, b^*] \), \( b^* > 0 \) be an open invex subset with respect to \( \eta : K \times K \to \mathbb{R} \) and \( a, b \in K \) with \( a < a + \eta(b, a) \). Suppose \( f : K \to \mathbb{R} \) is a differentiable mapping on \( K \) such that \( f' \in L([a, a + \eta(b, a)]) \). If \( |f'|^q \) is \( m \)-preinvex on \( K \) for \( q > 1 \), then we have the following inequality:

(2.7) \[
\left| f(a) + f(a + \eta(b, a)) - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) \, dx \right| \leq \frac{\eta(b, a)}{2} \left[ \left| f'(a) \right|^q + m \left| f' \left( \frac{b}{m} \right) \right|^q \right]^{\frac{1}{q}}.
\]

where \( \frac{1}{p} + \frac{1}{q} = 1. \)

**Proof.** By lemma 1 and using the well known Hölder’s integral inequality, we have

(2.8) \[
\left| f(a) + f(a + \eta(b, a)) - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) \, dx \right| \leq \frac{\eta(b, a)}{2} \left( \int_0^1 |1 - 2t|^p \, dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f'(a + t \eta(b, a)) \right|^q \, dt \right)^{\frac{1}{q}}.
\]

Since \( |f'|^q \) is \( m \)-preinvex on \( K \), for every \( a, b \in [a, b] \) with \( a < a + \eta(b, a) \) and \( m \in (0, 1] \), we have

\[
\left| f'(a + t \eta(b, a)) \right|^q \leq (1 - t) \left| f'(a) \right|^q + mt \left| f' \left( \frac{b}{m} \right) \right|^q.
\]

Hence

\[
\int_0^1 \left| f'(a + t \eta(b, a)) \right|^q \, dt \leq \int_0^1 \left[ (1 - t) \left| f'(a) \right|^q + mt \left| f' \left( \frac{b}{m} \right) \right|^q \right] \, dt
\]
\[= \frac{1}{2} \left| f'(a) \right|^q + \frac{m}{2} \left| f' \left( \frac{b}{m} \right) \right|^q.
\]
Moreover, by using basic calculus we have

\[
\int_0^1 |1 - 2t|^p \, dt = \int_0^{\frac{1}{2}} (1 - 2t)^p \, dt + \int_{\frac{1}{2}}^1 (2t - 1)^p \, dt = \frac{1}{p+1}.
\]

A usage of the last two inequalities in (2.8) gives the desired result. This completes the proof of theorem 5.

Corollary 2. If we take \((b, a) = b - a\) in theorem 5, then (2.7) becomes the following inequality:

\[
(2.9)
\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{b - a}{2(p + 1)^{\frac{1}{p}}} \left[ \left( \frac{f'(a)}{q} + m \left( \frac{b - a}{m} \right)^q \right)^{\frac{1}{q}} \right].
\]

A similar result may be stated as follows:

Theorem 6. Let \(K \subseteq [0, b^*], b^* > 0\) be an open invex subset with respect to \(\eta : K \times K \to \mathbb{R}\) and \(a, b \in K\) with \(a < a + \eta(b, a)\). Suppose \(f : K \to \mathbb{R}\) is a differentiable mapping on \(K\) such that \(f' \in L([a, a + \eta(b, a)])\). If \(|f'|^q\) is \(m\)-preinvex on \(K\) for \(q \geq 1\), then we have the following inequality:

\[
(2.10)
\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) \, dx \leq \frac{\eta(b, a)}{4} \left[ \left( \frac{f'(a)}{q} + m \left( \frac{b - a}{m} \right)^q \right)^{\frac{1}{q}} \right].
\]

