Weighted Norm Inequalities for Commutators of One-sided Discrete Square Functions

ZUNWEI FU

Abstract. The purpose of this paper is to prove the strong type inequalities with one-sided weights for commutators (with symbol $b \in \text{Lip}_\beta$) of one-sided discrete square functions. We also prove that $b \in \text{Lip}_\beta$ is a sufficient and necessary condition for the corresponding boundedness of commutators of one-sided maximal operators.

1 Introduction

A well known result of Coifman-Rochberg-Weiss\cite{4} states that the commutator

$$T_b f = T(bf) - bT(f)$$

(where $T$ is a Calderón-Zygmund singular integral operator) is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in \text{BMO}$. There are other links between the boundedness properties of the operator $T_b$ and the smoothness of $b$. A particular case of the result of Jason\cite{6} states that $T_b: L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is bounded, $1 < p < q < \infty$, if and only if $b \in \text{Lip}_\beta$, $1/p - 1/q = \beta/n$. Here, $\text{Lip}_\beta$ is the homogeneous Lipschitz space.

Many authors have studied strong and weak type inequalities for commutators with weights (see \cite{2}, \cite{9}, \cite{10}, \cite{19}). Furthermore, many of the results have been generalized to commutators of other operators, not only Calderón-Zygmund operators (see \cite{5}, \cite{21}, \cite{22}).

Very recently, Lorente-Riveros\cite{11} proved the strong type inequalities with one-sided weights for commutators (with symbol $b \in \text{BMO}$) of one-sided discrete square functions. Highly inspired by \cite{6} and \cite{11}, we shall prove the strong type inequalities with one-sided weights for commutators (with symbol $b \in \text{Lip}_\beta$) of one-sided discrete square functions. We also prove that $b \in \text{Lip}_\beta$ is a sufficient and necessary condition for the corresponding boundedness of commutators of one-sided maximal operators.

Throughout this paper the letter $C$ will be a positive constant, not necessarily the same at each occurrence. If $1 \leq p \leq \infty$, then its conjugate exponent will be denoted by $p'$ and $A_p$ will be the classical Muckenhoupt’s class of weights (see \cite{17}).

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2 Definitions and main results

Let \( f \) be a measurable function on \( \mathbb{R} \). For each \( n \in \mathbb{Z} \), let us define the operator \( A_n \) by
\[
A_n = \frac{1}{2^n} \int_{x+2^n}^{x+2^n} f(y) dy.
\]
It is a classical problem to study the different kinds of convergence of the \( \{A_n f\}_n \) when the function \( f \) belongs to \( L^p(\mathbb{R}, dx) \), being \( p \) in the range \( 1 \leq p < \infty \).

A method of measuring the speed of convergence of the sequence \( \{A_n f\}_n \) is to analyze the boundedness of the square function
\[
Sf(x) = \left( \sum_{n \in \mathbb{Z}} |A_n f(x) - A_{n-1} f(x)|^2 \right)^{1/2}.
\]
The square function \( S \) is of interest in ergodic theory and it has been extensively studied. In particular it has been proved in [7] that \( S \) (under the assumption of \( n \in \mathbb{Z} \) in the definition of \( S \), we denote \( S \) by \( S_{-} \)) is of weak type \((1, 1)\), maps \( L^p(\mathbb{R}) \) into itself, \( 1 < p < \infty \). For the Ergodic theory and connections with Analysis and Probability, we choose to refer to [3] and [8].

It is not difficult to see that \( Sf(x) = \|U^+ f(x)\|_2 \), where \( U^+ \) is the sequence valued operator
\[
U^+ f(x) = \int_{\mathbb{R}} H(x-y) f(y) dy,
\]
where
\[
H(x) = \left\{ \frac{1}{2^n} \chi(-2^n, 0) - \frac{1}{2^{n-1}} \chi(-2^{n-1}, 0) \right\}_{n \in \mathbb{Z}}.
\]
(See [23].)

