SOME APPLICATIONS OF FEJÉR’S INEQUALITY FOR CONVEX FUNCTIONS (I)

S.S. DRAGOMIR\textsuperscript{1,2} AND I. GOMM\textsuperscript{1}

Abstract. Some applications of Fejér’s inequality for convex functions are explored. Upper and lower bounds for the weighted integral

$$\int_a^b (b-x)(x-a) f(x) \, dx$$

under various assumptions for $f$ with applications to the trapezoidal quadrature rule are given. Some inequalities for special means are also provided.

1. Introduction

The Hermite-Hadamard integral inequality for convex functions $f : [a, b] \to \mathbb{R}$

\begin{equation}
\left(\text{HH}\right) \quad f \left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\end{equation}

is well known in the literature and has many applications for special means.

For related results, see for instance the research papers \cite{1, 8, 9, 10, 12, 11, 13, 14, 15}, the monograph online \cite{7} and the references therein.

In 1906, Fejér, while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

**Theorem 1.** Consider the integral $\int_a^b h(x) \, dx$, where $h$ is a convex function in the interval $(a, b)$ and $w$ is a positive function in the same interval such that

$$w(a+t) = w(b-t), \quad 0 \leq t \leq \frac{1}{2} (a+b),$$

i.e., $y = w(x)$ is a symmetric curve with respect to the straight line which contains the point $\left( \frac{1}{2} (a+b), 0 \right)$ and is normal to the $x$-axis. Under those conditions the following inequalities are valid:

\begin{equation}
(1.1) \quad h \left(\frac{a+b}{2}\right) \int_a^b w(x) \, dx \leq \int_a^b h(x) w(x) \, dx \leq \frac{h(a) + h(b)}{2} \int_a^b w(x) \, dx.
\end{equation}

If $h$ is concave on $(a,b)$, then the inequalities reverse in $(1.1)$.

Clearly, for $w(x) \equiv 1$ on $[a,b]$ we get \textup{HH}.

We observe that, if we take $w(x) = (b-x)(x-a)$, $x \in [a,b]$, then $w$ satisfies the conditions in Theorem 1.

$$\int_a^b (b-x)(x-a) \, dx = \frac{1}{6} (b-a)^3$$

1991 Mathematics Subject Classification. 26D15; 25D10.

Key words and phrases. Convex functions, Hermite-Hadamard inequality, Fejér’s Inequality, Special means.
and by [1.1] we have the following inequality

\[(1.2) \quad \frac{1}{6} h \left( \frac{a + b}{2} \right) (b - a)^3 \leq \int_a^b (b - x) (x - a) h (x) \, dx \]
\[\leq \frac{h(a) + h(b)}{12} (b - a)^3 , \]

for any convex function \( h : [a, b] \to \mathbb{R} \). If the function \( h \) is concave the inequalities in (1.2) reverse.

In this paper we establish amongst other some better bounds for the weighted integral

\[\int_a^b (b - x) (x - a) h (x) \, dx \]
in the case of convex functions \( h : [a, b] \to \mathbb{R} \). We also investigate the connection with the trapezoid rule and apply some of the obtained results for special means.

2. The Results

The following result holds.

**Theorem 2.** Let \( f : [a, b] \to \mathbb{R} \) be a twice differentiable function on \((a, b)\) and such that the second derivative \( f'' \) is convex on \((a, b)\). Then

\[(2.1) \quad \frac{1}{12} f'' \left( \frac{a + b}{2} \right) (b - a)^2 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f (x) \, dx \]
\[\leq \frac{f''(a) + f''(b)}{24} (b - a)^2 . \]

**Proof.** We know, see for instance [7, Lemma 4, p. 38], that

\[(2.2) \quad \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f (x) \, dx = \frac{1}{2 (b - a)} \int_a^b (x - a) (b - x) f'' (x) \, dx . \]

Since \( f'' \) is convex on \((a, b)\), then by (1.2) we have

\[(2.3) \quad \frac{1}{6} f'' \left( \frac{a + b}{2} \right) (b - a)^3 \leq \int_a^b (b - x) (x - a) f'' (x) \, dx \]
\[\leq \frac{f''(a) + f''(b)}{12} (b - a)^3 . \]