Proof. For \(q = 1\), the proof is the same as that of theorem 4. Suppose now that \(q > 1\). Using lemma 1 and the well-known power-mean integral inequality, we have

\[
(2.11)
\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) \, dx \leq \frac{\eta(b, a)}{2} \left( \int_0^1 |1 - 2t|^q \, dt \right)^{\frac{1-q}{q}} \left( \int_0^1 |1 - 2t| |f'(a + t\eta(b, a))|^q \, dt \right)^{\frac{1}{q}}.
\]
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Applying the $m$-preinvex convexity of $|f|^{q}$ on $K$ in the second integral on the right side of (2.11), we have

$$
(2.12) \int_{0}^{1} |1-2t| |f^{'} (a + t\eta (b,a))|^{q} dt \leq \int_{0}^{1} |1-2t| \left[ (1-t) \left| f^{'} (a) \right|^{q} + m t \left| f^{'} \left( \frac{b}{m} \right) \right|^{q} \right] dt \\
= \left| f^{'} (a) \right|^{q} \int_{0}^{1} |1-2t| (1-t) dt + m \int_{0}^{1} t |1-2t| dt \\
= \frac{1}{4} \left| f^{'} (a) \right|^{q} + \frac{m}{4} \left| f^{'} \left( \frac{b}{m} \right) \right|^{q}.
$$

Utilizing inequality (2.12) in (2.11), we get the inequality (2.10). This completes the proof of the theorem.

**Corollary 3.** Suppose $\eta (b,a) = b - a$, then one has the following inequality:

$$
(2.13) \left| \frac{f (a) + f (b)}{2} - \frac{\int_{a}^{b} f (x) dx}{b-a} \right| \leq \frac{b-a}{4} \left[ \left| f^{'} (a) \right|^{q} + m \left| f^{'} \left( \frac{b}{m} \right) \right|^{q} \right]^{\frac{1}{q}}.
$$

**Remark 2.** For $q = 1$, (2.13) reduces to the inequality proved in theorem 4. If $q = \frac{p}{p-1} (p > 1)$, we have $4p > p + 1$ for $p > 1$ and accordingly

$$
\frac{1}{4} < \frac{1}{2(p+1)^{\frac{1}{q}}}.
$$

This reveals that the inequality (2.10) is better than the one given by (2.7) in theorem 5.

Now we give our results for $(\alpha, m)$-preinvex functions.

**Theorem 7.** Let $K \subseteq [0, b^{*}]$, $b^{*} > 0$ be an open invex subset with respect to $\eta : K \times K \rightarrow \mathbb{R}$ and $a, b \in K$ with $a < a + \eta (b,a)$. Suppose $f : K \rightarrow \mathbb{R}$ is a differentiable mapping on $K$ such that $f^{'} \in L ([a, a + \eta (b,a)])$. If $|f^{'}|$ is $(\alpha, m)$-preinvex on $K$, then we have the following inequality:

$$
(2.14) \left| \frac{f (a) + f (a + \eta (b,a))}{2} - \frac{\int_{a}^{a+\eta(b,a)} f (x) dx}{\eta (b,a)} \right| \\
\leq \frac{\eta (b,a)}{2^{\frac{1+\alpha}{1+\alpha}}} \left[ \nu_2 \left| f^{'} (a) \right| + m \nu_1 \left| f^{'} \left( \frac{b}{m} \right) \right| \right],
$$

where $\nu_1 = \frac{1+\alpha}{2(1+\alpha)(2+\alpha)}$ and $\nu_2 = \frac{1}{2} - \nu_1$.

**Proof.** From lemma 1, we have

$$
(2.15) \left| \frac{f (a) + f (a + \eta (b,a))}{2} - \frac{\int_{a}^{a+\eta(b,a)} f (x) dx}{\eta (b,a)} \right| \\
\leq \frac{\eta (b,a)}{2} \int_{0}^{1} |1-2t| \left| f^{'} (a + t\eta (b,a)) \right| dt.
$$
Since \( f' \) is \((\alpha, m)\)-preinvex on \( K \), we have for every \( t \in [0,1] \) that

\[
(2.16) \quad \int_0^1 |1 - 2t| \left| f'(a + t\eta(b,a)) \right| \, dt \\
\leq \left| f'(a) \right| \int_0^1 |1 - 2t| (1 - t^\alpha) \, dt + m \left| f' \left( \frac{b}{m} \right) \right| \int_0^1 t^\alpha |1 - 2t| \, dt \\
= \left( \frac{1}{2} - \nu_1 \right) \left| f'(a) \right| + m\nu_1 \left| f' \left( \frac{b}{m} \right) \right|,
\]

where

\[
\int_0^1 |1 - 2t| t^\alpha \, dt = \frac{1 + \alpha \cdot 2^\alpha}{2^\alpha (1 + \alpha) (2 + \alpha)} = \nu_1
\]

and

\[
\int_0^1 |1 - 2t| (1 - t^\alpha) \, dt = \frac{1 - \frac{1 + \alpha \cdot 2^\alpha}{2^\alpha (1 + \alpha) (2 + \alpha)}}{2} = \frac{1}{2} - \nu_1.
\]

Utilizing (2.15) in (2.14), we get the required inequality and hence the proof of the theorem is completed.