**Definition 2.1** The one-sided Hardy-Littlewood maximal operators \( M^+ \) and \( M^- \) are defined for locally integrable functions \( f \) by
\[
M^+ f(x) = \sup_{h > 0} \frac{1}{h} \int_{x-h}^{x+h} |f| \quad \text{and} \quad M^- f(x) = \sup_{h > 0} \frac{1}{h} \int_{x-h}^x |f|.
\]
The good weights for these operators are the one-sided weights, \( A_p^+ \) and \( A_p^- \):
\[
\sup_{a < b < c} \frac{1}{(c-a)^p} \int_a^b \omega \left( \int_b^c \omega^1-p \right)^{p-1} < \infty, \quad 1 < p < \infty, \quad (A_p^+),
\]
\[
M^- \omega(x) < C \omega(x), \quad a.e. \quad (A_p^+),
\]
and
\[
A_p^+ = \cup_{p \geq 1} A_p^+. \quad (A_p^+)
\]
The classes \( A_p^- \) are defined in a similar way. It is interesting to note that \( A_p = A_p^+ \cap A_p^- \), \( A_p \subset A_p^+ \) and \( A_p \subset A_p^- \). \( M^+ \) is bounded on \( L^p(\omega) \) if and only if satisfies the \( A_p^+ \) condition. (See [13], [14], [20] for more definitions and results.)
It is proved in [23] that \( \omega \in A^+_p, 1 < p < \infty \), if and only if \( S \) is bounded from \( L^p(\omega) \) to \( L^p(\omega) \) and that \( \omega \in A^+_1 \) if and only if \( S \) is of weak-type \((1,1)\) with respect to \( \omega \).

**Definition 2.2** The one-sided fractional maximal operator \( M^+_\alpha, 0 < \alpha < 1 \), is defined for locally integrable functions \( f \) by

\[
M^+_\alpha f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_{x}^{x+h} |f|. 
\]

It is proved in [1] that \( M^+_\alpha \) is bounded from \( L^p(\omega^p) \) to \( L^q(\omega^q) \) if and only if \( \omega \in A^+(p,q) \), for \( 1 < p < q, 1/p - 1/q = \alpha \), where

\[
\left( \frac{1}{h} \int_{x-h}^{x} \omega^q \right)^{1/q} \left( \frac{1}{h} \int_{x}^{x+h} \omega^{-p'} \right)^{1/p'} \leq C, \quad (A^+(p,q))
\]

\[
\|\omega \chi_{[x-h,x]}\|_\infty \left( \frac{1}{h} \int_{x}^{x+h} \omega^{-p'} \right)^{1/p'} \leq C, \quad (A^+(p,\infty))
\]

for all \( h > 0 \) and \( x \in \mathbb{R} \).

**Definition 2.3** [18] The Lipschitz space \( Lip_\beta(\mathbb{R}) \) is the space of functions \( f \) satisfying

\[
\|f\|_{Lip_\beta(\mathbb{R}^n)} = \sup_{x,h \in \mathbb{R}, h \neq 0} \frac{|f(x+h) - f(x)|}{|h|^\beta} < \infty.
\]

Now we shall state our results.

**Theorem 2.4** Let \( b \in Lip_\beta(\mathbb{R}) \), and \( k \in \mathbb{N} \). The \( k \)-th order commutator of the one-sided discrete square function is defined by

\[
S^k_b f(x) = \left\| \int_{\mathbb{R}} (b(x) - b(y))^k H(x-y) f(y) dy \right\|_{l^2}.
\]

Then for \( \omega \in A^+(p,q), 1 < p < q < \infty, 1/p - 1/q = k/\beta \), we have

\[
\left( \int_{\mathbb{R}} |S^k_b f|^q \omega^q \right)^{1/q} \leq C \left( \int_{\mathbb{R}} |f|^p \omega^p \right)^{1/p},
\]

for all bounded \( f \) with compact support.