Utilising (2.2) and (2.3) we deduce the desired result (2.1). \( \square \)

**Theorem 3.** Let \( f : [a, b] \to \mathbb{R} \) be a twice differentiable function on \((a, b)\).

If there exists a real number \( m \) such that \( f'' (x) \geq m \) for any \( x \in (a, b) \), then

\[(2.4) \quad \frac{1}{6} f \left( \frac{a + b}{2} \right) (b - a)^3 + \frac{1}{240} m (b - a)^5 \]
\[\leq \int_a^b (b - x) (x - a) f (x) \, dx \]
\[\leq \frac{f(a) + f(b)}{12} (b - a)^3 - \frac{1}{60} m (b - a)^5 , \]
If there exists a real number $M$ such that $f''(x) \leq M$ for any $x \in (a, b)$, then

\begin{equation}
\frac{f(a) + f(b)}{12} (b - a)^3 - \frac{1}{60} M (b - a)^5
\leq \int_a^b (b - x) (x - a) f(x) \, dx
\leq \frac{1}{6} f\left(\frac{a + b}{2}\right) (b - a)^3 + \frac{1}{240} M (b - a)^5.
\end{equation}

**Proof.** Define the function $h_m : [a, b] \to \mathbb{R}$ by

$$h_m(x) := f(x) + \frac{1}{2} m (x - a) (b - x).$$

This function is twice differentiable and the second derivative is

$$h''_m(x) = f''(x) - m \geq 0, \ x \in (a, b)$$

showing that $h_m$ is convex on $[a, b]$.

If we apply the inequality (1.2) for $h_m$, then we have

\begin{equation}
\frac{1}{6} \left[ f\left(\frac{a + b}{2}\right) + \frac{1}{8} m (b - a)^2 \right] (b - a)^3
\leq \int_a^b (b - x) (x - a) f(x) \, dx + \frac{1}{2} m \int_a^b (b - x)^2 (x - a)^2 \, dx
\leq \frac{f(a) + f(b)}{12} (b - a)^3.
\end{equation}

Observe that

$$\frac{1}{6} \left[ f\left(\frac{a + b}{2}\right) + \frac{1}{8} m (b - a)^2 \right] (b - a)^3
= \frac{1}{6} f\left(\frac{a + b}{2}\right) (b - a)^3 + \frac{1}{48} m (b - a)^5.$$

We also have

$$\int_a^b (b - x)^2 (x - a)^2 \, dx = \frac{1}{3} (x - a)^3 (b - x)^2\bigg|_a^b + \frac{2}{3} \int_a^b (b - x) (x - a)^3 \, dx
= \frac{2}{3} \left( \frac{1}{4} (b - x) (x - a)^4 \bigg|_a^b + \frac{1}{4} \int_a^b (x - a)^4 \, dx \right)
= \frac{1}{30} (b - a)^5.$$

Then (2.6) becomes

$$\frac{1}{6} f\left(\frac{a + b}{2}\right) (b - a)^3 + \frac{1}{48} m (b - a)^5
\leq \int_a^b (b - x) (x - a) f(x) \, dx + \frac{1}{60} m (b - a)^5
\leq \frac{f(a) + f(b)}{12} (b - a)^3,$$

which is equivalent with (2.4).
Now define the function $h_M : [a, b] \to \mathbb{R}$ by

$$h_M(x) := -f(x) - \frac{1}{2} M(x - a)(b - x).$$

This function is twice differentiable and

$$h''_M(x) := M - f''(x) \geq 0, \quad x \in (a, b)$$

showing that $h_M$ is convex on $[a, b]$.