**Corollary 4.** If \( \eta(b,a) = b - a \) in theorem 7, the we have the inequality:

\[
(2.17) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{2} \left[ \nu_2 \left| f'(a) \right| + m\nu_1 \left| f' \left( \frac{b}{m} \right) \right| \right],
\]

where \( \nu_1 = \frac{1 + \alpha \cdot 2^\alpha}{2^\alpha (1 + \alpha) (2 + \alpha)} \) and \( \nu_2 = \frac{1}{2} - \nu_1 \).

**Theorem 8.** Let \( K \subseteq [0,b^*] \), \( b^* > 0 \) be an open invex subset with respect to \( \eta : K \times K \to \mathbb{R} \) and \( a, b \in K \) with \( a < a + \eta(b,a) \). Suppose \( f : K \to \mathbb{R} \) is a differentiable mapping on \( K \) such that \( f' \in L([a,a + \eta(b,a)]) \). If \( \left| f' \right|^q \) is \((\alpha, m)\)-preinvex on \( K \), \( q > 1 \), then we have the following inequality:

\[
(2.18) \quad \left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a + \eta(b,a)} f(x) \, dx \right| \\
\leq \frac{\eta(b,a)}{2(p + 1)^\frac{1}{p}} \left[ \frac{\alpha \left| f'(a) \right|^q + m \left| f' \left( \frac{b}{m} \right) \right|^q}{1 + \alpha} \right]^{\frac{1}{q}},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** Using lemma 1 and the Hölder’s integral inequality, we have

\[
(2.19) \quad \left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a + \eta(b,a)} f(x) \, dx \right| \\
\leq \frac{\eta(b,a)}{2} \left( \int_0^1 |1 - 2t|^p \, dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f'(a + t\eta(b,a)) \right|^q \, dt \right)^{\frac{1}{q}}.
\]

By the \((\alpha, m)\)-preinvexity of \( \left| f' \right|^q \), we have for every \( t \in [0,1] \)

\[
\left| f'(a + t\eta(b,a)) \right|^q \leq (1 - t^\alpha) \left| f'(a) \right|^q + mt^\alpha \left| f' \left( \frac{b}{m} \right) \right|^q.
\]
for \((\alpha, m) \in (0,1] \times (0,1]\). Hence
\[
\int_0^1 \left| f' (a + t\eta (b,a)) \right|^q dt \leq \left| f' (a) \right|^q \int_0^1 (1-t^\alpha) dt + m \left| f' \left( \frac{b}{m} \right) \right|^q \int_0^1 t^\alpha dt
\]
\[
= \frac{\alpha}{1+\alpha} \left| f' (a) \right|^q + \frac{m}{1+\alpha} \left| f' \left( \frac{b}{m} \right) \right|^q .
\]

An application of the above inequality in (2.19) and the fact
\[
\int_0^1 \left| 1-2t^p \right| dt = \frac{1}{p+1}
\]
gives the desired inequality. \(\square\)

**Corollary 5.** If in theorem 8, we take \(\eta (b,a) = b - a\), we get the following inequality:

\[
(2.20) \quad \left| f (a) + f (b) - \frac{1}{b-a} \int_a^b f (x) dx \right| 
\leq \frac{b-a}{2 (p+1)^{1/2}} \left[ \alpha \left| f' (a) \right|^q + \frac{m}{1+\alpha} \left| f' \left( \frac{b}{m} \right) \right|^q \right]^{1/2} ,
\]
where \(\frac{1}{p} + \frac{1}{q} = 1\).