Obviously, if \( \beta > 1 \), \( Lip_\beta(\mathbb{R}) \) contains only constants, \( S^k_b \equiv 0 \), so we will only concentrate our discussion on the cases \( 0 < \beta \leq 1 \) in what follows. It should be pointed out that if \( 0 < \beta < 1 \), \( Lip_\beta(\mathbb{R}) = \dot{\lambda}_\beta(\mathbb{R}) \), where \( \dot{\lambda}_\beta(\mathbb{R}) \) is the homogeneous Besov-Lipschitz space; but if \( \beta = 1 \), \( Lip_\beta(\mathbb{R}) \subset \subset \dot{\lambda}_\beta(\mathbb{R}) \).
Theorem 2.5  Let k-th order commutator of the one-sided maximal operator be defined by
\[ M^{+,k}_b f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |b(x) - b(y)|^k |f(y)| dy. \]
Then the following conditions are equivalent:
(i) $M^{+,k}_b$ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$ for pairs $(p, q)$, such that $1 < p < q < \infty, 1/p - 1/q = k\beta$ and $\omega \in A^+(p, q)$.
(ii) $M^{+,k}_b$ is bounded from $L^p(dx)$ to $L^q(dx)$ for some pair $(p, q)$, such that $1 < p < q < \infty, 1/p - 1/q = k\beta$.
(iii) $b \in \text{Lip}^\beta(\mathbb{R})$.

Theorem 2.6  Let k-th order commutator of the one-sided fractional maximal operator be defined by
\[ M^{+,k}_{\alpha,b} f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int_x^{x+h} |b(x) - b(y)|^k |f(y)| dy. \]
Then the following conditions are equivalent:
(i) $M^{+,k}_{\alpha,b}$ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$ for pairs $(p, q)$, such that $1 < p < q < \infty, 1/p - 1/q = \alpha + k\beta$ and $\omega \in A^+(p, q)$.
(ii) $M^{+,k}_{\alpha,b}$ is bounded from $L^p(dx)$ to $L^q(dx)$ for some pair $(p, q)$, such that $1 < p < q < \infty, 1/p - 1/q = \alpha + k\beta$.
(iii) $b \in \text{Lip}^\beta(\mathbb{R})$.

Similarly, it is not difficult to prove strong type inequalities with pairs of related weights for commutators of one-sided singular integral (given by a Calderón-Zygmund kernel with support in $(-\infty, 0)$, see [10]) and the weyl fractional integral.

3 Proof of main results

In order to prove our results, let us first introduce some lemmas and notations.

Lemma 3.1[18]  For any $x, y \in \mathbb{R}$, if $f \in \text{Lip}_\beta(\mathbb{R})$, $0 < \beta < 1$, then
\[ |f(x) - f(y)| \leq |x - y|^{\beta} \|f\|_{\text{Lip}_\beta}, \]
and given any interval $I$ in $\mathbb{R}$, there is
\[ \sup_{x \in I} |f(x) - f_I| \leq C|I|^{\beta} \|f\|_{\text{Lip}_\beta}, \]
if $I^* \subset I$, then
\[ |f_{I^*} - f_I| \leq C\|f\|_{\text{Lip}_\beta}|I|^{\beta}, \]
where
\[ f_I = \frac{1}{|I|} \int_I f. \]
Lemma 3.2 For $0 < \beta < 1$, $1 \leq q < \infty$, we have
\[
\|f\|_{Lip_{\beta}} \approx \sup_{I} \frac{1}{|I|^{1+\beta}} \int_{I} |f - f_{I}| \approx \sup_{I} \frac{1}{|I|^{1+\beta}} \left( \frac{1}{|I|} \int_{I} |f - f_{I}|^{q} \right)^{1/q},
\]
for $q = \infty$ the formula should be interpreted appropriately.

The main tool for proving our results is an extrapolation theorem that appeared in [12], with slight modifications.