If we apply the inequality (1.2) for $h_M$, then we have

$$\frac{1}{6} \left[ -f\left(\frac{a+b}{2}\right) - \frac{1}{8} M (b-a)^2 \right] (b-a)^3$$

$$\leq \int_a^b (b-x)(x-a) \left[ -f(x) - \frac{1}{2} M(x-a)(b-x) \right] dx$$

$$\leq \frac{-f(a) - f(b)}{12} (b-a)^3,$$

which, by multiplication with $-1$, produces

$$\frac{1}{6} f\left(\frac{a+b}{2}\right) (b-a)^3 + \frac{1}{48} M (b-a)^5$$

$$\geq \int_a^b (b-x)(x-a) f(x) dx + \frac{1}{2} M \int_a^b (x-a)^2 (b-x)^2 dx$$

$$\geq \frac{f(a) + f(b)}{12} (b-a)^3$$

that is equivalent with

$$\frac{f(a) + f(b)}{12} (b-a)^3 - \frac{1}{60} M (b-a)^5$$

$$\leq \int_a^b (b-x)(x-a) f(x) dx$$

$$\leq \frac{1}{6} f\left(\frac{a+b}{2}\right) (b-a)^3 + \frac{1}{240} M (b-a)^5$$

and the inequality (2.5) is proved. \hfill \square

**Corollary 1.** Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable function on $(a, b)$. If there exists a $K > 0$ such that $|f''(x)| \leq K$ for any $x \in (a, b)$, then

$$\int_a^b (b-x)(x-a) f(x) dx - \frac{1}{12} (b-a)^3 \left[ f\left(\frac{a+b}{2}\right) + f(a) + f(b) \right]$$

$$\leq \frac{1}{96} K (b-a)^5.$$

**Proof.** If we write the inequality (2.4) for $m = -K$ and the inequality (2.5) for $M = K$ we have
Some Applications of Fejér's Inequality

(2.8) \[
\frac{1}{6} f \left( \frac{a+b}{2} \right) (b-a)^3 - \frac{1}{240} K (b-a)^5 \\
\leq \int_a^b (b-x) (x-a) f(x) \, dx \\
\leq \frac{f(a) + f(b)}{12} (b-a)^3 + \frac{1}{60} K (b-a)^5,
\]

and

(2.9) \[
\frac{f(a) + f(b)}{12} (b-a)^3 - \frac{1}{60} K (b-a)^5 \\
\leq \int_a^b (b-x) (x-a) f(x) \, dx \\
\leq \frac{1}{6} f \left( \frac{a+b}{2} \right) (b-a)^3 + \frac{1}{240} K (b-a)^5.
\]

If we add the inequality (2.8) with (2.8) and divide the sum by 2 we get

\[
\frac{1}{12} f \left( \frac{a+b}{2} \right) (b-a)^3 + \frac{f(a) + f(b)}{24} (b-a)^3 - \frac{1}{96} K (b-a)^5 \\
\leq \int_a^b (b-x) (x-a) f(x) \, dx \\
\leq \frac{1}{12} f \left( \frac{a+b}{2} \right) (b-a)^3 + \frac{f(a) + f(b)}{24} (b-a)^3 + \frac{1}{96} K (b-a)^5,
\]

which is equivalent with the desired result (2.7).

\[\square\]

Remark 1. We observe that the case \( m > 0 \) in the inequality (2.4) produces a better result than (1.2).

For twice differentiable functions we can provide the following perturbed trapezoid quadrature rule

(2.10) \[
\int_a^b f(x) \, dx \approx \frac{f(a) + f(b)}{2} (b-a) \\
- \frac{1}{24} (b-a)^3 \left[ f'' \left( \frac{a+b}{2} \right) + \frac{f''(a) + f''(b)}{2} \right].
\]

Denote \( R_{P,T}(f; a, b) \) the error in approximating the integral as in (2.10), namely

\[
R_{P,T}(f; a, b) := \int_a^b f(x) \, dx - \frac{f(a) + f(b)}{2} (b-a) \\
+ \frac{1}{24} (b-a)^3 \left[ f'' \left( \frac{a+b}{2} \right) + \frac{f''(a) + f''(b)}{2} \right].
\]

The following result that provides an a priori error bound for functions whose forth derivatives are bounded, holds.