**Theorem 9.** Let \(K \subseteq [0,b^*], b^* > 0\) be an open invex subset with respect to \(\eta : K \times K \to \mathbb{R}\) and \(a, b \in K\) with \(a < a + \eta (b,a)\). Suppose \(f : K \to \mathbb{R}\) is a differentiable mapping on \(K\) such that \(f' \in L ([a,a + \eta (b,a)])\). If \(|f'|^q\) is \((\alpha, m)\)-preinvex on \(K\), \(q \geq 1\), then we have the following inequality:

\[
(2.21) \quad \left| f (a) + f (a + \eta (b,a)) \right| - \frac{1}{\eta (b,a)} \int_a^{a+\eta(b,a)} f (x) dx 
\leq \frac{\eta (b,a)}{2} \left( \frac{1}{\nu_2} \right) ^{1/2} \left[ \nu_1 \left| f' (a) \right|^q + m \nu_1 \left| f' (b) \right|^q \right] ^{1/2} ,
\]
where \(\nu_2 = \frac{1}{2} - \nu_1\) and \(\nu_1 = \frac{1+\alpha 2^\alpha}{2+1+\alpha [2+\alpha]}\).

**Proof.** For \(q = 1\), the proof is similar to that of theorem 7. Suppose that \(q > 1\). Using lemma 1, we have that the following inequality holds:

\[
(2.22) \quad \left| f (a) + f (a + \eta (b,a)) \right| - \frac{1}{\eta (b,a)} \int_a^{a+\eta(b,a)} f (x) dx 
\leq \frac{\eta (b,a)}{2} \left( \int_0^1 \left| 1-2t \right| dt \right) ^{1/2} \left( \int_0^1 \left| f' (a + t\eta (b,a)) \right|^q \right) ^{1/2} .
\]
By the \((\alpha, m)\)-preinvexity of \(f_0^q\) on \(K\), for every \(t \in [0,1]\) and \((\alpha, m) \in (0, 1] \times (0, 1]\) we have

\[
\int_0^1 |1 - 2t| f'(a + t\eta(b, a)) dt \\
\leq \int_0^1 |1 - 2t| \left[ (1 - t)\alpha f'(a) + mt^\alpha f'(b) \right] dt \\
= |f'(a)| \int_0^1 |1 - 2t| (1 - t)\alpha dt + m |f'(b)| \int_0^1 |1 - 2t| t^\alpha dt \\
= \nu_2 |f'(a)| + m \nu_1 |f'(b)|^q.
\]

Using (2.23) in (2.22), we get the required inequality (2.21). This completes the proof of the theorem. \(\square\)

**Corollary 6.** Suppose \(\eta(b, a) = b - a\) in theorem 9, then one has the inequality:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right| \\
\leq \frac{b - a}{2} \left( \frac{1}{2} \right)^{1 - \frac{q}{2}} \left[ \nu_2 |f'(a)|^q + m \nu_1 |f'(b)|^q \right]^\frac{1}{q},
\]

where \(\nu_2 = \frac{1}{2} - \nu_1\) and \(\nu_1 = \frac{1 + m 2^\alpha}{2^{\alpha+1}(1 + \alpha)(2 + \alpha)}\).

**Remark 3.** If we take \(m = 1\) in theorem 4 and theorem 5 or if we take \(\alpha = m = 1\) in theorem 7 and theorem 8 we get those results proved in theorem 2 and theorem 3 respectively. This shows that our results are more general than those proved in [3].

**Remark 4.** If we take \(m = 1\) in theorem 4 and theorem 5 or if we take \(\alpha = m = 1\) in theorem 7 and theorem 8 with \(\eta(b, a) = b - a\), we get those results proved in [6] and [25].

### 3. An Extension to Functions of Several Variables

In this section we will extend Corollary 1 and corollary 4 to functions of several variables defined on an invex subset of \(\mathbb{R}^n\). To this end, we need the following property of invex functions.