Lemma 3.3 Let $1 < p_{0} < \infty$ and $T$ be a sublinear operator defined in $C_{c}^{\infty}$. Assume that for all $\omega \in A^{+}(p_{0}, \infty)$ there exists $C = C(\omega)$ such that
\[
\|\omega Tf\|_{\infty} \leq C\|f\omega\|_{p_{0}}.
\]
Then for all pairs $(p, q)$ such that $1 < p < p_{0}$, $1/p - 1/q = 1/p_{0}$ and all $\omega \in A^{+}(p, q)$, there exists $C = C(\omega)$ such that
\[
\|\omega Tf\|_{q} \leq C\|f\omega\|_{p},
\]
provided the left hand side is finite.

We will also need the following result of Martín-Reyes and de la Torre (theorem 4 in [16]):

Lemma 3.4 Let $1 < p < \infty$. If $\omega \in A_{p}^{+}$ and $M^{+}f \in L^{p}(\omega)$, then there exists $C = C(\omega)$ such that
\[
\int_{\mathbb{R}} (M^{+}f)^{p} \omega \leq C \int_{\mathbb{R}} (f^{z^{+}})^{p} \omega,
\]
where
\[
f^{z^{+}}(x) = \sup_{h > 0} \frac{1}{h} \int_{x}^{x+h} \left( f(y) - \frac{1}{h} \int_{x}^{x+2h} f \right)^{+} dy
\]
and $z^{+} = \max(z, 0)$.

It is proved in [16] that
\[
f^{z^{+}}(x) \leq \sup_{h > 0, a \in \mathbb{R}} \frac{1}{h} \int_{x}^{x+h} (f(y) - a)^{+} dy + \frac{1}{h} \int_{x}^{x+2h} (a - f(y))^{+} dy \leq C\|f\|_{BMO}.
\]

Lemma 3.5 Let $\omega \in A_{1}^{+}$. Then there exists $s > 1$ such that $\omega^{r} \in A_{1}^{r}$, for all $r$ such that $1 < r \leq s$. Let $\omega \in A^{+}(p, q)$. Then $\omega^{q} \in A_{p}^{q}$ and $\omega^{p} \in A_{p}^{p}$, where $1 < p < q < \infty$.

Applying Hölder’s inequality in the definition of $A^{+}(p, q)$, we can get the following Lemma.
Lemma 3.6 Let \( \omega \in A^+(p, q) \). Then \( \omega \in A^+(p_0, q) \) and \( \omega \in A^+(p, p_0) \), where \( 1 < p < p_0 < q < \infty \).

Proof of Theorem 2.4. Let \( \omega^q \in A^+_q \). By Lemma 3.4, we have
\[
\int_{\mathbb{R}} |S^j b f|^q \omega^q \leq C \int_{\mathbb{R}} |M^+(S^j b f)|^q \omega^q \leq C \int_{\mathbb{R}} |(S^j b f)^{1+}|^q \omega^q.
\]
To prove the theorem for any \( b \in Lip_\beta \), we proceed in the same way as in [11]. We will control \( (S^j b f)^{1+} \) by some one-sided maximal operators. Using Lemma 3.3, we shall prove that they are bounded from \( L^p(\omega^p) \) to \( L^q(\omega^q) \).