Proposition 1. Let \( f : [a, b] \to \mathbb{R} \) be a four time differentiable function on \((a, b)\). If there exists a \( K > 0 \) such that \( |f^{(4)}(x)| \leq K \) for any \( x \in (a, b) \), then

(2.11) \[
|R_{P,T}(f; a, b)| \leq \frac{1}{192} K (b-a)^5.
\]
The following result that improves the inequality (1.2) also holds.

**Theorem 4.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a convex function. Then

\[
\frac{1}{6} f \left( \frac{a + b}{2} \right) (b - a)^3 \leq 2 \int_a^b \left( x - \frac{a + b}{2} \right)^2 f \left( \frac{x + \frac{a + b}{2}}{2} \right) dx
\]

\[
\leq \int_a^b (b - x) (x - a) f (x) dx
\]

\[
\leq \int_a^b \left( x - \frac{a + b}{2} \right)^2 f (x) dx + \frac{(b - a)^3}{12} f \left( \frac{a + b}{2} \right)
\]

\[
\leq \frac{f (a) + f (b)}{2} (b - a)^3.
\]

**Proof.** Denote, as usual, \( F (x) := \int_a^x f (t) dt, \) \( x \in [a, b] \). By the Hermite-Hadamard inequality we have for any \( x \in [a, b], \) \( x \neq \frac{a + b}{2} \) that

\[
f \left( \frac{x + \frac{a + b}{2}}{2} \right) \leq \frac{F (x) - F \left( \frac{a + b}{2} \right)}{x - \frac{a + b}{2}} \leq \frac{1}{2} \left[ f (x) + f \left( \frac{a + b}{2} \right) \right],
\]

which, by multiplication with \( \left( x - \frac{a + b}{2} \right)^2 \geq 0 \) implies

\[
f \left( \frac{x + \frac{a + b}{2}}{2} \right) \left( x - \frac{a + b}{2} \right)^2
\]

\[
\leq \left[ F (x) - F \left( \frac{a + b}{2} \right) \right] \left( x - \frac{a + b}{2} \right)
\]

\[
\leq \frac{1}{2} \left[ f (x) + f \left( \frac{a + b}{2} \right) \right] \left( x - \frac{a + b}{2} \right)^2.
\]
that holds for any \( x \in [a, b] \).

Integrating the inequality \( (2.13) \) on the interval \([a, b]\) we get

\[
\int_a^b \left(x - \frac{a + b}{2}\right)^2 \left(\frac{x + a + b}{2}\right) \, dx
\leq \int_a^b \left[F(x) - F\left(\frac{a + b}{2}\right)\right] \left(x - \frac{a + b}{2}\right) \, dx
\]

\[
\leq \frac{1}{2} \int_a^b \left[f(x) + f\left(\frac{a + b}{2}\right)\right] \left(x - \frac{a + b}{2}\right)^2 \, dx
\]

\[
= \frac{1}{2} \left[\int_a^b \left(x - \frac{a + b}{2}\right)^2 f(x) \, dx + f\left(\frac{a + b}{2}\right) \frac{(b - a)^3}{12}\right].
\]

Now, observe that

\[
\int_a^b \left[F(x) - F\left(\frac{a + b}{2}\right)\right] \left(x - \frac{a + b}{2}\right) \, dx
\]

\[
= \int_a^b F(x) \left(x - \frac{a + b}{2}\right) \, dx = \frac{1}{2} \int_a^b F(x) d\left(x - \frac{a + b}{2}\right)^2
\]

\[
= \frac{1}{2} \left[F(x) \left(x - \frac{a + b}{2}\right)^2\right]_a^b - \int_a^b \left(x - \frac{a + b}{2}\right)^2 f(x) \, dx
\]

\[
= \frac{1}{2} \left[\frac{(b - a)}{2} \int_a^b f(x) - \int_a^b \left(x - \frac{a + b}{2}\right)^2 f(x) \, dx\right]
\]

\[
= \frac{1}{2} \int_a^b \left(b - x\right) \left(x - a\right) f(x) \, dx
\]

and by \( (2.14) \) we have

\[
\int_a^b \left(x - \frac{a + b}{2}\right)^2 \left(\frac{x + a + b}{2}\right) \, dx
\]

\[
\leq \frac{1}{2} \int_a^b \left(b - x\right) \left(x - a\right) f(x) \, dx
\]

\[
= \frac{1}{2} \left[\int_a^b \left(x - \frac{a + b}{2}\right)^2 f(x) \, dx + f\left(\frac{a + b}{2}\right) \frac{(b - a)^3}{12}\right],
\]

which proves the second and the third inequality in \( (2.12) \).