**Condition C** [34]: Let \(K \subseteq \mathbb{R}^n\) be an open invex subset with respect to \(\eta : K \times K \to \mathbb{R}^n\). For any \(x, y \in K\) and any \(t \in [0, 1]\),

\[
\eta (y, y + t\eta(x, y)) = -t\eta (x, y)
\]

and

\[
\eta (x, y + t\eta(x, y)) = (1 - t) \eta (x, y).
\]

It is to be noted from **Condition C** that for every \(x, y \in K\) and every \(t_1, t_2 \in [0, 1]\), we have

\[
\eta (y + t_2\eta(x, y), y + t_1 \eta(x, y)) = (t_2 - t_1) \eta (x, y).
\]

**Proposition 1.** Let \(K \subseteq \mathbb{R}^n\) be an invex set with respect to \(\eta : K \times K \to \mathbb{R}^n\) and \(f : K \to \mathbb{R}\) is a function. Suppose that \(f\) satisfies **Condition C** on \(K\). Then
for every \(x, y \in K\) the function \(f\) is \(m\)-preinvex with respect to \(\eta\) on \(\eta\)-path \(P_{xv}\), \(v = x + \eta(x, y)\), if and only if the function \(\varphi : [0, 1] \rightarrow \mathbb{R}\) defined by
\[
\varphi(t) := f(x + t\eta(y, x))
\]
is \(m\)-convex on \([0, 1] \, , \, m \in (0, 1]\).

**Proof.** Suppose that \(\varphi\) is \(m\)-convex on \([0, 1]\) and \(z_1 := x + t_1\eta(y, x) \in P_{xv}\) and \(z_2 := x + t_2\eta(y, x) \in P_{xv}\). Fix \(\lambda \in [0; 1]\). Since \(f\) satisfies **Condition C**, by (3.1) we have
\[
\begin{align*}
  f(z_1 + \lambda\eta(z_2, z_1)) &= f(x + (1 - \lambda)t_1 + \lambda t_2))\eta(y, x) \\
  &= \varphi((1 - \lambda)t_1 + \lambda t_2) \\
  &\leq (1 - \lambda)\varphi(t_1) + m\lambda\varphi\left(\frac{t_2}{m}\right) \\
  &= (1 - \lambda)f(z_1) + m\lambda f\left(\frac{z_2}{m}\right).
\end{align*}
\]

Conversely, let \(x, y \in K\) and the function \(f\) be \(m\)-preinvex with respect to \(\eta\) on \(\eta\)-path \(P_{xv}\). Suppose that \(t_1, t_2 \in [0, 1]\). Then for every \(\lambda \in [0, 1], \, m \in (0, 1]\) and using (3.1), we have
\[
\begin{align*}
  \varphi((1 - \lambda)t_1 + \lambda t_2) &= f(x + ((1 - \lambda)t_1 + \lambda t_2)\eta(y, x)) \\
  &= f(x + t_1\eta(y, x) + \lambda(t_2 - t_1)\eta(y, x)) \\
  &= f(x + t_1\eta(y, x) + \lambda\eta(x + t_2\eta(x, y), x + t_1\eta(x, y))) \\
  &\leq (1 - \lambda)f(x + t_2\eta(y, x)) + m\lambda f\left(\frac{x + t_2\eta(x, y)}{m}\right) \\
  &= (1 - \lambda)\varphi(t_1) + m\lambda\varphi\left(\frac{t_2}{m}\right).
\end{align*}
\]
Hence \(\varphi\) is \(m\)-preinvex function on \([0, 1]\). \(\square\)

**Proposition 2.** Let \(K \subseteq \mathbb{R}^n\) be an invex set with respect to \(\eta : K \times K \rightarrow \mathbb{R}^n\) and \(f : K \rightarrow \mathbb{R}\) is a function. Suppose that \(\eta\) satisfies **Condition C** on \(K\). Then for every \(x, y \in K\) the function \(f\) is \((\alpha, m)\)-preinvex with respect to \(\eta\) on \(\eta\)-path \(P_{xv}\), \(v = x + \eta(x, y)\), if and only if the function \(\varphi : [0, 1] \rightarrow \mathbb{R}\) defined by
\[
\varphi(t) := f(x + t\eta(y, x))
\]
is \((\alpha, m)\)-convex on \([0, 1], \, (\alpha, m) \in (0, 1] \times (0, 1]\).