Let \( \lambda \) be an arbitrary constant. Then \( b(x) - b(y) = (b(x) - \lambda) - (b(y) - \lambda) \) and
\[
S^j b f(x) = \left\| \int_{\mathbb{R}} (b(x) - b(y))^k H(x - y) f(y) dy \right\|_2
\]
\[
= \left\| \sum_{j=0}^k C_{j,k}(b(x) - \lambda)^j \int_{\mathbb{R}} (b(y) - \lambda)^{k-j} H(x - y) f(y) dy \right\|_2
\]
\[
\leq \left\| \int_{\mathbb{R}} (b(y) - \lambda)^k H(x - y) f(y) dy \right\|_2
\]
\[
+ \left\| \sum_{j=1}^k C_{j,k}(b(x) - \lambda)^j \int_{\mathbb{R}} (b(y) - \lambda)^{k-j} H(x - y) f(y) dy \right\|_2
\]
\[
\leq S((b - \lambda)^k f)(x)
\]
\[
+ \sum_{j=1}^k \sum_{s=0}^{s+j} C_{j,k,s}(b(x) - \lambda)^{s+j} \int_{\mathbb{R}} (b(x) - b(y))^{k-j-s} H(x - y) f(y) dy \right\|_2
\]
\[
\leq S((b - \lambda)^k f)(x) + \sum_{m=0}^{k-1} C_{k,m} b(x) - \lambda|^{k-m} S^m b f(x),
\]
where \( C_{j,k} \) (respectively \( C_{j,k,s} \)) are absolute constants depending only on \( j \) and \( k \) (respectively \( j, k \) and \( s \)). Let \( x \in \mathbb{R}, h > 0 \). Let \( i \in \mathbb{Z} \) be such that \( 2^i \leq h < 2^{i+1} \) and set \( J = [x, x + 2^{i+3}] \). Then, write \( f = f_1 + f_2 \), where \( f_1 = f \chi_J \) and set \( \lambda = b_J \). Then
\[
\frac{1}{h} \int_{x}^{x+h} |S^j b f(y) - S((b - b_J)^k f_2)(x)| dy
\]
\[
\leq \frac{1}{h} \int_{x}^{x+h} |S((b - b_J)^k f_1)(y)| dy
\]
\[
+ \frac{1}{h} \int_{x}^{x+h} |S((b - b_J)^k f_2)(y) - S((b - b_J)^k f_2)(x)| dy
\]
\[
+ \sum_{m=0}^{k-1} C_{k,m} \frac{1}{h} \int_{x}^{x+h} |b(y) - b_J|^{k-m} |S^m b f(y)| dy
\]
\[
= I(x) + II(x) + III(x).
\]
For $II(x)$, we have

$$II(x) \leq \frac{1}{h} \int_x^{x+2h+3} ||U^+((b-b_J)^k f_2)(y) - U^+((b-b_J)^k f_2)(x)||_2 dy,$$

and

$$||U^+((b-b_J)^k f_2)(y) - U^+((b-b_J)^k f_2)(x)||_2 \leq \int_{x+2h+3}^{\infty} |b(t)-b_J|^k |f(t)||H(y-t)-H(x-t)||_2 dt.$$

Consider the following sublinear operators defined in $C_c^\infty$:

$$M_1^+ f(x) = \sup_{i \in \mathbb{Z}} \frac{1}{2^i} \int_x^{x+2i+2} |S((b-b_J)^k f \chi_J)(y)| dy;$$

$$M_2^+ f(x) = \sup_{i \in \mathbb{Z}} \frac{1}{2^i} \int_x^{x+2i+3} \int_{x+2i+3}^{\infty} |b(t)-b_J|^k |f(t)||H(y-t)-H(x-t)||_2 dt dy;$$

and

$$M_{3,m}^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+2h} |b(y) - b_{[x,x+8h]}|^k |f(y)| dy, \quad 0 \leq m \leq k-1, k \geq 1.$$

The above definitions give that

$$(S_b^k f)^{\leq+} \leq C \left(M_1^+ f(x) + M_2^+ f(x) + \sum_{m=0}^{k-1} M_{3,m}^+ (S_b^m f(x))\right).$$

We shall prove, using Lemma 3.3, that these operators are bounded from $L^p(\omega^q)$ to $L^q(\omega^p)$, $\omega \in A^+(p,q)$, $1 < p < q < \infty$, $1/p - 1/q = k\beta$.

Boundedness of $M_1^+$: Let $\omega \in A^+(1/k\beta, \infty)$, then $\omega^{-1/(1-k\beta)} \in A_1^{-}$. Therefore, there exists $t > 1$ such that $\omega^{-t/(1-k\beta)} \in A_1^{-}$. Let $s > 1, r > 1$ be such that $s = t/(1-k\beta)$ and $1/r - 1/s = k\beta$. Then, using Hölder’s inequality and the fact that $S$ maps $L^r(\mathbb{R})$ into itself, we get

$$\frac{1}{2^i} \int_x^{x+2i+2} |S((b-b_J)^k f \chi_J)(y)| dy$$

$$\leq \left(\frac{1}{2^i} \int_x^{x+2i+2} |S((b-b_J)^k f \chi_J)(y)|^r dy\right)^{1/r}$$

$$\leq \left(\frac{1}{2^i} \int_x^{x+2i+3} |(b-b_J)^k f(y)|^r dy\right)^{1/r}$$

$$\leq C \sup_{y \in J} |b(y) - b_J|^k \left(\frac{1}{2^i} \int_x^{x+2i+3} |f(y)|^r dy\right)^{1/r}.$$
by Hölder’s inequality, we get

\[ C^{2i\beta} ||b||_{Lip_\beta} \left( \frac{1}{2^i} \int_{x^{2i+3}} \omega^{r'} \omega^{-s} dy \right)^{1/r} \]

\[ \leq C^{2i\beta} ||b||_{Lip_\beta} \left( \frac{1}{2^i} \int_{x^{2i+3}} |f(y)|^{1/k\beta} dy \right) k\beta \left( \frac{1}{2^i} \int_{x^{2i+3}} \omega^{-s} dy \right)^{1/s} \]

\[ \leq C ||b||_{Lip_\beta} ||f\omega||_{1/k\beta\omega^{-1}(x)} \]

The last inequality is deduced by the fact \( \omega^{-s} \in A_1^- \).

As a consequence,

\[ ||\omega M^+_1 f||_\infty \leq C ||b||_{Lip_\beta} ||f\omega||_{1/k\beta}. \]

Then, by Lemma 3.3, for all \( \omega \in A^+(p, q) \), \( 1/p - 1/q = k\beta \),

\[ ||M^+_1 f||_{\omega^q, q} \leq C ||b||_{Lip_\beta} ||f\omega||_{p, p}. \]

Boundedness of \( M^+_2 \): Set \( I_j = [x, x + 2^{j+1}] \), we have that

\[ \int_{x^{2i+3}}^{\infty} |b(t) - b_{I_j}|^k |f(t)||H(y - t) - H(x - t)||_2 dt \]

\[ \leq C \sum_{j=1}^{\infty} \int_{x^{2i+3}}^{\infty} |b(t) - b_{I_j}|^k |f(t)||H(y - t) - H(x - t)||_2 dt \]

\[ + C \sum_{j=1}^{\infty} |b_{I_j} - b_{I_j}|^k \int_{x^{2i+3}}^{\infty} |f(t)||H(y - t) - H(x - t)||_2 dt \]

\[ = II_1(x) + II_2(x). \]

We proceed in the same way as in the estimates of \( M^+_1 \), choose \( r' \) such that \( 1/r + 1/r' = 1 \), by Hölder’s inequality, we get

\[ II_1(x) \leq C \sum_{j=1}^{\infty} \left( \int_{I_j} |(b - b_{I_j})^k f|^r \right)^{1/r} \left( \int_{x^{2i+3}}^{\infty} ||H(y - t) - H(x - t)||_{2}^{r'} dt \right)^{1/r'} \]

\[ \leq C \sum_{j=1}^{\infty} \sup_{I_j} |b - b_{I_j}|^k \left( \int_{I_j} |f|^r \right)^{1/r} \left( \int_{x^{2i+3}}^{\infty} ||H(y - t) - H(x - t)||_{2}^{r'} dt \right)^{1/r'} \]

\[ \leq C ||b||_{Lip_\beta} ||f\omega||_{1/k\beta\omega^{-1}(x)} \sum_{j=1}^{\infty} 2^{j/r} \left( \int_{x^{2i+3}}^{\infty} ||H(y - t) - H(x - t)||_{2}^{r'} dt \right)^{1/r'}. \]

It is proved in theorem 1.6 of [23] that for all \( y \in [x, x + 2^{i+3}] \) the kernel \( H \) satisfies

\[ \left( \int_{x^{2i+3}}^{\infty} ||H(y - t) - H(x - t)||_{2}^{r'} dt \right)^{1/r'} \leq C \frac{2^{i/r'}}{2^i}. \]
Then we get
\[
I_1(x) \leq C\|b\|_{Lip_p}^k \|f\omega\|_{1/k\beta}\omega^{-1}(x) \sum_{j=i+3}^{\infty} \left( \frac{2^j}{2^i} \right)^{1/r'}
\leq C\|b\|_{Lip_p}^k \|f\omega\|_{1/k\beta}\omega^{-1}(x).
\]

Observe that \(|b_{I_j} - b_I| \leq C 2^{j/\beta} \|b\|_{Lip_p}^\beta\), similar to the estimates of \(I_1(x)\), we can get
\[
I_2(x) \leq C\|b\|_{Lip_p}^k \|f\omega\|_{1/k\beta}\omega^{-1}(x).
\]
As a consequence,
\[
\|\omega M_2^+ f\|_\infty \leq C\|b\|_{Lip_p}^k \|f\omega\|_{1/k\beta}.
\]

Then, by Lemma 3.3, for all \(\omega \in A^+(p, q)\), \(1/p - 1/q = k\beta\),
\[
\|M_2^+ f\|_{\omega^q, q} \leq C\|b\|_{Lip_p}^k \|f\|_{\omega^q, q}.
\]

**Boundedness of \(M_3^+\):** We shall prove that \(M_3^+\) are bounded from \(L^{p_0}(\omega^{p_0})\) to \(L^q(\omega^q)\), \(\omega \in A^+(p_0, q)\), \(1 < p < p_0 < q < \infty\), \(1/p_0 - 1/q = (k - m)\beta\).

Let \(\omega \in A^+(\frac{1}{(k-m)\beta}, \infty)\), then \(\omega^{-1/(1-(k-m)\beta)} \in A_1^{-}\). Therefore, there exists \(t_0 > 1\) such that \(\omega^{-t_0/(1-(k-m)\beta)} \in A_1^{-}\). Let \(s_0 > 1, r_0 > 1\) be such that \(s_0 = t_0/(1 - (k - m)\beta)\) and \(1/r_0 - 1/s_0 = (k - m)/\beta\). Then, using Hölder’s inequality, we get
\[
\frac{1}{h} \int_x^{x+2h} |b(y) - b_{[x,x+8h]}|^{k-m} |f(y)| dy \\
\leq \left( \frac{1}{h} \int_x^{x+2h} |f(y)|^{\alpha_0} \omega^{\alpha_0} \omega^{-\alpha_0} dy \right)^{1/r_0} \left( \frac{1}{h} \int_x^{x+2h} |b(y) - b_{[x,x+8h]}|^{(k-m)r_0'} dy \right)^{1/r_0'} \\
\leq C h^{(k-m)\beta} \|b\|_{Lip_p}^{k-m} \left( \frac{1}{h} \int_x^{x+2h} |f(y)| \omega^{1/(k-m)\beta} dy \right)^{(k-m)\beta} \left( \frac{1}{h} \int_x^{x+2h} \omega^{-s_0} dy \right)^{1/s_0} \\
\leq C\|b\|_{Lip_p}^{k-m} \|f\omega\|_{1/(k-m)\beta}\omega^{-1}(x).
\]
The last inequality is deduced by the fact \(\omega^{-s_0} \in A_1^{-}\).

As a consequence,
\[
\|\omega M_3^+ f\|_\infty \leq C\|b\|_{Lip_p}^{k-m} \|f\omega\|_{1/(k-m)\beta}.
\]

Then, by Lemma 3.3, for all \(\omega \in A^+(p_0, q)\), \(1/p_0 - 1/q = (k - m)\beta\),
\[
\|M_3^+ f\|_{\omega^q, q} \leq C\|b\|_{Lip_p}^{k-m} \|f\|_{\omega^{p_0}, p_0}.
\]
Specially, when \(k = 1\), by all the above estimates, we can deduce
\[
\|S_b f\|_{\omega^q, q} \leq C\|b\|_{Lip_p} \|f\|_{\omega^q, 1}.
\]
where $\omega \in A^+(p_1, q)$, $1/p_1 - 1/q = \beta$.