The function \( g(x) := f\left(\frac{x + a + b}{2}\right) \) is convex on \([a, b]\) and \( w(x) := (x - \frac{a + b}{2})^2 \) is nonnegative and symmetric on \([a, b]\). Applying Fejér’s first inequality we have

\[
f\left(\frac{a + b}{2}\right) \int_a^b \left(x - \frac{a + b}{2}\right)^2 \, dx \leq \int_a^b f\left(\frac{x + a + b}{2}\right) \left(x - \frac{a + b}{2}\right)^2 \, dx
\]
i.e.
\[
\frac{(b-a)^3}{12} f \left( \frac{a+b}{2} \right) \leq \int_a^b \left( x - \frac{a+b}{2} \right)^2 f \left( \frac{x + \frac{a+b}{2}}{2} \right) dx,
\]
which proves the first inequality in (2.12).

From the Fejér’s second inequality for the convex function \( f \) function and the weight \( w(x) := (x - \frac{a+b}{2})^2 \) we also have
\[
\int_a^b \left( x - \frac{a+b}{2} \right)^2 f(x) dx \leq \frac{f(a) + f(b)}{2} \int_a^b \left( x - \frac{a+b}{2} \right)^2 dx
\]
\[
= \frac{f(a) + f(b)}{24} (b-a)^3,
\]
which proves the fourth inequality in (2.12).

The last inequality is obvious. \( \square \)

**Corollary 2.** Let \( f : [a, b] \to \mathbb{R} \) be a twice differentiable function on \( (a, b) \) and such that the second derivative \( f'' \) is convex on \( (a, b) \). Then
\[(2.15)\]
\[
\frac{1}{12} f'' \left( \frac{a+b}{2} \right) (b-a)^2 \leq \int_a^b \left( x - \frac{a+b}{2} \right)^2 f'' \left( \frac{x + \frac{a+b}{2}}{2} \right) dx
\]
\[
\leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx
\]
\[
\leq \frac{1}{2} \int_a^b \left( x - \frac{a+b}{2} \right)^2 f''(x) dx + \frac{(b-a)^3}{24} f'' \left( \frac{a+b}{2} \right)
\]
\[
\leq \frac{(b-a)^3}{24} \left[ f'' \left( \frac{a+b}{2} \right) + \frac{f''(a) + f''(b)}{2} \right]
\]
\[
\leq \frac{f''(a) + f''(b)}{24} (b-a)^3.
\]

We observe that the inequality (2.15) is a better result than (2.1).

### 3. Applications for Special Means

Let us recall the following means for two positive numbers.

1. **The Arithmetic mean**
   \[
   A = A(a, b) := \frac{a+b}{2}, \; a, b > 0;
   \]

2. **The Geometric mean**
   \[
   G = G(a, b) := \sqrt{ab}, \; a, b > 0;
   \]

3. **The Harmonic mean**
   \[
   H = H(a, b) := \frac{2ab}{a+b}, \; a, b > 0;
   \]

4. **The Logarithmic mean**
   \[
   L = L(a, b) := \begin{cases} 
   a & \text{if } a = b \\
   \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b;
   \end{cases}, \; a, b > 0,
   \]
The Identric mean

\[ I = I(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left( \frac{b^e}{a^e} \right)^{\frac{1}{e}} & \text{if } a \neq b \end{cases}, \quad a, b > 0; \]

The p-Logarithmic mean

\[ L_p = L_p(a, b) := \begin{cases} a & \text{if } a = b \\ \left[ \frac{b^{p+1-a} - a^{p+1}}{(p+1)(b-a)} \right]^\frac{1}{p} & \text{if } a \neq b \end{cases}, \quad a, b > 0. \]

The following inequality is well known in the literature:

\[ H \leq G \leq L \leq I \leq A. \]

It is also known that \( L_p \) is monotonically increasing over \( p \in \mathbb{R} \), denoting \( L_0 = I \) and \( L_{-1} = L \).