**Proof.** The proof is similar to that of the proof of proposition 1, therefore we omit the details. \(\square\)

**Theorem 10.** Let \(K \subseteq \mathbb{R}^n\) be an invex set with respect to \(\eta : K \times K \rightarrow \mathbb{R}^n\) and \(f : K \rightarrow \mathbb{R}^+\) is a function. Suppose that \(\eta\) satisfies Condition C on \(K\). Suppose that for every \(x, y \in K\) the function \(f\) is \(m\)-preinvex with respect to \(\eta\) on \(\eta\)-path \(P_{xv}, \, m \in (0, 1]\). Then for every \(a, b \in (0, 1)\) with \(a < b\) the following inequality
holds:

\[(3.2) \quad \left| \frac{1}{2} \left( \int_0^a f(x + s\eta(y, x))ds + \int_0^b f(x + s\eta(y, x))ds \right) \right|
\]
\[\quad - \frac{1}{b-a} \int_a^b \left( \int_0^s f(x + t\eta(y, x))dt \right) ds \leq \frac{b-a}{8} \left[ (\phi'(a) + m\phi'\left(\frac{b}{m}\right)) \right].\]

\[\quad \leq \left| \int_0^a f(x + s\eta(y, x))ds + \int_0^b f(x + s\eta(y, x))ds \right|\]

Proof. Let \(x, y \in K\) and \(a, b \in (0, 1)\) with \(a < b\). Since \(f : K \rightarrow \mathbb{R}^+\) is \(m\)-preinvex with respect to \(\eta\) on \(\eta\)-path \(P_{xy}, m \in (0, 1]\), by proposition 1 the function \(\varphi : [0, 1] \rightarrow \mathbb{R}^+\) defined by

\[\varphi(t) := f(x + t\eta(y, x))\]

is \(m\)-convex on \([0, 1]\). Now we define function \(\phi : [0, 1] \rightarrow \mathbb{R}^+\) as

\[\phi(t) := \int_0^a \varphi(s)ds = \int_0^t f(x + s\eta(y, x))ds.\]

It is clear that for every \(t \in (0, 1)\) we have

\[\phi'(t) = \varphi(t) = f(x + t\eta(y, x)) \geq 0,\]

hence \(\phi'(t) = \phi'(t)\). Applying corollary 1 to the function \(\phi\), we get

\[(3.3) \quad \left| \frac{\phi(a) + \phi(b)}{2} - \frac{1}{b-a} \int_a^b \phi(s) ds \right| \leq \frac{b-a}{8} \left[ (\phi'(a) + m\phi'\left(\frac{b}{m}\right)) \right],\]

we deduce from (3.3) that (3.2) holds. This completes the proof of the theorem. \(\square\)

**Theorem 11.** Let \(K \subseteq \mathbb{R}^n\) be an invex set with respect to \(\eta : K \times K \rightarrow \mathbb{R}^n\) and \(f : K \rightarrow \mathbb{R}^+\) is a function. Suppose that \(\eta\) satisfies Condition C on \(K\). Suppose that for every \(x, y \in K\) the function \(f\) is \((\alpha, m)\)-preinvex with respect to \(\eta\) on \(\eta\)-path \(P_{xy}, (\alpha, m) \in (0, 1]\). Then for every \(a, b \in (0, 1)\) with \(a < b\) the following inequality holds:

\[(3.4) \quad \left| \frac{1}{2} \left( \int_0^a f(x + s\eta(y, x))ds + \int_0^b f(x + s\eta(y, x))ds \right) \right|
\]
\[\quad - \frac{1}{b-a} \int_a^b \left( \int_0^s f(x + t\eta(y, x))dt \right) ds \leq \frac{b-a}{8} \left[ \nu_2 f(x + a\eta(y, x)) + m\nu_1 f \left( x + \frac{b}{m}\eta(y, x) \right) \right],\]

where \(\nu_1 = \frac{1+\alpha^2}{2^{\alpha}(1+\alpha)(2+\alpha)}\) and \(\nu_2 = \frac{1}{2} - \nu_1\).

Proof. The proof of is similar to that of theorem 10 using corollary 4 so we omit the details to the readers. \(\square\)

**Remark 5.** Let \(\varphi(t) : [0, 1] \rightarrow \mathbb{R}^+\) be a function and \(q\) be a positive real number. Then \(\varphi\) is \(m\)-convex or \((\alpha, m)\)-convex function if and only if \(\varphi(t)^q : [0, 1] \rightarrow \mathbb{R}^+\) is \(m\)-convex or \((\alpha, m)\)-convex respectively. Hence similar results can be stated as
those of proposition 1 and proposition 2 by using corollary 2, corollary 3, corollary 5 and corollary 6 and we omit the details for the interested reader.

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