Using the induction principle, let $\omega \in A^+(p, p_0)$, $1 < p < p_0 < \infty$, $1/p - 1/p_0 = m\beta$. Then $1/p - 1/q = k\beta$, $1 < p < q < \infty$, by Lemma 3.6, we get that, for all $\omega \in A^+(p, q)$,

$$\|M^+_{\beta,m}(S^n_b f)\|_{\omega,q} \leq C\|b\|_{\operatorname{Lip}}^{k-m} \|S^n_b f\|_{\omega,p_0} \leq C\|b\|_{\operatorname{Lip}}^{k} \|f\|_{\omega,p}.$$  

**Proof of Theorem 2.5.**

(iii)$\Rightarrow$(i) By Lemma 3.1, we have

$$\frac{1}{h} \int_x^{x+h} |b(x) - b(y)|^k |f(y)|dy \leq C \|b\|_{\operatorname{Lip}}^{k} \int_x^{x+h} |f(y)|dy.$$  

Using the fact that $M^+_{k\beta}$ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$ if and only if $\omega \in A^+(p, q)$, for $1 < p < q$, $1/p - 1/q = k\beta$, we can get the desired result.

(i)$\Rightarrow$(ii) Given an appropriate pair $(p, q)$, set $\omega \equiv 1$.

(ii)$\Rightarrow$(iii) Set $I = (a, b)$, $I^+ = (b, c)$, and $|I| = |I^+|$. Then

$$\frac{1}{|I|^{1+\beta}} \int_I |b(y) - b_I|dy \leq \frac{C}{|I|^{1+\beta}} \int_I |b(y) - b_{I^+}|dy$$

$$\leq \frac{C}{|I|^{\beta}} \left( \frac{1}{|I|} \int_I |b(y) - b_I| dy \right)^{1/k}$$

$$\leq \frac{C}{|I|^{\beta}} \left( \frac{1}{|I|} \int_I \left| \int_{I^+} (b(y) - b(x))dx \right|^k dy \right)^{1/k}$$

$$\leq \frac{C}{|I|^{\beta}} \left( \frac{1}{|I|} \int_I \left( \frac{1}{|I^+|} \int_{I^+} |b(y) - b(x)|^k dx \right) dy \right)^{1/k}.$$  

Observe that, for $y \in I$,

$$\frac{1}{|I^+|} \int_{I^+} |b(y) - b(x)|^k dx = \frac{1}{|I^+|} \int_y^c |b(y) - b(x)|^k \chi_{I^+}(x)dx \leq CM_{I^+} b^+ \chi_I^+ (y).$$  

Then by Hölder’s inequality and (ii),

$$\frac{1}{|I|^{1+\beta}} \int_I |b(y) - b_I|dy \leq \frac{C}{|I|^{\beta}} \left( \frac{1}{|I|} \int_I M_{I^+} b^+ \chi_I^+ (y)dy \right)^{1/k}$$

$$\leq \frac{C}{|I|^{\beta}} \left( \frac{1}{|I|} \int_I |M_{I^+} b^+ \chi_I^+ (y)|^q dy \right)^{1/qk}$$

$$\leq \frac{C}{|I|^{\beta}} \left( \int_{\mathbb{R}} |\chi_I^+ (y)|^pdy \right)^{1/pk}$$

$$\leq C \frac{|I|^{1/qk}}{|I|^{\beta+1/qk}} = C.$$  

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So, by Lemma 3.2, we get $b \in \text{Lip}_\beta$.

Similar to Theorem 2.5, we can finish the proof of Theorem 2.6 easily. We omit the details here.

References


Zunwei Fu
Department of Mathematics
Linyi Normal University,
Linyi Shandong, 276005,
P.R. China
e-mail: lyfzw@tom.com