Consider the function \( f : [a, b] \subset (0, \infty) \rightarrow (0, \infty) \), \( f(x) = x^p \) for \( p \geq 3 \). We have the fourth derivative of the function given by

\[ f^{(4)}(x) = p(p-1)(p-2)(p-3)x^{p-4} \]

which shows that the second derivative \( f'' \) is convex on \([a, b]\). Applying the inequality [2.1] we have

\[ \frac{1}{12} p(p-1) \left( \frac{a+b}{2} \right)^{p-2} (b-a)^2 \leq \frac{a^p + b^p}{2} - \frac{1}{b-a} \int_a^b x^p dx \]

\[ \leq p(p-1) \frac{a^{p-2} + b^{p-2}}{24} (b-a)^2, \]

which in terms of the special means define above can be written as

\[ \frac{1}{12} p(p-1) A^{p-2} (a, b) (b-a)^2 \leq A(a^p, b^p) - L_p^p (a, b) \]

\[ \leq \frac{1}{12} p(p-1) A(a^{p-2}, b^{p-2}) (b-a)^2, \]

that holds for any \( a, b > 0 \) and \( p \geq 3 \).

Consider the function \( f : [a, b] \subset (0, \infty) \rightarrow (0, \infty) \), \( f(x) = \frac{1}{x} \). Then \( f''(x) = \frac{2}{x^3} \)

and \( f^{(4)}(x) = \frac{24}{x^5} \) showing that the second derivative is convex on \([a, b]\). Applying the inequality [2.1] we have

\[ \frac{1}{6} \frac{(b-a)^2}{A^3(a,b)} \leq \frac{1}{a + b} \frac{1}{2} \frac{\ln b - \ln a}{b-a} \]

\[ \leq \frac{2}{a^2} + \frac{2}{b^2} (b-a)^2, \]

which is equivalent with

\[ \frac{1}{6} \frac{(b-a)^2}{A^3(a,b)} \leq \frac{L(a,b) - H(a,b)}{L(a,b) H(a,b)} \leq \frac{1}{6} \frac{(b-a)^2}{H(a^3, b^3)} \]

that holds for any \( a, b > 0 \).
Consider the function $f : [a, b] \subset (0, \infty) \to (0, \infty)$, $f(x) = -\ln x$. Then $f''(x) = \frac{1}{x^2}$ and $f^{(4)}(x) = \frac{2}{x^3}$ showing that the second derivative is convex on $[a, b]$. Applying the inequality (2.1) we have
\[ \frac{1}{12} \frac{(b-a)^2}{A^2(a,b)} \leq -\frac{\ln a - \ln b}{2} + \frac{1}{b-a} \int_a^b \ln x \, dx \]
\[ \leq \frac{1}{24} + \frac{1}{24} (b-a)^2. \]

Observe that
\[ \frac{1}{b-a} \int_a^b \ln x \, dx = \frac{1}{b-a} \left[ x \ln x \bigg|_a^b - (b-a) \right] = \left[ \ln \left( \frac{b}{a} \right)^{1/(b-a)} - 1 \right] = \ln I(a,b), \]
and
\[ -\frac{\ln a - \ln b}{2} = \ln \frac{1}{G(a,b)}. \]

Then we get
\[ \frac{1}{12} \frac{(b-a)^2}{A^2(a,b)} \leq \ln \left( \frac{I(a,b)}{G(a,b)} \right) \leq \frac{1}{12} \frac{(b-a)^2}{H(a^2,b^2)} \]
that holds for any $a, b > 0$.

The interested reader may apply the inequality (2.11) or (2.15) to obtain other similar results. However, the details are omitted here.

**References**


1Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

2School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa.